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Properties of derivations on some convolution algebras

Research Article

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Abstract: For all convolution algebras $L^1[0,1)$, L^1_{loc} and $A(\omega)=\bigcap_n L^1(\omega_n)$, the derivations are of the form $D_\mu f = Xf*\mu$ for suitable measures μ, where $(Xf)(t)=tf(t)$. We describe the (weakly) compact as well as the (weakly) Montel derivations on these algebras in terms of properties of the measure μ. Moreover, for all these algebras we show that the extension of D_μ to a natural dual space is weak-star continuous.

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1. Introduction

The aim of this paper is to study various properties of derivations on some convolution Banach and Fréchet algebras. A starting point for the paper is the characterisation in [6, Theorem 4.1] of (weak) compactness of derivations on the weighted convolution Banach algebras $L^1(\omega)$. Other inspirations include the recent papers [2, 3, 13] on (weak) compactness and weak-star continuity of derivations from some Banach algebras to their dual spaces.

The algebras that we will consider are $L^1[0,1)$, L^1_{loc} and $A(\omega)=\bigcap_n L^1(\omega_n)$ (see the relevant sections for the definitions). These are all convolution algebras on $[0,1)$ or $\mathbb{R}^+=[0,\infty)$ with the usual convolution product

$$(f*g)(t)=\int_0^t f(s)g(t-s)\,ds$$

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for f and g in the algebra in question and $t \in [0,1)$ or $t \in \mathbb{R}^+$. On each of these algebras $\mathcal{B}$ all derivations are continuous and are of the form $D_\mu f = Xf * \mu$ for μ in a suitable class of measures, where X is the operator defined by $(Xf)(t) = tf(t)$ for $t \in [0,1)$ or $t \in \mathbb{R}^+$ and $f \in \mathcal{B}$. For derivations on $L^1[0,1)$ we obtain a characterisation of (weak) compactness in terms of the measure μ similar to the one for $L^1(\omega)$ in [6]. For the Fréchet algebras L^1_{loc} and $A(\omega)$ we will see that there are no non-zero weakly compact derivations, and in these cases the following seems to be a more useful notion: A linear operator between two Fréchet spaces is called (*weak*) *Montel* (see [1]) if it maps bounded sets to (weakly) relatively compact sets. On Banach spaces this notion agrees with the one of (weakly) compact operators, and generally a (weakly) compact operator is (weakly) Montel. For the Fréchet algebras L^1_{loc} and $A(\omega)$ we give characterisations of (weak) Montel derivations similar to the characterisations of (weak) compactness for $L^1(\omega)$ and $L^1[0,1)$.

We also study weak-star continuity of the extension $\overline{D}_\mu$ of D_μ to a natural dual space containing the algebra in question. In all cases we prove weak-star continuity of $\overline{D}_\mu$ by showing that $\overline{D}_\mu$ is the adjoint of the continuous linear operator T_μ defined by

$$(T_\mu h)(t) = t \int h(t+s)\, d\mu(s),$$

for h belonging to the predual and $t \in [0,1)$ or $t \in \mathbb{R}^+$, and where the integrals are over $[0,1-t)$ or $\mathbb{R}^+$.

2. Derivations on $L^1[0,1)$

Let $L^1[0,1)$ be the Volterra algebra of (equivalence classes of) integrable functions f on $[0,1)$ with convolution product and the norm $\|f\| = \int_0^1 |f(t)|\, dt$. Similarly, $M[0,1)$ denotes the Banach algebra of finite, complex Borel measures on $[0,1)$. Also, we let $C_0[0,1)$ be the space of continuous functions h on $[0,1]$ with $h(1) = 0$. It is well known that $\langle h, \mu \rangle = \int_{[0,1)} h(t)\, d\mu(t)$ for $h \in C_0[0,1)$ and $\mu \in M[0,1)$ identifies $M[0,1)$ isometrically isomorphically with the dual space of $C_0[0,1)$.

The continuous derivations on $L^1[0,1)$ were described as follows by Kamowitz and Scheinberg [11, Theorem 2]: Let μ be a measure on $[0,1)$ for which $t|\mu|([0,1-t))$ is bounded as $t \to 0_+$. Then

$$D_\mu f = Xf * \mu, \qquad f \in L^1[0,1),$$

defines a continuous derivation on $L^1[0,1)$, and conversely every continuous derivation on $L^1[0,1)$ is of this form. Subsequently, Jewell and Sinclair [10] proved that derivations on $L^1[0,1)$ are automatically continuous.

We first show that the derivations D_μ extend to weak-star continuous derivations on the measure algebra $M[0,1)$.

Proposition 2.1.
Let μ be a measure on $[0,1)$ for which $t|\mu|([0,1-t))$ is bounded as $t \to 0_+$. Then

$$\overline{D}_\mu \nu = X\nu * \mu, \qquad \nu \in M[0,1),$$

extends D_μ to a weak-star continuous, bounded derivation $\overline{D}_\mu$ on $M[0,1)$.

