Gromov hyperbolicity of planar graphs

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Abstract: We prove that under appropriate assumptions adding or removing an infinite amount of edges to a given planar graph preserves its non-hyperbolicity, a result which is shown to be false in general. In particular, we make a conjecture that every tessellation graph of $\mathbb{R}^2$ with convex tiles is non-hyperbolic; it is shown that in order to prove this conjecture it suffices to consider tessellation graphs of $\mathbb{R}^2$ such that every tile is a triangle and a partial answer to this question is given. A weaker version of this conjecture stating that every tessellation graph of $\mathbb{R}^2$ with rectangular tiles is non-hyperbolic is given and partially answered. If this conjecture were true, many tessellation graphs of $\mathbb{R}^2$ with tiles which are parallelograms would be non-hyperbolic.

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1. Introduction

Hyperbolic spaces play an important role in geometric group theory and in the geometry of negatively curved spaces. The concept of Gromov hyperbolicity grasps the essence of both negatively curved spaces like the classical hyperbolic space or Riemannian manifolds of negative sectional curvature, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. It is remarkable that a simple concept leads to such rich general theory [1, 16, 17].

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The theory of Gromov spaces was initially for the study of finitely generated groups (see [17] and the references therein), where its practical importance was discussed. This theory was mainly applied to the study of automatic groups [31], which appear in computational science. The concept of hyperbolicity appears also in discrete mathematics, in particular, a number of algorithmic problems in hyperbolic spaces and hyperbolic graphs has been considered in recent papers [12, 13, 15, 28]. Another application of these spaces is secure transmission of information on the internet [21–23], playing a significant role in the spread of viruses through the network [21, 23]. It has been shown empirically in [42] that the internet topology embeds with better accuracy into a hyperbolic space than into an Euclidean space of comparable dimension. Hyperbolicity is also useful in the study of DNA data [7].

The study of mathematical properties of Gromov hyperbolic spaces and its applications is a topic of recent and increasing interest in graph theory; see, for instance [3–5, 7, 8, 10, 14, 21–25, 27, 29, 30, 33–36, 39–41, 43, 44].

In recent years several researchers have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For instance, the Gehring–Osgood $j$-metric is Gromov hyperbolic; and the Vuorinen $j$-metric is not Gromov hyperbolic except in the punctured space [18]. The study of Gromov hyperbolicity of the quasihyperbolic and the Poincaré metrics is the subject of [2, 6, 19, 20, 36–38, 40, 41]. In particular, in [36, 40, 41, 43] it is proved the equivalence of the hyperbolicity of many negatively curved surfaces and the hyperbolicity of a very simple graph. Deciding whether a space is hyperbolic is a difficult problem since the location of geodesics is unknown, and hence, it is useful to know hyperbolicity criteria for graphs. This will be the topic of our further discussion.

One of the main questions in the study of any mathematical property is to find transformations which preserve that property. In [8, Theorem 3.15] the authors prove that adding or removing any finite amount of edges of a graph preserves its non-hyperbolicity (or hyperbolicity). It is thus natural to consider what happens if the amount of edges is infinite. Theorem 3.1 below gives a positive answer to this question under some appropriate hypotheses for planar graphs; Theorem 3.6 shows that the general answer is negative, even for planar graphs.

The papers [9, 35] study the hyperbolicity of some type of planar graphs. In particular, in [9], the authors conjectured that every tessellation graph of $\mathbb{R}^2$ with convex tiles is non-hyperbolic. Sections 4 and 5 deal with this open problem. Theorem 5.1 shows that in order to prove this conjecture, it suffices to consider tessellation graphs of $\mathbb{R}^2$ such that every tile is a triangle. A weaker conjecture is stated here, namely that every tessellation graph of $\mathbb{R}^2$ with rectangular tiles is non-hyperbolic. Theorem 4.6 gives a partial answer to this question. Finally, Theorem 4.2 shows that if this weaker conjecture is true, then any tessellation graphs of $\mathbb{R}^2$ with tiles which are parallelograms are non-hyperbolic.

2. Background on Gromov hyperbolic spaces

Let $(X, d)$ be a metric space and let $y : [a, b] \to X$ be a continuous function. The curve $y$ is a geodesic if $L(y)_{[s, t]} = d(y(s), y(t))$ for every $s, t \in [a, b]$, where $L$ denotes the length of a curve; a geodesic line is a geodesic with domain $\mathbb{R}$, and a geodesic ray is a geodesic with domain $[0, \infty)$. $X$ is a geodesic metric space if for every $x, y \in X$ there exists a geodesic joining $x$ and $y$; denote by $[xy]$ any of such geodesics (since uniqueness of geodesics is not required, this notation is ambiguous, but convenient). It is clear that every geodesic metric space is path-connected. If the metric space $X$ is a graph, $[u, v]$ denotes the edge joining the vertices $u$ and $v$.

In order to consider a graph $G$ as a geodesic metric space, one must identify any edge $[u, v] \in E(G)$ with the real interval $[0, l]$ (if $l = L([u, v])$); therefore, any point in the interior of any edge is a point of $G$ and, if the edge $[u, v]$ is considered as a graph with just one edge, then it is isometric to $[0, l]$. A connected graph $G$ is naturally equipped with a distance defined on its points, induced by taking shortest paths in $G$, inducing in $G$ the structure of a metric graph. Note that edges can have arbitrary lengths.

Throughout the paper only simple, connected and locally finite graphs are considered (i.e., graphs without loops or multiple edges and so that each ball contains a finite number of edges); these properties guarantee graphs are geodesic metric spaces. The study of the hyperbolicity of graphs with loops and multiple edges can be reduced to the study of the hyperbolicity of simple graphs [5, Theorems 8 and 10].

If $X$ is a geodesic metric space and $J = \{J_1, J_2, \ldots, J_n\}$ is a polygon with sides $J_i \subseteq X$, then $J$ is said to be $\delta$-thin if for every $x \in J_i$ one has that $d\left(x, \bigcup_{j \neq i} J_j\right) \leq \delta$. The sharp thin constant of $J$, $\delta(J)$, is then $\delta(J) = \inf\{\delta \geq 0 : J \text{ is } \delta\text{-thin}\}$. If $x_1, x_2, x_3$ are points in $X$, a geodesic triangle $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2], [x_2x_3]$ and $[x_3x_1]$. 

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The space $X$ is $\delta$-hyperbolic (or satisfies the Rips condition with constant $\delta$) if every geodesic triangle in $X$ is $\delta$-thin. Denote by $\delta(X)$ the sharp hyperbolicity constant of $X$, i.e., $\delta(X) = \sup \{ \delta(T) : T$ is a geodesic triangle in $X \}$. The space $X$ is hyperbolic if $X$ is $\delta$-hyperbolic for some $\delta > 0$; in this case, $\delta(X) = \inf \{ \delta \geq 0 : X$ is $\delta$-hyperbolic $\}$.

Trivially, every bounded metric space $X$ is $\delta$-hyperbolic for some $\delta > 0$; every simply connected complete Riemannian manifold with sectional curvature verifying $K \leq -e^2 < 0$ is hyperbolic. More background and further results are given, e.g., [1, 16]. The spaces $X$ with $\delta(X) = 0$ are precisely the metric trees, and the hyperbolicity constant of a geodesic metric space can be viewed as a measure of how "tree-like" the space is.

There are several definitions of Gromov hyperbolicity, all equivalent in the sense that if $X$ is $\delta$-hyperbolic with respect to definition $A$, then it is $\delta'$-hyperbolic with respect to definition $B$ for some $\delta'$, see, e.g., [1, 16].