Proof. Let $\nu \in M[0,1)$ and $h \in C_0[0,1)$. Then

$$\left| \int_{[0,1)} h(t)\, d(X\nu * \mu)(t) \right| = \left| \int_{[0,1)} \int_{[0,1-t)} h(t+s)\, d\mu(s)\, t\, d\nu(t) \right|$$
$$\le \int_{[0,1)} \int_{[0,1-t)} |h(t+s)|\, d|\mu|(s)\, t\, d|\nu|(t) \le \|h\| \int_{[0,1)} t|\mu|([0,1-t))\, d|\nu|(t) < \infty.$$

This shows that $X\nu * \mu \in M[0,1)$ and that $\overline{D}_\mu$ is a bounded, linear operator on $M[0,1)$. A direct calculation shows that X is a derivation on $M[0,1)$ and the same thus holds for $\overline{D}_\mu$.

Let $h \in C_0[0,1)$ and let

$$(T_\mu h)(t) = t \int_{[0,1-t)} h(t+s)\, d\mu(s), \qquad t \in [0,1].$$

The continuity of $T_\mu h$ is relatively standard (see, for instance, [5, Theorem 3.3.15]), but there is a slight complication because we are integrating over $[0,1-t)$. For $0 \le t, t_0 \le 1$ we have

$$\begin{aligned}\big|(T_\mu h)(t) - (T_\mu h)(t_0)\big| &= \left| t \int_{[0,1-t)} h(t+s)\, d\mu(s) - t_0 \int_{[0,1-t_0)} h(t_0+s)\, d\mu(s) \right| \\ &\le |t-t_0| \int_{[0,1-t)} |h(t+s)|\, d|\mu|(s) + t_0 \int_{[0,1-t)} |h(t+s) - h(t_0+s)|\, d|\mu|(s) \\ &\qquad + t_0 \left| \int_{[0,1-t)} h(t_0+s)\, d\mu(s) - \int_{[0,1-t_0)} h(t_0+s)\, d\mu(s) \right|.\end{aligned}$$

The first two terms tend to zero as $t \to t_0$. For $t \to t_0$ with $t \ge t_0$ the third term $t_0 \big| \int_{[1-t,1-t_0)} h(t_0+s)\, d\mu(s) \big| \to 0$, since $\bigcap_{t>t_0} [1-t, 1-t_0) = \emptyset$. Also, for $t \to t_0$ with $t \le t_0$ the third term $t_0 \big| \int_{[1-t_0,1-t)} h(t_0+s)\, d\mu(s) \big| \to t_0 |h(1)\mu(\{1-t_0\})| = 0$, since $h(1) = 0$. Hence $T_\mu h \in C_0[0,1)$ and it follows that T_μ is a continuous linear operator on $C_0[0,1)$. Moreover, for $\nu \in M[0,1)$ and $h \in C_0[0,1)$ we have

$$\langle h, \overline{D}_\mu \nu \rangle = \int_{[0,1)} h(t)\, d(X\nu * \mu)(t) = \int_{[0,1)} \int_{[0,1-t)} h(t+s)\, d\mu(s)\, t\, d\nu(t) = \langle T_\mu h, \nu \rangle.$$

Hence $\overline{D}_\mu = T_\mu^*$ and, in particular, $\overline{D}_\mu$ is weak-star continuous. □

The following characterisation of (weakly) compact derivations on $L^1[0,1)$ and its proof are strongly inspired by [6, Theorem 4.1].

Theorem 2.2.

Let μ be a measure on $[0,1)$ for which $t|\mu|([0,1-t))$ is bounded as $t \to 0_+$. Then the following conditions are equivalent:

(a) *D_μ is a compact derivation on $L^1[0,1)$.*

(b) *D_μ is a weakly compact derivation on $L^1[0,1)$.*

(c) *μ is absolutely continuous and $t|\mu|([0,1-t)) \to 0$ as $t \to 0_+$.*

(d) *$\overline{D}_\mu$ is a compact derivation on $M[0,1)$.*

(e) *$\overline{D}_\mu$ is a weakly compact derivation on $M[0,1)$.*

Proof. The implications (d) ⇒ (a) ⇒ (b) and (d) ⇒ (e) ⇒ (b) are obvious.

(b) ⇒ (c): Let $t > 0$, δ_t be the Dirac point measure at t and let (e_k) be a bounded approximate identity for $L^1[0,1)$. Since D_μ is weakly compact there exist a subsequence (e_{k_j}) and $f \in L^1[0,1)$ such that

$$D_\mu\big(\delta_t * e_{k_j}\big) \to f \qquad \text{weakly in } L^1[0,1) \text{ as } j \to \infty.$$

Let $g \in L^1[0,1)$. Since $L^1[0,1)^*$ is a $L^1[0,1)$ module we have $D_\mu(\delta_t * e_{k_j}) * g \to f * g$ weakly in $L^1[0,1)$ as $j \to \infty$. Also,

$$D_\mu\big(\delta_t * e_{k_j}\big) * g = \overline{D}_\mu(\delta_t) * e_{k_j} * g + \delta_t * D_\mu(e_{k_j}) * g \to \overline{D}_\mu(\delta_t) * g$$

in $L^1[0,1)$ as $j \to \infty$, since $D_\mu(e_k) * g = D_\mu(e_k * g) - e_k * D_\mu(g) \to D_\mu(g) - D_\mu(g) = 0$ in $L^1[0,1)$ as $k \to \infty$. Hence $\overline{D}_\mu(\delta_t) * g = f * g$ for all $g \in L^1[0,1)$, so we deduce that $t\delta_t * \mu = \overline{D}_\mu(\delta_t) = f \in L^1[0,1)$. Since this holds for all $t > 0$, it follows that μ is absolutely continuous.

Since the set $\{\delta_t * e_k : t > 0,\ k \in \mathbb{N}\}$ is bounded, it follows that $\{D_\mu(\delta_t * e_k) : t > 0,\ k \in \mathbb{N}\}$ is weakly relatively compact in $L^1[0,1)$. Also, for $t > 0$ we saw above that $t\delta_t * \mu = \overline{D}_\mu(\delta_t)$ is a weak cluster point of the sequence $(D_\mu(\delta_t * e_k))$, so we deduce that $\{t\delta_t * \mu : t > 0\}$ is weakly relatively compact in $L^1[0,1)$. From the Dunford–Pettis characterisation [7, Theorem 4.21.2] of weakly relatively compact subsets of $L^1[0,1)$ (or, as in [6], the Dieudonné–Grothendieck characterisation [7, Theorem 4.22.1 (4)] of weakly relatively compact subsets of $M[0,1)$) it then follows that $\{t\delta_t * |\mu| : t > 0\}$ is weakly relatively compact in $L^1[0,1)$. Let (t_i) be any net in $(0,1)$ with $t_i \to 0$. Then there exist a subnet (t_{i_j}) and $f \in L^1[0,1)$ such that $t_{i_j}\delta_{t_{i_j}} * |\mu| \to f$ weakly in $L^1[0,1)$. Let $a > 0$. Clearly, $t_{i_j}\delta_{t_{i_j}} * \delta_a * |\mu| \to \delta_a * f$ weakly in $L^1[0,1)$, but $t_{i_j}\delta_{t_{i_j}} \to 0$ in $M[0,1)$ and $\delta_a * |\mu| \in L^1[0,1)$, so $t_{i_j}\delta_{t_{i_j}} * \delta_a * |\mu| \to 0$ in $L^1[0,1)$, and we deduce that $\delta_a * f = 0$. Since this holds for all $a > 0$ we have $f = 0$, so we conclude that $t\delta_t * |\mu| \to 0$ weakly in $L^1[0,1)$ as $t \to 0_+$. The constant function with value 1 belongs to $L^\infty[0,1) = L^1[0,1)^*$, so we have $t|\mu|([0,1-t)) = \langle t\delta_t * |\mu|, 1\rangle \to 0$ as $t \to 0_+$.

(c) $\Rightarrow$ (d): We first prove that

$$E = \{t\delta_t * \mu : t \in [0,1)\} = \{\overline{D}_\mu(\delta_t) : t \in [0,1)\}$$

is compact in $M[0,1)$. Let $(t_n\delta_{t_n} * \mu)$ be a sequence in E. We may assume that there exists $t_0 \in [0,1]$ such that $t_n \to t_0$ as $n \to \infty$. Assume that $t_0 = 0$. Since $\|\overline{D}_\mu(\delta_t)\| = t|\mu|([0,1-t))$ we have

$$t_n\delta_{t_n} * \mu = \overline{D}_\mu(\delta_{t_n}) \to 0 \in E$$

as $n \to \infty$. Now assume that $0 < t_0 \le 1$. Choose t with $0 < t < t_0$. Since $t\delta_t * \mu = \overline{D}_\mu(\delta_t) \in M[0,1)$ and since μ is absolutely continuous, we have $\delta_t * \mu \in L^1[0,1)$. Since (δ_s) is strongly continuous on $L^1[0,1)$, it follows that

$$t_n\delta_{t_n} * \mu = t_n\delta_{t_n - t}\delta_t * \mu \ \to\ t_0\delta_{t_0 - t}\delta_t * \mu = t_0\delta_{t_0} * \mu \in E$$

as $n \to \infty$ (with $\delta_1 * \mu = 0 \in E$ in case $t_0 = 1$). Consequently, E is compact, so by Mazur's theorem [4, Theorem VI.4.8] the closed convex hull $\overline{\mathrm{co}}(E)$ is compact. Let $F \subseteq M[0,1)$ consist of those finite point measures