Let $(X, d_X)$ and $(Y, d_Y)$ be two metric spaces. A map $f : X \to Y$ is said to be an $((\alpha, \beta))$-quasi-isometric embedding, with constants $\alpha \geq 1$, $\beta \geq 0$ if for every $x, y \in X$:

$$\alpha^{-1} d_X(x, y) - \beta \leq d_Y(f(x), f(y)) \leq \alpha d_X(x, y) + \beta.$$

The function $f$ is $\varepsilon$-full if for each $y \in Y$ there exists $x \in X$ with $d_Y(f(x), y) \leq \varepsilon$. A map $f : X \to Y$ is said to be a quasi-isometry, if there exist constants $\alpha \geq 1$, $\beta \geq 0$ such that $f$ is an $\varepsilon$-full $((\alpha, \beta))$-quasi-isometric embedding. In that case we say that $X$ and $Y$ are quasi-isometric. Note that a quasi-isometric embedding, in general, is not continuous. Let $X$ be a metric space, $Y$ a non-empty subset of $X$ and $\varepsilon$ a positive number. The $\varepsilon$-neighborhood of $Y$ in $X$, denoted by $Y_\varepsilon (Y)$ is defined as the set $\{ x \in X : d_X(x, y) \leq \varepsilon \}$.

A fundamental property of hyperbolic spaces is the following

**Theorem 2.1 (invariance of hyperbolicity).**

Let $f : X \to Y$ be an $((\alpha, \beta))$-quasi-isometric embedding between the geodesic metric spaces $X$ and $Y$. If $Y$ is hyperbolic, then $X$ is hyperbolic. Besides, if $f$ is $\varepsilon$-full for some $\varepsilon \geq 0$ (a quasi-isometry), then $X$ is hyperbolic if and only if $Y$ is hyperbolic.

If $D$ is a closed subset of $X$, the inner metric considered in $D$ is defined as

$$d_\delta(z, w) = \inf \{ L_\chi(y) : y \in D$ is a continuous curve joining $z$ and $w \} \geq d_\chi(z, w).$$

Consequently, $L_\delta(y) = L_\chi(y)$ for every curve $y \in D$.

In an informal way, a tessellation, $T$, on a complete Riemannian surface, $X$, is a partition of $X$ by geometric shapes (called tiles) with no overlaps and no gaps. The tessellation graph associated to $T$ is the union of the boundaries of the tiles. More precisely, for $n \geq 1$, an $n$-cell is a topological space homeomorphic to the open ball in $\mathbb{R}^n$. A 0-cell is a singleton space. A tessellation on a complete Riemannian surface, $X$, is a CW 2-complex on $X$ such that every point on $X$ is contained in some $n$-cell of the complex for some $n \in \{ 0, 1, 2 \}$. A tessellation graph is the 1-skeleton (the set of 0-cells and 1-cells). The edges (1-cells) of a tessellation graph are just rectifiable paths in $X$ and have the length induced by the metric on $X$ (these paths may or may not be geodesics in $X$). Throughout the paper $X = \mathbb{R}^2$ with the exceptions of Theorem 3.4 and the proof of Theorem 3.6, where $X$ will stand for the hyperbolic plane.

Along the paper, given a set $E$ contained in a Riemannian surface $X$, we denote by $A_X(E)$ its area and by $\overline{E}$ its closure.

### 3. Hyperbolicity of tessellation graphs

If $G_0$ is a non-hyperbolic tessellation graph of $\mathbb{R}^2$, a natural question is whether this non-hyperbolicity will be preserved when adding to it any number (possibly infinite) of vertices and edges. Theorem 3.1 gives an affirmative answer to this question under some regularity hypotheses on $G_0$ (Theorem 3.6 below will show this result to be false in general). It will also show a connection between the continuous and discrete frames, see e.g., [26].
Theorem 3.1.

Let $G_0$ be the 1-skeleton of a tessellation of $\mathbb{R}^2$ with tiles $\{F_n\}_{n \in \mathbb{N}}$. Assume that there exists a partition $I = \Lambda_1 \cup \Lambda_2$ of the set of indices and positive constants $c_1, c_2$, verifying the following properties:

(i) $\text{diam}_{G_0}(\partial F_n) \leq c_1$ and $A_{G_0}(F_n) \geq c_2$ for every $n \in \Lambda_1$,
(ii) $d_{F_n}(x, y) \leq c_1 d_{F_n}(x, y)$ for every $x, y \in \partial F_n$ and for every $n \in \Lambda_2$.

Then $G_0$ is not hyperbolic. Moreover, any 1-skeleton $G$ of a tessellation of $\mathbb{R}^2$ which contains $G_0$ as a subgraph is not hyperbolic.

Proof. It will be proven that the inclusion $i : G_0 \rightarrow \mathbb{R}^2$ is a quasi-isometric embedding; in fact, it is shown that

$$d_{\mathbb{R}^2}(x, y) \leq d_{G_0}(x, y) \leq \left(2c_1^2 + c_1\right)d_{\mathbb{R}^2}(x, y) + \pi c_1^2$$

for every $x, y \in G_0$.

First of all, it is clear that $d_{\mathbb{R}^2}(x, y) = d_{\mathbb{R}^2}((x), (y)) \leq d_{G_0}(x, y)$ for every $x, y \in G_0$. Fix now $x, y \in G_0$ and let $\sigma$ be the Euclidean segment joining $x$ and $y$ in $\mathbb{R}^2$. If $n \in \Lambda_1$ and $F_n \cap \sigma \neq \emptyset$, then $F_n \subseteq \mathcal{V}_{\text{diam}}(\sigma) \subseteq \mathcal{V}_1(\sigma)$. Since $\mathcal{V}_1(\sigma)$ is the union of two half-disks and a rectangle, clearly, $A_{\mathbb{R}^2}(\mathcal{V}_1(\sigma)) = 2c_1 L(\sigma) + \pi c_1^2$. Let $\mathcal{N}(\sigma)$ denote the number of $F_n$ with $n \in \Lambda_1$ that crosses $\sigma$, then

$$c_2 \mathcal{N}(\sigma) \leq A_{\mathbb{R}^2}(\mathcal{V}_1(\sigma)) = 2c_1 L(\sigma) + \pi c_1^2.$$

Therefore,

$$\mathcal{N}(\sigma) \leq c_2^{-1}\left(2c_1 d_{\mathbb{R}^2}(x, y) + \pi c_1^2\right).$$

Consider $\sigma$ as an oriented segment from $x$ to $y$. A finite set of points will be inductively defined as follows: let $y_1$ be the first point on $\sigma$ with $y_1 \in \bigcup_{n \in \Lambda_1} F_n$; then $y_1 \in F_{r_1}$ for some $r_1 \in \Lambda_1$; take $y_2$ to be the last point on $\sigma \cap F_{r_1}$. Proceeding this way, assume that $\{y_1, \ldots, y_{j-1}\}$ have been defined with $y_{2s-1}$ the first point and $y_{2s}$ the last on $\sigma \cap F_{r_s}$ for $s = 1, \ldots, j$. If $\sigma \setminus [y_1, y_2]$ does not intersect $\bigcup_{n \in \Lambda_1 \setminus \{r_1, \ldots, r_j\}} F_n$, then this process is stopped. If $\sigma \setminus [y_1, y_2]$ intersects $\bigcup_{n \in \Lambda_1 \setminus \{r_1, \ldots, r_j\}} F_n$, then define $y_{2j+1} = \pi/c_{\mathbb{R}^2}$ as the first point in $(\sigma \setminus [y_1, y_2]) \cup \{y_2\}$ with $y_{2j+1} \in \bigcup_{n \in \Lambda_1 \setminus \{r_1, \ldots, r_j\}} F_n$, then $y_{2j+1} \in F_{r_{j+1}}$ for some $r_{j+1} \in \Lambda_1$ and define $y_{2j+2}$ as the last point in $\sigma \cap F_{r_{j+1}}$. Eventually this process will finish and a finite set $\{y_1, \ldots, y_{2N}\}$ will be obtained.