$$\nu = \sum_{k=1}^{K} \alpha_k \delta_{t_k}, \qquad K \in \mathbb{N}, \quad 0 \le t_1 < t_2 < \ldots < t_K < 1,$$

for which $\|\nu\| = \sum_{k=1}^{K} |\alpha_k| \le 1$. Then $\overline{D}_\mu(F) \subseteq \mathrm{co}(E)$. Let $\nu \in M[0,1)$ with $\|\nu\| \le 1$. As in the proof of [6, Theorem 4.1] there exists a net (ν_i) in F with $\nu_i \to \nu$ strongly in $M[0,1)$. The net $(\overline{D}_\mu(\nu_i))$ belongs to the compact set $\overline{\mathrm{co}}(E)$ and thus has a convergent subnet $(\overline{D}_\mu(\nu_{i_j}))$ with limit $\rho \in \overline{\mathrm{co}}(E)$. Also, for $f \in L^1[0,1)$ we have

$$\overline{D}_\mu(\nu_{i_j}) * f = D_\mu(\nu_{i_j} * f) - \nu_{i_j} * D_\mu(f) \ \to\ D_\mu(\nu * f) - \nu * D_\mu(f) = \overline{D}_\mu(\nu) * f,$$

so we deduce that $\overline{D}_\mu(\nu) = \rho \in \overline{\mathrm{co}}(E)$. Hence $\overline{D}_\mu$ is compact. □

3. Derivations on L^1_{loc}

We denote by L^1_{loc} the space of locally integrable functions on $\mathbb{R}^+$ and by M_{loc} the space of Radon measures on $\mathbb{R}^+$, that is, locally finite, complex Borel measures on $\mathbb{R}^+$. For $n \in \mathbb{N}$ we define the restriction map $R_n\colon M_{\mathrm{loc}} \to M[0,n)$ and the inclusion map $S_n\colon M[0,n) \to M_{\mathrm{loc}}$ in the obvious way. Equipped with the seminorms $\mu \mapsto \|R_n\mu\|$, $\mu \in M_{\mathrm{loc}}$, for $n \in \mathbb{N}$ it is well known that L^1_{loc} and M_{loc} become Fréchet convolution algebras on $\mathbb{R}^+$. These algebras can also be regarded as the projective limits of the spaces $L^1[0,n)$ and respectively $M[0,n)$.

The multipliers and derivations on L^1_{loc} were described in [8, Theorems 2.14 and 3.1]: For $\mu \in M_{\mathrm{loc}}$ the linear map $M_\mu f = \mu * f$, $f \in L^1_{\mathrm{loc}}$, defines a continuous multiplier on L^1_{loc} and conversely every multiplier on L^1_{loc} is of this form. Similarly, for $\mu \in M_{\mathrm{loc}}$ the linear map $D_\mu f = Xf * \mu$, $f \in L^1_{\mathrm{loc}}$, defines a continuous derivation on L^1_{loc} and conversely every derivation on L^1_{loc} is of this form. (In particular, multipliers and derivations on L^1_{loc} are automatically continuous.) Moreover, $\overline{D}_\mu \nu = X\nu * \mu$, $\nu \in M_{\text{loc}}$, extends D_μ to a continuous derivation on M_{loc}.

Let C_c be the space of compactly supported, continuous functions on $\mathbb{R}^+$. We regard C_c as the inductive limit of the spaces $C_0[0,n)$ and equip it with the corresponding inductive limit topology. It follows as in the proof of [12, Proposition 3.3] (see also [9]) that

$$\langle h, \mu \rangle = \int_{\mathbb{R}^+} h(t)\, d\mu(t), \qquad h \in C_c, \quad \mu \in M_{\text{loc}},$$

identifies M_{loc} with the dual space of C_c.

Proposition 3.1.

Let $\mu \in M_{\text{loc}}$. Then the derivation $\overline{D}_\mu$ is weak-star continuous on M_{loc}.

Proof. Let $h \in C_c$ and let

$$(T_\mu h)(t) = t \int_{\mathbb{R}^+} h(t+s)\, d\mu(s), \qquad t \in \mathbb{R}^+.$$

It follows as in [5, Theorem 3.3.15] or the proof of Proposition 2.1 that $T_\mu h$ is continuous. Also, if $\operatorname{supp} h \subseteq [0,n]$ for some $n \in \mathbb{N}$, then $\operatorname{supp} T_\mu h \subseteq [0,n]$, so T_μ maps C_c into C_c. Moreover, for $h \in C_c$ and $\nu \in M_{\text{loc}}$ a calculation similar to the one in the proof of Proposition 2.1 shows that $\langle h, \overline{D}_\mu \nu \rangle = \langle T_\mu h, \nu \rangle$. Hence $\overline{D}_\mu = T^*_\mu$ and in particular $\overline{D}_\mu$ is weak-star continuous. □

It follows from [14, Proposition 8.4.30] that the dual space of L^1_{loc} can be identified with the inductive limit of the dual spaces $L^1[0,n)^* = L^\infty[0,n)$, which again can be identified with the space L^∞_c of measurable functions on $\mathbb{R}^+$ with compact support (with the inductive limit topology).