Let $[x_1, x_2], [x_3, x_4], \ldots, [x_{2m-1}, x_{2m}]$ be the Euclidean segments contained in the closure of $\sigma \setminus \bigcup_{n \in \Lambda_1} [y_{2n-1}, y_{2n}]$. Notice that $[x_{2j-1}, x_{2j}] \subseteq \sigma \cap \bigcup_{n \in \Lambda_1} F_n$, and thus

$$d_{G_0}(x, y) \leq \sup_{n \in \Lambda_1} \left\{\text{diam}_{G_0}(\partial F_n)\right\} \mathcal{N}(\sigma) + \sum_{j=1}^{m} d_{G_0}(x_{2j-1}, x_{2j}) \leq c_1 \mathcal{N}(\sigma) + c_1 \sum_{j=1}^{m} d_{\mathbb{R}^2}(x_{2j-1}, x_{2j}) \leq \pi (2c_1^2 + c_1) d_{\mathbb{R}^2}(x, y) + \pi c_1^2,$$

which completes the proof of (1).

We shall show next that $G_0$ is not hyperbolic. To this end it will be first proven that

$$\delta(G_0) \geq \frac{1}{2} \sup_{\alpha} \text{diam}_{\mathbb{R}^2}(\partial F_n).$$

(2)

For any fixed $n$, let us consider the set $A_n$ of closed curves in $G_0$ freely homotopic to $\partial F_n$ in $\mathbb{R}^2 \setminus F_n$. Choose a closed curve $\alpha_n \in A_n$ with $L(\alpha_n) = \min \left\{L(\alpha) : \alpha \in A_n\right\}$; it is clear that $L(\alpha_n) \geq 2 \text{diam}_{\mathbb{R}^2}(\partial F_n)$ and that $d_{\mathbb{R}^2}(x, y) = d_{G_0}(x, y)$ for every $x, y \in \alpha_n$ since $\alpha_n$ is a shortcut curve in $A_n$. Let $x_n, y_n$ be points in $\alpha_n$ with $d_{\mathbb{R}^2}(x_n, y_n) = d_{\mathbb{R}^2}(x_n, y_n) = L(\alpha_n)/2$. Then there are two different geodesics $\alpha_n^1, \alpha_n^2$ in $G_0$ joining $x_n$ and $y_n$ with $\alpha_n^1 \cap \alpha_n^2 = \alpha_n$. Therefore the set $B_n = \{\alpha_n^1, \alpha_n^2\}$ is a geodesic bigon (a geodesic triangle having two of its vertices to be the same point). If $u_n$ is the midpoint of $\alpha_n^1$, then $\delta(B_n) \geq d_{G_0}(u_n, \alpha_n^1) = d_{G_0}(u_n, \{x_n, y_n\}) = L(\alpha_n)/4 \geq (1/2) \text{diam}_{\mathbb{R}^2}(\partial F_n)$. Taking the supremum on $n$, (2) follows.
If \( \sup_n \text{diam}_{\mathbb{R}^2} \partial F_n = \infty \), by (2), \( G_0 \) is not hyperbolic. If \( \sup_n \text{diam}_{\mathbb{R}^2} \partial F_n = c^*_1 < \infty \), then the inclusion \( i: G_0 \to \mathbb{R}^2 \) is a \( c^*_1 \)-full \( (a_0, b_0/a_0) \)-quasi-isometry, with \( a_0 = 2c^*_1 c_1^2 + c_1 > c_1 \geq 1 \) and \( b_0 = \pi c_1^2 c_2^2 \), since \( \text{diam}_{\mathbb{R}^2} F_n \leq \text{diam}_{G_0} \partial F_n \). In this case, Theorem 2.1 implies \( G_0 \) is not hyperbolic.

Let us finally show that any 1-skeleton \( G \) of a tessellation of \( \mathbb{R}^2 \) which contains \( G_0 \) as a subgraph will not be hyperbolic. Clearly, \( d_C(x, y) \leq d_{G_0}(x, y) \) for every \( x, y \in G_0 \), and also \( d_{\mathbb{R}^2}(x, y) \leq d_C(x, y) \) for every \( x, y \in G \). By (1),

\[
d_C(x, y) \leq d_{G_0}(x, y) \leq a_0 d_{\mathbb{R}^2}(x, y) + b_0 \leq a_0 d_C(x, y) + b_0
\]

or, equivalently,

\[
\frac{1}{a_0} d_{G_0}(x, y) - \frac{b_0}{a_0} \leq d_C(x, y) \leq d_{G_0}(x, y).
\]

for every \( x, y \in G_0 \). Thus the inclusion \( i_0: G_0 \to G \) is an \( (a_0, b_0/a_0) \)-quasi-isometric embedding. Therefore, since \( G_0 \) is not hyperbolic, by Theorem 2.1 one obtains that \( G \) is not hyperbolic. \( \square \)

The arguments just given in the proof of Theorem 3.1 have the following consequences.

**Theorem 3.2.**

Let \( G_0 \) be the 1-skeleton of a tessellation of \( \mathbb{R}^2 \) such that there exist non-negative constants \( a_0, b_0 \) so that \( d_{G_0}(x, y) \leq a_0 d_{\mathbb{R}^2}(x, y) + b_0 \) for every \( x, y \in G_0 \). Then any 1-skeleton \( G \) of a tessellation of \( \mathbb{R}^2 \) which contains \( G_0 \) as a subgraph is not hyperbolic.

**Theorem 3.3.**

Let \( G \) be the 1-skeleton of a tessellation of \( \mathbb{R}^2 \) with tiles \( \{F_n\} \). Then \( \delta(G) \geq (1/2) \sup_n \text{diam}_{\mathbb{R}^2} \partial F_n \).

A direct consequence of Theorem 3.3 is that, if \( \sup_n \text{diam}_{\mathbb{R}^2} \partial F_n = \infty \), then \( G \) is not hyperbolic. It will be shown in Theorem 3.6, one of the main results of this section, that if \( \sup_n \text{diam}_G \partial F_n = \infty \), this is false. The following results on hyperbolicity be needed there.

**Theorem 3.4 ([35, Theorem 3.1 and Remark 3.2]).**

Let \( G \) be the 1-skeleton of a tessellation of the hyperbolic plane \( \mathbb{H} \) with tiles \( \{F_n\} \). If for some positive constants \( c_1, c_2 \), one has \( \text{diam}_G \partial F_n \leq c_1 \) and \( A_{\mathbb{H}}(F_n) \geq c_2 \) for every \( n \), then \( G \) is hyperbolic.

Let us denote by \( G \setminus \{v\} \) the metric space obtained by removing the point \( \{v\} \) from the metric space \( G \). A vertex \( v \) of a graph \( G \) is a cut vertex if \( G \setminus \{v\} \) is not connected. Note that in a tree, any vertex with degree greater than one is a cut vertex. Finally, let us denote by \( \{G_r\} \), the closures in \( G \) of the connected components of the set \( G \setminus \{v \in V(G) : v \text{ is a cut vertex of } G\} \).

The set \( \{G_r\} \), is the canonical T-decomposition of \( G \).

**Example.**

Let us consider two cycle graphs \( \Gamma_1, \Gamma_2 \), and \( x_1 \in V(\Gamma_1), x_2 \in V(\Gamma_2) \). Define the graph \( G \) as the graph with \( V(G) = V(\Gamma_1) \cup V(\Gamma_2) \) and \( E(G) = E(\Gamma_1) \cup E(\Gamma_2) \cup \{x_1, x_2\} \). Then \( \{\Gamma_1, \Gamma_2, [x_1, x_2]\} \) is the canonical T-decomposition of \( G \).

**Theorem 3.5 ([5, Theorem 5]).**

If \( \{G_r\} \), is the canonical T-decomposition of \( G \), then \( \delta(G) = \sup_r \delta(G_r) \).
The next result will deal with periodic graphs. The tessellation graph $G$ of $\mathbb{R}^2$ is periodic if there exists $(u, v) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ such that $T(G) = G$, where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as $T(x, y) = (x, y) + (u, v)$. Recall that a geodesic line is a geodesic with domain $\mathbb{R}$. By Euclidean line we mean an straight line in $\mathbb{R}^2$, i.e., a geodesic line in the Euclidean plane.

**Theorem 3.6.**
There exists a periodic hyperbolic 1-skeleton $G$ of a tessellation of $\mathbb{R}^2$ with tiles $\{F_n\}$ verifying $\sup_n \text{diam}_G \partial F_n = \infty$ and containing infinitely many Euclidean lines. Furthermore, there exists a periodic non-hyperbolic subgraph $G_0$ of $G$ which is also a tessellation graph of $\mathbb{R}^2$.

**Remark 3.7.**
The main idea in the construction of such a tessellation is to include in $\mathbb{R}^2$ a tessellation graph quasi-isometric to a periodic model of the hyperbolic plane. The example given in Theorem 3.6 shows that it is not possible to replace $\sup_n \text{diam}_{\mathbb{R}^2} \partial F_n$ by $\sup_n \text{diam}_G \partial F_n$ in Theorem 3.3. Theorem 4.6 shows a large class of non-hyperbolic tessellation graphs containing infinitely many Euclidean lines.

**Proof.** The structure of the proof is as follows: first, a hyperbolic graph $G_1$, which is a tessellation of $\mathbb{H}$, will be defined; based on $G_1$, define a new hyperbolic graph $G_0$ which is a tessellation of $\mathbb{R}^2$; finally, the graph $G$ satisfying all conditions in the statement will be defined from $G_0$.

Let us consider the hyperbolic plane $\mathbb{H}$ with its Fermi coordinates, see, e.g., [11, p. 247], i.e., the plane $\mathbb{R}^2$ with the Riemannian metric $ds^2 = \cosh^2 y \, dx^2 + dy^2$ (thus $dA = \cosh y \, dx \, dy$).

Let $[t]$ stand for the integer part of $t$. Consider the segments $I_{m,n}$ in $\mathbb{H}$ given by $I_{m,n} = \{(x, y) \in \mathbb{H} : m/\cosh m \leq x \leq (m + 1)/\cosh m, y = m\}$ for $m = 0, 1, 2, \ldots$ and $0 \leq n \leq \lfloor \cosh m \rfloor - 1$, and $I_{m,n} = \{(x, y) \in \mathbb{H} : x = n/\cosh m, m \leq y \leq \lfloor m + 1\rfloor\}$ for $m \geq 0$ and $0 \leq n \leq \lfloor \cosh m \rfloor$.

Let $G_1 = \bigcup_{m,n} \{I_{m,n} \cup J_{m,n}\}$, $S_0(x, y) = (x, -y)$ and $G_2 = G_1 \cup S_0(G_1)$. Also, let $R_k(x, y) = (x + k, y)$ for $k \in \mathbb{Z}$. The graph $G_3$ is now defined as $G_3 = \bigcup_k R_k(G_2)$. Clearly, $G_3$ is a tessellation graph of $\mathbb{H}$. Let us check that it verifies the hypotheses in Theorem 3.4. To this end, let $\{F_r^*\}$ be the tiles of the tessellation $G_3$. Since $S_0$ and $R_k$ are isometries of $\mathbb{H}$, $\text{diam}_{G_3} \partial F_r^* \leq L_{\mathbb{H}}(\partial F_r^*)$. A standard computation gives

$$L_{\mathbb{H}}(I_{m,n}) = \int_{m/\cosh m}^{(m+1)/\cosh m} \cosh m \, dx = \cosh m \leq 2, \quad L_{\mathbb{H}}(J_{m,n}) = \int_m^{m+1} dy = 1,$$

$$A_{\mathbb{H}} \left\{(x, y) \in \mathbb{H} : \frac{n}{\cosh m} \leq x \leq \frac{n+1}{\cosh m}, m \leq y \leq m+1 \right\} = \int_{m/\cosh m}^{(m+1)/\cosh m} \int_m^{m+1} \cosh y \, dy \, dx \geq \int_m^{(m+1)/\cosh m} \cosh m \, dx = \frac{\cosh m}{\cosh m} \geq 1.$$

Therefore,

$$\text{diam}_{G_3} \partial F_r^* \leq L_{\mathbb{H}}(\partial F_r^*) \leq 4e + 4, \quad A_{\mathbb{H}}(F_r^*) \geq 1,$$

for every $r$, and Theorem 3.4 allows to conclude that $G_3$ is hyperbolic.

Consider now the graph $G_3$ embedded in the Euclidean plane $\mathbb{R}^2$. Let us define $K_{0,0} = I_{0,0}$; for $m \geq 1$ and $0 \leq n \leq \lfloor \cosh m \rfloor - 1$, let $K_{n,m}$ be a polygonal curve joining the endpoints of $I_{m,n}$ which is contained in the rectangle $\{(x, y) \in \mathbb{R}^2 : n/\cosh m \leq x \leq (n+1)/\cosh m, m - 1/6 \leq y \leq m + 1/6\}$, where $L_{\mathbb{H}}(K_{n,m}) = L_{\mathbb{H}}(I_{m,n})$ and

$$\bigcup_{n=0}^{\lfloor \cosh (m-1) \rfloor} \left( \frac{n}{\cosh (m-1)}, m \right) \subset \bigcup_{n=0}^{\lfloor \cosh m \rfloor - 1} K_{n,m}.$$
Set $G_4 = \bigcup_{n=0}^{m} \{ K_{m,n} \cup J_{m,n} \}$, $G_5 = G_4 \cup S_0(G_4)$ and define the next graph $G_6$ as $G_6 = \bigcup_{n} R_{n}(G_5)$. Clearly, $G_6$ is a tessellation graph of $\mathbb{R}^2$. Since the graphs $G_3$ (in $\mathbb{H}$) and $G_6$ (in $\mathbb{R}^2$) are isometric, the graph $G_6$ is hyperbolic.