Lemma 3.2.

The weak topology on L^1_{loc} coincides with the topology τ on L^1_{loc} obtained as the projective limit of the weak topologies on $L^1[0,n)$.

Proof. The topology τ is the coarsest topology on L^1_{loc} making all the restrictions $R_n \colon L^1_{\text{loc}} \to (L^1[0,n), \text{weak})$ continuous. For $n \in \mathbb{N}$ and $\varphi \in L^\infty[0,n) = L^1[0,n)^*$, we let $U_{n,\varphi} = \{g \in L^1[0,n) : |\langle g, \varphi \rangle| < 1\}$. Then $\{U_{n,\varphi} : \varphi \in L^\infty[0,n)\}$ is a base for the 0-neighbourhoods in the weak topology on $L^1[0,n)$. Hence $\{R_n^{-1}(U_{n,\varphi}) : n \in \mathbb{N},\ \varphi \in L^\infty[0,n)\}$ is a base for the 0-neighbourhoods in the τ topology. Moreover, $R_n^{-1}(U_{n,\varphi}) = \{f \in L^1_{\text{loc}} : |\langle R_n f, \varphi \rangle| < 1\} = \{f \in L^1_{\text{loc}} : |\langle f, S_n \varphi \rangle| < 1$. Since $(L^1_{\text{loc}})^* = L^\infty_c$ as the inductive limit of $L^\infty[0,n)$, it follows that τ equals the weak topology on L^1_{loc}. □

It follows from the definition of the projective limit topology on L^1_{loc} that a linear map $T \colon L^1_{\text{loc}} \to L^1_{\text{loc}}$ is continuous if and only if for every $n \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and a constant K such that $\|R_n T f\| \le K \|R_m f\|$ for every $f \in L^1_{\text{loc}}$. For weakly compact operators on L^1_{loc} we have the following general description.

Proposition 3.3.

For a continuous, linear operator T on L^1_{loc} the following conditions are equivalent:

(a) *T is weakly compact.*

(b) *There exists $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ the operator $T_{nm} = R_n T S_m \colon L^1[0,m) \to L^1[0,n)$ is weakly compact and moreover $Tf = 0$ for every $f \in L^1_{\text{loc}}$ with $f = 0$ on $[0,m)$.*

(c) *There exists $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ the operator $T_{nm} = R_n T S_m \colon L^1[0,m) \to L^1[0,n)$ is weakly compact and satisfies $R_n T = T_{nm} R_m$.*

Proof. (a) ⇒ (b): Clearly, T_{nm} is weakly compact for all $m, n \in \mathbb{N}$. Moreover, there exists a neighbourhood U of 0 in L^1_{loc} for which $T(U)$ is weakly relatively compact in L^1_{loc}. For every $n \in \mathbb{N}$ it follows that $R_n T(U)$ is weakly relatively compact, in particular weakly bounded, and thus bounded in $L^1[0,n)$ by the principle of uniform boundedness. There exist $m \in \mathbb{N}$ and $\delta > 0$ such that $V = \{f \in L^1_{\text{loc}} : \|R_m f\| < \delta\} \subseteq U$. It follows that for every $n \in \mathbb{N}$ there exists a constant K_n such that $\|R_nTf\| \le K_n\|R_mf\|$ for $f \in L^1_{\text{loc}}$. In particular, if $f \in L^1_{\text{loc}}$ with $f = 0$ on $[0, m)$, then $Tf = 0$ on $[0, n)$ for every $n \in \mathbb{N}$ and hence $Tf = 0$.

(b) $\Rightarrow$ (c): Since $1 - S_mR_m$ is the projection onto $\{f \in L^1_{\text{loc}} : f = 0 \text{ on } [0, m)\}$, we have $T(1 - S_mR_m) = 0$. Hence $R_nT = R_nTS_mR_m = T_{nm}R_m$ for every $n \in \mathbb{N}$.

(c) $\Rightarrow$ (a): Let $V = \{f \in L^1_{\text{loc}} : \|R_mf\| < 1\}$. Then V is a neighbourhood of 0 in L^1_{loc} and $R_m(V)$ is the unit ball in $L^1[0, m)$, so $R_n(T(V)) = T_{nm}(R_m(V))$ is weakly relatively compact in $L^1[0, n)$ for every $n \in \mathbb{N}$. By Lemma 3.2 and [15, p. 85] it follows that $T(V)$ is weakly relatively compact in L^1_{loc}, so T is weakly compact. □

A slightly simpler version of the proof above shows the following result.