Finally, let us define $\delta$. For $m \geq 0$, $0 \leq n \leq \lceil \cosh m \rceil - 1$ and $0 \leq s \leq m$, let $M_{m,n,s}$ be the cycle graph which is the union of the four Euclidean segments joining the points

$$
\left( \frac{n + s/(4m + 4)}{\cosh m}, m + \frac{1}{2} \right), \quad \left( \frac{n + (2s + 1)/(8m + 8)}{\cosh m}, m + \frac{2}{3} \right),
$$

$$\left( \frac{n + (s + 1)/(4m + 4)}{\cosh m}, m + \frac{1}{2} \right), \quad \left( \frac{n + (2s + 1)/(8m + 8)}{\cosh m}, m + \frac{1}{3} \right).
$$

Set $G_7 = \bigcup_{m,n,s} M_{m,n,s}$, $G_8 = G_7 \cup S_0(G_7)$, $G_9 = \bigcup_{k} R_{k}(G_8)$ and $G = G_6 \cup G_9$. Clearly, $G$ is a tessellation graph of $\mathbb{R}^2$. Note that the sets $M_{m,n,s}$, their images by $S_0$ and $R_k$, and $G_6$, are the canonical T-decomposition of $G$; hence, Theorem 3.5 gives that $\delta(G) = \max \{ \delta(G_6), \sup_{m,n,s} \delta(M_{m,n,s}) \}$. One can check that $\delta(M_{m,n,s}) = L_{2\mathbb{R}}(M_{m,n,s})/4 \leq 1$; since $\delta(G) \leq \max \{ \delta(G_6), 1 \} < \infty$, the graph $G$ is hyperbolic.

Let us check the condition on the tiles of this graph. Denote by $\{ F_r \}$ the tiles of $G$; if $\partial F_r$ contains $M_{m,n,0}, M_{m,n,1}, \ldots, M_{m,n,m}$, then

$$\text{diam}_G \partial F_r \geq \frac{1}{2} \sum_{s=0}^{m} L_{2\mathbb{R}}(M_{m,n,s}) \geq \frac{m + 1}{2} \frac{2}{3} \geq \frac{m + 1}{3},$$

and one concludes that $\sup_{\partial F_r} \text{diam}_G \partial F_r = \infty$. Furthermore, the graph $G$ is periodic and contains infinitely many Euclidean lines by construction.

Finally, let us construct a periodic non-hyperbolic subgraph $G_0$ of $G$ which is also a tessellation graph of $\mathbb{R}^2$. Let us define $K_n = \bigcup_{m=0}^{\lceil \cosh m \rceil - 1} K_{m,n}$, $G_{10} = \bigcup_{m} \{ K_n \cup J_{m,0} \cup J_{m,\lceil \cosh m \rceil} \}$, $G_{11} = G_{10} \cup S_0(G_{10})$ and $G_0 = \bigcup_{k} R_{k}(G_{11})$. It is clear that $G_0$ is a tessellation graph of $\mathbb{R}^2$ and a subgraph of $G$. For each $m \geq 0$, consider the midpoint $p_m$ of $K_n$, i.e., the point with $d_{G_0}(p_m, (0, m)) = d_{G_0}(p_m, (1, m)) = L_{2\mathbb{R}}(K_n)/2$, and the geodesic bigon $B_m$ in $G_0$ with two different geodesics $y_m^1, y_m^2$, joining $p_m$ and $p_{m+1}$; then $y_m^1 \cup y_m^2 = K_n \cup K_{n+1} \cup J_{m,0} \cup J_{m,\lceil \cosh m \rceil}$. If $q_m$ is the midpoint of $y_m^1$, then

$$\delta(B_m) \geq d_{G_0}(q_m, y_m^2) = d_{G_0}(q_m, \{ p_m, p_{m+1} \}) = \frac{1}{2} L_{2\mathbb{R}}(y_m^1) = \frac{1}{2} \left( \frac{1}{2} L_{2\mathbb{R}}(K_n) + L_{2\mathbb{R}}(J_{m,0}) + \frac{1}{2} L_{2\mathbb{R}}(K_{n+1}) \right)$$

$$= \frac{1}{2} \frac{1}{2} \left( \frac{\cosh m + \cosh(m + 1) + 2}{\cosh m + 1} \right) \geq \frac{1}{4} \left( \frac{\cosh m + \cosh(m + 1) + 2}{\cosh m + 1} \right),$$

and one concludes $\delta(G_0) \geq \sup_{\partial F_r} \delta(B_m) = \infty$.

A corollary for 2-quasiperiodic graphs follows. Recall that the tessellation graph $G$ of $\mathbb{R}^2$ is 2-periodic if there exist two linearly independent vectors $(u_1, v_1), (u_2, v_2) \in \mathbb{R}^2$ such that $T_j(G) = G$, $j = 1, 2$, where $T_j : \mathbb{R}^2 \to \mathbb{R}^2$ are defined as

$$T_j(x, y) = (x, y) + (u_j, v_j), \quad j = 1, 2.$$ 

The graph $G$ is 2-quasiperiodic if there exists a 2-periodic subgraph $G_0$ of $G$.

**Corollary 3.8.**

If $G$ is 2-quasiperiodic then $G$ is not hyperbolic.

**Proof.** If $G_0$ is a 2-periodic subgraph with tiles $\{ F_r \}_{r \in \mathbb{Z}}$, then one can take the partition $\Lambda_1 = \{ 1 \}, \Lambda_2 = \emptyset$ of the set of indices in the statement of Theorem 3.1. Therefore, $G$ is not hyperbolic. □
4. Tessellations with parallelograms and rectangles

In this section it is shown that the hyperbolicity of certain tessellations with parallelograms is equivalent to the hyperbolicity of tessellations with rectangles. It is also shown that under some hypotheses rectangular tessellations are not hyperbolic.

4.1. Tessellations with parallelograms

Next it will be shown that considering tessellations of parallelograms with bounded inclinations is equivalent to considering rectangular tessellations with sides parallel to the axis in order to study hyperbolicity.

Consider the standard basis in \( \mathbb{R}^2 \) defined by \( \{ \mathbf{e}_1, \mathbf{e}_2 \} \) and, given \( \alpha, \beta \), let \( \mathbf{U}_\alpha \) and \( \mathbf{V}_\beta \) be the vectors defined by \( \mathbf{U}_\alpha = (\cos \alpha, \sin \alpha) \), \( \mathbf{V}_\beta = (\sin \beta, \cos \beta) \). Fix real numbers \( a \) and \( b \) with \( a < b < \pi + a \). A tessellation \( T \) is a \( p \)-tessellation of \( \mathbb{R}^2 \) if its tiles satisfy the following conditions:

- \( T \) is a parallelogram, for all \( T \in T \).
- For each \( T \in T \) there exist a pair of angles \( \alpha, \beta \) such that \( \alpha \in (a, b), \beta \in (-\pi/2 - a, \pi/2 - b) \), and the sides of \( T \) are parallel to \( \mathbf{U}_\alpha \) and \( \mathbf{V}_\beta \), respectively.

Notice that the second condition above implies that if two adjacent tiles in \( T \) partially share a side, then (for both of them) there is a side that is either parallel to \( \mathbf{U}_\alpha \) (for some \( a \)) or to \( \mathbf{V}_\beta \) (for some \( \beta \)).

**Theorem 4.1.**

Given a \( p \)-tessellation \( T \) of \( \mathbb{R}^2 \), there exist a tessellation \( T \) of \( \mathbb{R}^2 \) with rectangular tiles and a bijective continuous function \( f : T \rightarrow T \) such that \( f \mid _{S} \) is an isometry from the 1-skeleton \( S \) of \( T \) to the 1-skeleton \( G \) of \( T \).

**Proof.** Applying a rotation, without loss of generality, we may assume \( \alpha \in (-c, c) \) where \( c = (b - a)/2 \). Then the vectors \( \mathbf{U}_\alpha \) and \( \mathbf{V}_\beta \) give, respectively, the “almost-horizontal” and “almost-vertical” directions of a tile. In fact, if \( P \) is a vertex of a tile, \( T \in T \), the lines with directions \( \mathbf{U}_\alpha \) and \( \mathbf{V}_\beta \) through \( P \) divide the plane in four sectors. Since the sides of every \( T \in T \) are parallel to \( \mathbf{U}_\alpha \) and \( \mathbf{V}_\beta \) with \( \alpha \in (-c, c) \) and \( \beta \in (-\pi/2 + c, \pi/2 - c) \), no more than four tiles can share a vertex and therefore \( T \) has the structure of a rectangular tessellation.

For a given \( \mathbf{U}_\alpha \) and \( \mathbf{V}_\beta \), let \( S(\alpha, \beta) = \{ T : T \in T \ is a \ parallel \ of \ angles \ \alpha, \beta \} \). The tessellation \( T \) induces a “partition” of \( \mathbb{R}^2 \), \( S \), given by the connected components of \( S(\alpha, \beta) \), with \( \alpha \in (-c, c) \), \( \beta \in (-\pi/2 + c, \pi/2 - c) \). If \( B \neq \emptyset \) is a connected component of \( S(\alpha, \beta) \) then \( B \) is a union of closures of tiles of \( T \). If \( B = \mathbb{R}^2 \), then there exist \( \alpha \) and \( \beta \) such that all the tiles in \( T \) have sides parallel to \( \mathbf{U}_\alpha \) and \( \mathbf{V}_\beta \). In this situation define \( f = f_{\mathbf{U}_\alpha} \) where \( f_{\mathbf{U}_\alpha} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is the linear map such that \( f_{\mathbf{U}_\alpha}(\mathbf{U}_\alpha) = \mathbf{e}_1 \) and \( f_{\mathbf{U}_\alpha}(\mathbf{V}_\beta) = \mathbf{e}_2 \). Clearly, \( f \) is an isometry from the 1-skeleton \( S \) of \( T \) to the 1-skeleton \( G \) of \( T \).

In what follows it is assumed that \( B \neq \mathbb{R}^2 \). Since each \( B \neq \emptyset \) is the union of parallelograms whose sides have a fixed inclination, then its boundary components are polygonal lines with two possible angles. Moreover, \( B \) is a convex set and therefore it is either a parallelogram (if \( B \) bounded) or otherwise, it is a generalized parallelogram with a side at infinity, that is, a half-strip (if there is one side at infinity), a strip or a sector (if there are two) or half-plane (if there are three). Indeed, if it is not convex, there is a tile \( T \in T, T \in B \), that shares two sides with \( B \) and therefore \( T \) is a parallelogram with sides parallel to those of \( B \), thus \( T \in B \). The same argument implies that if \( B' \neq B \) are two connected components of \( S(\alpha, \beta) \) such that \( B \cap B' \neq \emptyset \), then \( B \) and \( B' \) share a whole side or a vertex.

The function \( f : T \rightarrow T \) will be a piecewise linear function defined on the sets \( B \) inductively and so that if \( B \in S(\alpha, \beta) \), then

\[
f(x) - f(y) = f_{\alpha}(x) - f_{\alpha}(y) = f_{\beta}(x - y) \quad \text{for all } x, y \in B,
\]

where \( f_{\alpha} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is the linear map such that \( f_{\alpha}(\mathbf{U}_\alpha) = \mathbf{e}_1 \) and \( f_{\alpha}(\mathbf{V}_\beta) = \mathbf{e}_2 \). To start the induction, let \( O = (0, 0) \) and define \( f(O) = O \). Denote by \( B_0 \) one of the sets which contains \( O \), and let \( \alpha_0 \) and \( \beta_0 \) be such that \( B_0 \in S(\alpha_0, \beta_0) \).
If \( x \in B_0 \) then
\[
f(x) = f_{0y0}(x) = f(O) + f_{0y0}(x - O).
\]
Notice that, for all \( x, y \in B_0 \), relation (3) trivially holds by the linearity of \( f_{0y0} \). Let \( C_0 = B_0 \). Assume now that \( f \) is defined and continuous on a connected set \( C_n \) which is a finite union of blocks \( B \in S \) defined as \( C_n = \{ B \in S : B \cap C_{n-1} \neq \emptyset \} \) and that (3) holds for every set \( B \in C_n \). Extend \( f \) from \( C_n \) onto \( C_{n+1} = \{ B \in S : B \cap C_n \neq \emptyset \} \) in the following way: for \( B \in C_{n+1} \setminus C_n \), \( B \in S(\alpha, \beta) \), take any point \( P \in \partial B \cap C_n \), and define
\[
f(x) = f(P) + f_{\alpha\beta}(x - P), \quad x \in B.
\]
Notice that (3) holds for points \( x, y \in B \) by the linearity of \( f_{\alpha\beta} \). We are left to show that the extension is well defined. Indeed, since no more than four tiles of \( \mathcal{T} \) can meet at a vertex, and since different \( B \)'s share a whole side, at each vertex exactly four different sets \( B \in S \) meet. The function \( f \) straightens the sides of each \( B \) and places it adjacent to the images of its neighbors. Concretely, if \( B \in C_{n+1} \setminus C_n \) and \( x \in \partial B \cap C_n \) then considering \( x \) as a point on \( B \in C_{n+1} \setminus C_n \), \( f_{\theta}(x) = f(P) + f_{\alpha\beta}(x - P) \), for a point \( P \in \partial B \cap C_n \) where \( f(P) \) was already defined. If \( x \neq P \) there is nothing to prove. If \( x = P \), then there exists \( B' \in C_n \) such that \( x \in \partial B' \), thus \( B \) and \( B' \) share a side the one with \( P \) and \( x \). By (3), \( f_{\theta}(x) = f(P) + f_{\alpha\beta}(x - P) \). Since both \( x, P \in \partial B' \cap \partial B \) then, \( f_{\alpha\beta}(x - P) = f_{\alpha\beta}(x - P) \) and therefore \( f \) is well defined on \( B \cap C_n \). To see that it is well defined on \( C_{n+1} \), consider now \( B, B' \in C_{n+1} \setminus C_n \) such that \( B \cap B' \neq \emptyset \). Then, there exists a point \( Q \in B \cap B' \cap C_n \) and by (3),
\[
f_B(x) = f(Q) + f_{\alpha\beta}(x - Q); \quad f_{B'}(x) = f(Q) + f_{\alpha\beta'}(x - Q).
\]
Since \( f \) is well defined on \( C_n \), \( f(Q) \) is the same in both definitions, and since \( x, Q \in B \cap B' \) then \( f_{\alpha\beta}(x - Q) = f_{\alpha\beta'}(x - Q) \). Thus \( f_B(x) = f_{B'}(x) = f(x) \) and \( f \) is well defined on \( C_{n+1} \). An induction argument gives that \( f \) is continuous in \( \mathbb{R}^2 \).
Notice that, by construction, \( f \) maps each \( B \) to a rectangle with sides parallel to the axes, and each \( \mathcal{T} \in B \) to a rectangle inside \( f(B) \) also with sides parallel to the axes. Also if \( B_1 \) and \( B_2 \) are adjacent to \( B \) on opposite sides (that is, \( B \cap B_i \neq \emptyset \), \( i = 1, 2 \), and \( B_1 \cap B_2 = \emptyset \)), then \( f(B_1) \) and \( f(B_2) \) are also adjacent to \( f(B) \) on opposite sides. Therefore, the function \( f \) is both injective and surjective. Finally, since \( f \) is linear on \( B \) each \( \mathcal{T} \in B \) is mapped to a rectangle and its side lengths are preserved. That is, when restricted to the 1–skeleton \( \mathcal{S} \) of \( B \) the function \( f \) is an isometry.

The next result is a consequence of the previous theorem.