Proposition 3.4.

For a continuous, linear operator T on L^1_{loc} the following conditions are equivalent:

(a) *T is compact.*

(b) *There exists $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ the operator $T_{nm} = R_nTS_m : L^1[0, m) \to L^1[0, n)$ is compact and moreover $Tf = 0$ for every $f \in L^1_{\text{loc}}$ with $f = 0$ on $[0, m)$.*

(c) *There exists $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ the operator $T_{nm} = R_nTS_m : L^1[0, m) \to L^1[0, n)$ is compact and satisfies $R_nT = T_{nm}R_m$.*

Since the derivations D_μ and the multipliers T_μ described in the beginning of the section are all 1-1 we obtain the next two results as consequences of Proposition 3.3.

Corollary 3.5.

There are no non-zero weakly compact derivations on L^1_{loc}.

Corollary 3.6.

There are no non-zero weakly compact multipliers on L^1_{loc}.

Motivated by Corollary 3.5 we will now consider the weaker notions of (weakly) Montel derivations on L^1_{loc} for which we have the following result similar to Theorem 2.2.

Theorem 3.7.

For $\mu \in M_{\text{loc}}$ the following conditions are equivalent:

(a) *D_μ is a Montel derivation on L^1_{loc}.*

(b) *D_μ is a weakly Montel derivation on L^1_{loc}.*

(c) *μ is absolutely continuous.*

(d) *$\overline{D}_\mu$ is a Montel derivation on M_{loc}.*

(e) *$\overline{D}_\mu$ is a weakly Montel derivation on M_{loc}.*

Proof. The implications (d) $\Rightarrow$ (a) $\Rightarrow$ (b) and (d) $\Rightarrow$ (e) $\Rightarrow$ (b) are obvious.

(b) $\Rightarrow$ (c): Let $m \in \mathbb{N}$ and let B_m be the closed unit ball in $L^1[0, m)$. Then

$$S_m(B_m) = \left\{ f \in L^1_{\text{loc}} : \operatorname{supp} f \subseteq [0, m] \text{ and } \|R_mf\| \le 1 \right\},$$

so $R_n(S_m(B_m)) \subseteq B_n$ for every $n \in \mathbb{N}$. Hence $S_m(B_m)$ is bounded in L^1_{loc}. It thus follows that $D_\mu(S_m(B_m))$ is weakly relatively compact in L^1_{loc}, so $R_m(D_\mu(S_m(B_m)))$ is weakly relatively compact in $L^1[0, m)$. Consequently, $R_mD_\mu S_m$ is weakly compact. For $g \in L^1[0, m)$ and $t \in [0, m)$ we have

$$(R_mD_\mu S_m)(g)(t) = \int_{[0,t)} (t-s)(S_mg)(t-s)\, d\mu(s) = \int_{[0,t)} (t-s)g(t-s)\, d(R_m\mu)(s). \tag{1}$$

It is an easy corollary to the description of the continuous derivations on $L^1[0,1)$ mentioned in Section 2 and to Theorem 2.2 that the continuous derivations on $L^1[0,m)$ are exactly the maps $\widetilde{D}_\mu g = Xg*\mu$, $g \in L^1[0,m)$, for some measure μ on $[0,m)$ with $t|\mu|([0,m-t))$ bounded as $t \to 0_+$, and that $\widetilde{D}_\mu$ is (weakly) compact if and only if μ is absolutely continuous and $t|\mu|([0,m-t)) \to 0$ as $t \to 0_+$. From (1) we see that

$$R_m D_\mu S_m = \widetilde{D}_{R_m\mu}, \tag{2}$$

so we deduce that $R_m\mu$ is absolutely continuous. Since $m \in \mathbb{N}$ was arbitrary, this shows that μ is absolutely continuous on $\mathbb{R}^+$.

(c) $\Rightarrow$ (d): Let $m \in \mathbb{N}$. It follows from the equivalent of (2) for $\overline{D}_\mu$ and the comments in the proof of (b) $\Rightarrow$ (c) that $R_m\overline{D}_\mu S_m$ is compact. For $\nu \in M_{\mathrm{loc}}$ we observe that $R_m\overline{D}_\mu\nu$ only depends on $S_mR_m\nu$, that is $R_m\overline{D}_\mu\nu = R_m\overline{D}_\mu S_mR_m\nu$, so $R_m\overline{D}_\mu = R_m\overline{D}_\mu S_mR_m$. Let B be a bounded set in M_{loc}. Then $R_m(B)$ is bounded in $M[0,m)$, so

$$R_m(\overline{D}_\mu(B)) = (R_m\overline{D}_\mu S_m)(R_m(B))$$

is relatively compact in $M[0,m)$. Since $m \in \mathbb{N}$ was arbitrary it follows from [15, p. 85] that $\overline{D}_\mu(B)$ is relatively compact in M_{loc}. Hence $\overline{D}_\mu$ is Montel. □