**Theorem 4.2.**
All \( p \)-tessellation graphs of \( \mathbb{R}^2 \) are non-hyperbolic if and only if all tessellation graphs of \( \mathbb{R}^2 \) whose tiles are rectangles are non-hyperbolic.

### 4.2. Tessellations with infinitely many parallel rays

**Lemma 4.3.**
Let \( T \) be a rectangular tessellation in \( \mathbb{R}^2 \approx \mathbb{C} \) with tiles parallel to the coordinate axes and with infinitely many vertical rays in the upper-half plane. If \( F \) is any tile on the tessellation, \( L \) and \( l \) are the lengths of its longest and shortest sides respectively, consider the following two conditions:

(i) There exists an increasing function \( g : \mathbb{R}^+ \to \mathbb{R}^+ \) such that for every tile \( F \), \( L/l \leq g(d_{\mathbb{R}}(F, i\mathbb{R})) \), where \( i\mathbb{R} \) denotes the imaginary axis.

(ii) There exists a constant \( C \) so that for every tile \( F \), \( ld_{\mathbb{C}}(F, 0) \geq C \).

If (i) or (ii) hold, then for any point \( x \) lying on a vertical ray and any other vertical ray \( y \) with \( \text{Re} \ x < \text{Re} \ y \), there is a geodesic advancing always rightwards and upwards which joins \( x \) to \( y \).

**Remark.**
Note that any curve advancing always rightwards and upwards is a geodesic.
Theorem 3.6 shows that the existence of infinitely many vertical rays (or even infinitely many vertical lines) does not guarantee the non-hyperbolicity of a tessellation graph.

Proof. Let \( y_0 \) and \( y_1 \) be any two of these vertical rays, and without loss of generality suppose \( y_0 \) lies on the left of \( y_1 \). Let \( D \) be defined as \( D = d(y_0, iR) \geq 0 \). Let \( \sigma \) be the geodesic ray starting in \( x \) defined as follows: \( \sigma(t) = x + it \) for \( t \in [0, t_0] \), where \( t_0 = \max \{ 0, \inf_{z \in [y_1 \cup \gamma]} \Im \chi - \Im x \} \); after that \( \sigma \) advances rightwards when it is possible and otherwise upwards. Denote by \( \{ F_k \} \) a choice of (ordered) tiles with \( \sigma \subset \bigcup_j \partial F_k \).

For any tile \( F_k \), let \( h_k \) denote the length of its horizontal side, and \( v_k \) the length of its vertical side. The goal is to show that, in any case, there exists \( N \) such that

\[
\sum_{k=1}^N h_k \geq d(y_0, y_1).
\]

Suppose not. Then, there exists \( C_1 \) such that \( \sum_{k=1}^\infty h_k \leq C_1 \). It will be shown that this implies that there exists \( C_2 \) such that \( \sum_{k=1}^\infty v_k \leq C_2 \), contradicting the fact that \( T \) is a tessellation.

Assume (i) holds. Without loss of generality, we can assume that \( g(t) \geq 1 \) for every \( t > 0 \); then \( v_k \leq h_k g(d(F_k, iR)) \) for all \( k \). Therefore,

\[
\sum_{k=1}^\infty v_k \leq \sum_{k=1}^\infty h_k g(d(F_k, iR)) \leq \sum_{k=1}^\infty h_k g \left( \sum_{n=1}^{k-1} h_n + D \right) \leq \sum_{k=1}^\infty h_k g(C_1 + D) \leq C_1 g(C_1 + D) = C_2.
\]

Assume (ii) holds. Without loss of generality, by Theorem 3.3 we can assume that \( \sup_k L(\partial F_k) = C_0 < \infty \). Then,

\[
d_C(F_k, 0) \leq d_C(F_1, 0) + \sum_{j=1}^{k-1} (v_j + h_j) \leq d_C(F_1, 0) + \sum_{j=1}^{k-1} C_0 \leq d_C(F_1, 0) + C_0(k-1).
\]

Thus, by hypothesis,

\[
h_k \geq \frac{C}{d_C(F_k, 0)} \geq \frac{C}{d_C(F_1, 0) + C_0(k-1)},
\]

and therefore one concludes \( \sum_{j=1}^\infty h_k = \infty \).

As it was mentioned above, there are several equivalent definitions of hyperbolicity. For the proof of the next result, the one involving uniformity in the divergence of the geodesics is used. Namely

**Definition 4.4.**

A function \( e : \mathbb{R}^+ \to \mathbb{R}^+ \) is a divergence function for the geodesic metric space \( X \) if for all \( x \in X \), all \( R \in \mathbb{R}^+ \) and all geodesics \( y = [xy], y' = [xz] \), \( e \) satisfies the following condition: if \( r > 0 \), \( R + r \leq \min \{ d(x,y), d(x,z) \} \), \( d(y(R), y'(R)) \geq e(0) > 0 \) and \( \alpha \) is a path in \( X \backslash B([x, R+r]) \) from \( y(R+r) \) to \( y'(R+r) \), then \( L(\alpha) \geq e(r) \). Let \( X \) be a geodesic metric space. The geodesics diverge in \( X \) if there is a divergence function \( e(r) \) such that \( \lim_{r \to \infty} e(r) = \infty \).

In [1, 32] the following result was proved.

**Theorem 4.5.**

A geodesic metric space \( X \) is hyperbolic if and only if geodesics diverge in \( X \).

**Theorem 4.6.**

Let \( T \) be a rectangular tessellation of \( \mathbb{R}^2 \approx \mathbb{C} \) with tiles parallel to the coordinate axes and with infinitely many vertical rays. For \( F \) any tile on the tessellation, if (i) or (ii) of Lemma 4.3 hold, then the 1-skeleton of \( T \) is not hyperbolic.

**Remark.**

Theorem 3.6 shows that the existence of infinitely many vertical rays (or even infinitely many vertical lines) does not guarantee the non-hyperbolicity of a tessellation graph.
Proof. Seeking for a contradiction assume that the 1-skeleton $G$ of the tessellation $T$ is hyperbolic. Then, there exists a divergence function, $e: \mathbb{R}^+ \to \mathbb{R}^+$.

Denote by $\{y_k\}_k$ the vertical rays in $G$. Without loss of generality we can assume that $\text{Re } y_k$ increases with $k$ and that $\lim_{k \to \infty} \text{Im } y_k(t) = \infty$ for every $k$. Let $x \in y_0$. Let $\eta$ be a geodesic starting at $x$ which is the union of horizontal and vertical displacements and such that $\eta \cap y_k \neq \emptyset$ for every $k \geq 0$ (recall Lemma 4.3). Denote by $y_n$ the segment of $\eta$ which starts at the point $x$ and finishes at the first point $z_n$ of $y_k$. Fix $n$ to be so that $\|d(y_0, y_n)\| > e(0)$. Let $R$ be such that $\eta_n(R) = z_n$; then $\|d(C(y_0, y_n)) \geq d(C(y_0, y_n)) > e(0)$.

Consider a new geodesic $\mu$ which starts at $y_0(R)$ and which is the union of horizontal and vertical displacements, and such that $\mu \subset y_n \neq \emptyset$; let us fix $w_n \in \mu \cap y_n$ and let $\mu_n$ be the segment of $\mu$ which finishes at $w_n \in y_n$. Denote by $\eta_n, 1$ and $\eta_n, 2$ the vertical rays starting at $z_n$ and $w_n$, respectively. The curve $\Gamma_1$ given by the geodesic segments $\{y_0(R) \cup \mu_n \cup \eta_n, 2\}$ is a geodesic; similarly, the curve $\Gamma_2$ defined as $\Gamma_2 = \eta_n \cup \eta_n, 1$ is also a geodesic. Note that if $w_n = \Gamma_1(t_0)$, then $\Gamma_1(t) = \Gamma_2(t)$ for every $t \geq t_0$, and $\|d(C(\Gamma_1(R), \Gamma_2(R))) > e(0)$. This contradicts the hyperbolicity assumption.