4. Derivations on $A(\omega)$

In [12] we studied the following class of weighted convolution Fréchet algebras (see [12] for further details). Let ω be an algebra weight on $\mathbb{R}^+$, that is, a positive Borel function satisfying: ω and $1/\omega$ are locally bounded on $\mathbb{R}^+$, ω is right continuous on $\mathbb{R}^+$, ω is submultiplicative, that is $\omega(t+s) \le \omega(t)\omega(s)$ for $t,s \in \mathbb{R}^+$, and $\omega(0)=1$. We then define $L^1(\omega)$ as the weighted space of functions f on $\mathbb{R}^+$ for which $f\omega \in L^1(\mathbb{R}^+)$ with the norm

$$\|f\|_\omega = \int_0^\infty |f(t)|\,\omega(t)\,dt.$$

It is well known that $L^1(\omega)$ with convolution product is a commutative Banach algebra. Similarly, we let $M(\omega)$ be the Banach algebra of locally finite complex Borel measures μ on $\mathbb{R}^+$ for which

$$\|\mu\|_\omega = \int_{\mathbb{R}^+} \omega(t)\,d|\mu|t < \infty.$$

We consider an increasing sequence $\omega = (\omega_n)$ of algebra weights on $\mathbb{R}^+$ satisfying

(a) $\omega_n(t) \to \infty$ as $t \to \infty$ for every $n \in \mathbb{N}$,

(b) $\lim\limits_{t\to\infty} \omega_n^{1/t}(t) = 1$ for every $n \in \mathbb{N}$,

(c) $\sup\limits_{t\in\mathbb{R}^+} \omega_{n+1}(t)/\omega_n(t) = \infty$ for every $n \in \mathbb{N}$.

Let

$$A(\omega) = \bigcap_n L^1(\omega_n) \qquad \text{and} \qquad B(\omega) = \bigcap_n M(\omega_n)$$

and equip $A(\omega)$ and $B(\omega)$ with the increasing sequence of norms $\|\mu\|_n = \|\mu\|_{\omega_n}$, $\mu \in B(\omega)$. In this way $A(\omega)$ and $B(\omega)$ become Fréchet algebras, which can be viewed as projective limits of $L^1(\omega_n)$ and $M(\omega_n)$, respectively.

In [12] we obtained the following characterisation of the derivations on $A(\omega)$.

Theorem 4.1 ([12, Theorem 4.1]).

(a) *Suppose that for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that*

$$\sup_{t\in\mathbb{R}^+} \frac{t\,\omega_n(t)}{\omega_m(t)} < \infty. \tag{3}$$

Then $D_\mu(f) = (Xf)*\mu$, $f \in A(\omega)$, defines a continuous derivation on $A(\omega)$ for every $\mu \in B(\omega)$ and conversely every derivation on $A(\omega)$ has this form. Also, $\overline{D}_\mu(\nu) = (X\nu)*\mu$ for $\nu \in B(\omega)$ extends D_μ to a continuous derivation on $B(\omega)$.

(b) *If condition (3) is not satisfied, then there are no non-zero derivations on $A(\omega)$.*

(In particular, derivations on $A(\omega)$ are automatically continuous.)

As described in [12, Proposition 3.3] the algebra $B(\omega)$ can be identified with the dual space of the space $D(1/\omega) = \bigcup_{n\in\mathbb{N}} C_0(1/\omega_n)$ with the inductive limit topology (where $C_0(1/\omega_n)$ is space of continuous functions h on $\mathbb{R}^+$ for which h/ω_n is vanishes at infinity). We will first show that the derivations $\overline{D}_\mu$ are weak-star continuous.

Proposition 4.2.

Assume that condition (3) is satisfied and let $\mu \in B(\omega)$. Then the derivation $\overline{D}_\mu$ is weak-star continuous on $B(\omega)$.

Proof. Let $h \in D(1/\omega)$ and let

$$(T_\mu h)(t) = t \int_{\mathbb{R}^+} h(t+s)\,d\mu(s), \qquad t \in \mathbb{R}^+.$$