$\square$

5. Tessellations with convex tiles

In [9], the authors conjectured that every tessellation graph of $\mathbb{R}^2$ with convex tiles is non-hyperbolic. Our next result shows that in order to prove this conjecture, it suffices to consider tessellation graphs of $\mathbb{R}^2$ with triangular tiles.

Theorem 5.1.

All tessellation graphs of $\mathbb{R}^2$ whose tiles are convex polygons are non-hyperbolic if and only if all tessellation graphs of $\mathbb{R}^2$ whose tiles are triangles are non-hyperbolic.

Proof. Let $G$ be a tessellation graph of $\mathbb{R}^2$ whose tiles are convex polygons, and consider its tiles $F_n$. If $\sup_{n} \text{diam}_{F_n} \partial F_n = \infty$, then $G$ is non-hyperbolic and the conclusion holds. Therefore, assume that $c = \sup_{n} \text{diam}_{F_n} \partial F_n < \infty$. For each $n$, let $P_{n, 1}$ and $P_{n, 2}$ be two vertices of $F_n$ accomplishing the maximum Euclidean distance between the vertices of $F_n$. Let us consider a new tessellation graph of $\mathbb{R}^2$, $G'$, obtained from $G$ by adding in each tile $F_n$ new edges which join each vertex of $F_n$ with $P_{n, 1}$ by the Euclidean segment between them. That is, all the tiles of $G'$ are triangles and, therefore, by hypothesis, $G'$ is non-hyperbolic. We shall show that the inclusion $\iota: G \to G'$ is a $c$-full $(1 + \pi/2, 0)$-quasi-isometry and, therefore, by Theorem 2.1, $G$ will also be non-hyperbolic.

Let us consider a tile $F_n$ and its corresponding vertices $P_{n, 1}$ and $P_{n, 2}$. Then $F_n$ is contained in the closure of the Euclidean circle with center $P_{n, 2}$ and radius equal to the Euclidean distance between $P_{n, 1}$ and $P_{n, 2}$. Without loss of generality one can assume that $P_{n, 1}$ is the origin of coordinates and $P_{n, 2}$ is the point with coordinates $(1, 0)$. Let $P_n$ be a point of $\partial F_n$; since $F_n$ is a convex polygon, $P_n$ is contained in the right half-plane, i.e., if $(r, \theta)$ are the polar coordinates of $P_n$, then $0 \leq r \leq 1$ and $-\pi/2 \leq \theta \leq \pi/2$. Let $P_n'$ be the projection of $P_n$ over the circumference $\{x^2 + y^2 = 1\}$. The goal is to compare the Euclidean distance between $P_n$ and $P_{n, 1}$ and the sum of the Euclidean distance between $P_n$ and $P_n'$ plus the length of the arc of the circumference $\{x^2 + y^2 = 1\}$ between $P_n'$ and $P_{n, 1}$. To this end, one needs to bound the function

$$f(r, \theta) = \frac{\sqrt{1 + r^2 - 2r \cos \theta}}{1 + r^2 - 2r \cos \theta}, \quad 0 \leq r \leq 1, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$ 

Let us consider the functions

$$g_1(r, \theta) = \frac{\theta^2}{1 + r^2 - 2r \cos \theta}, \quad g_2(r, \theta) = \frac{(1 - r)^2}{1 + r^2 - 2r \cos \theta}.$$ 

For fixed $\theta$, the function $g(r) = 1 + r^2 - 2r \cos \theta$ attains its minimum value when $r = \cos \theta$, therefore $g_1(r, \theta) \leq \theta^2 / \sin^2 \theta$.

Since the ratio $\theta / \sin \theta$ increases for $\theta \in [0, \pi/2]$, $g_1(r, \theta) \leq \pi^2 / 4$ for $0 \leq r \leq 1$, $-\pi/2 \leq \theta \leq \pi/2$. Also, since $-2r \leq -2r \cos \theta$, $g_2(r, \theta) \leq 1$, $0 \leq r \leq 1$, $-\pi/2 \leq \theta \leq \pi/2$. Therefore it follows

$$\sup_{0 \leq r \leq 1} \frac{\theta^2}{1 + \pi^2 / 4 \theta^2 \sin^2 \pi / 2} \leq \frac{1 + \pi}{2}.$$ 

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The tile $F_\alpha$ is convex, thus

$$d_{\partial \Gamma_n}(P_\alpha, P_{n,1}) \leq \theta + 1 - r \leq \left(1 + \frac{\pi}{2}\right) \sqrt{1 + r^2 - 2r \cos \theta} = \left(1 + \frac{\pi}{2}\right) d_{\partial \Gamma_n}(P_\alpha, P_{n,1}) = \left(1 + \frac{\pi}{2}\right) d_{\partial \Gamma}(P_\alpha, P_{n,1}). \quad (4)$$

For any points $P, Q$ on the graph $G$, let us consider a geodesic $\gamma$ in $G'$ joining $P$ and $Q$. Let $\gamma_\alpha = \gamma \cap F_\alpha$, where $F_\alpha$ is the subgraph of $G'$ obtained by adding to $\partial F_\alpha$ the new edges joining the corresponding point $P_{n,1}$ with the other vertices of $\partial F_\alpha$. If $\gamma_\alpha$ is contained in $\partial F_\alpha$, then the length of $\gamma_\alpha$ in $G'$ coincides with its length in $G$. If $\gamma_\alpha$ is not contained in $\partial F_\alpha$, then $\gamma_\alpha = \gamma'_\alpha \cup \gamma''_\alpha$ where $\gamma'_\alpha = \gamma_\alpha \cap \partial F_\alpha$, $\gamma''_\alpha = \gamma_\alpha \setminus \gamma'_\alpha$. Note that the closure of $\gamma''_\alpha$ is connected and its endpoints are vertices in $\partial F_\alpha \cap V(G)$. Let $\sigma_\alpha$ be a geodesic in $G'$ joining the endpoints of $\gamma''_\alpha$; since $F_\alpha$ is convex, $\sigma_\alpha$ is contained in $\partial F_\alpha$. From (4) one gets

$$d_{G'}(P, Q) = L(\gamma) = \sum_{\alpha} L(\gamma_\alpha) = \sum_{\alpha} [L(\gamma'_\alpha) + L(\gamma''_\alpha)] \geq \sum_{\alpha} \left[ L(\gamma'_\alpha) + \left(1 + \frac{\pi}{2}\right)^{-1} L(\sigma_\alpha) \right] \geq \left(1 + \frac{\pi}{2}\right)^{-1} d_{G'}(P, Q).$$

In any case one concludes that $(1 + \pi/2)^{-1} d_{G'}(P, Q) \leq d_{G'}(P, Q) \leq d_{G'}(P, Q)$, which means that the inclusion $\iota: G \to G'$ is a $(1 + \pi/2, 0)$-quasi-isometric embedding. It is clear that $\iota$ is $c$-full, with $c = \sup_{\alpha} \text{diam}_{\partial F_\alpha} < \infty$. By hypothesis the graph $G'$ is non-hyperbolic, and by Theorem 2.1 it follows that $G$ is also non-hyperbolic. \hfill\Box

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## References