As in the proofs of Propositions 2.1 and 3.1 it follows that $T_\mu h$ is continuous on $\mathbb{R}^+$. Choose $n \in \mathbb{N}$ such that $h \in C_0(1/\omega_n)$, and then choose $m \in \mathbb{N}$ and $C > 0$ such that $t\,\omega_n(t) \le C\omega_m(t)$ for $t \in \mathbb{R}^+$. Let ε be a decreasing function with $\varepsilon(t) \to 0$ as $t \to \infty$ such that $|h(t)| \le \varepsilon(t)\,\omega_n(t)$ for $t \in \mathbb{R}^+$. We then have

$$|(T_\mu h)(t)| \le t \int_{\mathbb{R}^+} \varepsilon(t+s)\,\omega_n(t+s)\,d|\mu|(s) \le \varepsilon(t)\,t\,\omega_n(t) \int_{\mathbb{R}^+} \omega_n(s)\,d|\mu|(s) \le C\varepsilon(t)\,\omega_m(t)\,\|\mu\|_n$$

for $t \in \mathbb{R}^+$. Hence $T_\mu h \in C_0(1/\omega_m)$, and we deduce that T_μ is a continuous linear operator on $D(1/\omega)$. Moreover, for $\nu \in B(\omega)$ and $h \in D(1/\omega)$ a calculation similar to the one in the proof of Proposition 2.1 shows that $\langle h, \overline{D}_\mu \nu\rangle = \langle T_\mu h, \nu\rangle$. Hence $\overline{D}_\mu = T_\mu^*$ and in particular $\overline{D}_\mu$ is weak-star continuous. □

An argument similar to the one above shows that for a derivation D_μ on $L^1(\omega)$, the extension $\overline{D}_\mu$ is weak-star continuous on $M(\omega)$. As for L^1_{loc} in the previous section we will show that the zero operator is the only weakly compact derivation on $A(\omega)$.

Theorem 4.3.

There are no non-zero weakly compact derivations on $A(\omega)$.

Proof. We have $A(\omega)^* = \bigcup_n L^\infty(1/\omega_n)$ by [12, Corollary 2.2], and it follows as in the proof of Lemma 3.2 that the weak topology on $A(\omega)$ coincides with the topology τ obtained as the projective limit of the weak topologies on $L^1(\omega_n)$.

Let D be a weakly compact derivation on $A(\omega)$. There exists a neighbourhood U in $A(\omega)$ for which $D(U)$ is weakly relatively compact in $A(\omega)$. By the above it follows that $D(U)$ is weakly relatively compact in $L^1(\omega_n)$ for every $n \in \mathbb{N}$. In particular, $D(U)$ is weakly bounded and by the principle of uniform boundedness thus bounded in $L^1(\omega_n)$ for every $n \in \mathbb{N}$. There exists $m \in \mathbb{N}$ and $\delta > 0$ such that $V = \{f \in A(\omega) : \|f\|_{L^1(\omega_m)} < \delta\} \subseteq U$. Let $n \in \mathbb{N}$. It follows that there exists a constant K_n such that $\|Df\|_{L^1(\omega_n)} \le K_n \|f\|_{L^1(\omega_m)}$ for $f \in A(\omega)$. Since $A(\omega)$ is dense in $L^1(\omega_n)$ we deduce that D extends to a continuous linear operator $D_n \colon L^1(\omega_m) \to L^1(\omega_n)$. In particular, D_m is a derivation on $L^1(\omega_m)$, so by the Singer–Wermer theorem [5, Corollary 2.7.20 or Theorem 5.2.48] we have $D_m = 0$ and thus $D = 0$. □

We finish the paper by showing that under a slightly stronger assumption than (3), the Montel derivations D_μ on $A(\omega)$ correspond to absolutely continuous measures μ (as for L^1_{loc}).

Theorem 4.4.

Suppose that for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that

$$\frac{t\,\omega_n(t)}{\omega_m(t)} \to 0 \qquad \text{as} \quad t \to \infty,$$

and let $\mu \in B(\omega)$. Then the following conditions are equivalent:

(a) *D_μ is a Montel derivation on $A(\omega)$.*

(b) *D_μ is a weakly Montel derivation on $A(\omega)$.*

(c) *μ is absolutely continuous.*

(d) *$\overline{D}_\mu$ is a Montel derivation on $B(\omega)$.*

(e) *$\overline{D}_\mu$ is a weakly Montel derivation on $B(\omega)$.*

Proof. The implications (d) ⇒ (a) ⇒ (b) and (d) ⇒ (e) ⇒ (b) are obvious.

(b) ⇒ (c): Let (e_k) be a bounded approximate identity for $A(\omega)$ and let $t > 0$. Then $\{\delta_t * e_k : k \in \mathbb{N}\}$ is bounded in $A(\omega)$, so there exist a subsequence (e_{k_j}) and $f \in A(\omega)$ such that $D_\mu(\delta_t * e_{k_j}) \to f$ weakly in $A(\omega)$ as $j \to \infty$. Now, proceed as in the proof of Theorem 2.2, (b) ⇒ (c).

(c) ⇒ (d): Let E be a bounded set in $B(\omega)$. Let $n \in \mathbb{N}$ and choose $m \in \mathbb{N}$ such that $\overline{D}_\mu$ extends to a continuous linear map $\overline{D}_{mn}\colon M(\omega_m) \to M(\omega_n)$. We may assume that $m \in \mathbb{N}$ is chosen so that $t\,\omega_n(t)/\omega_m(t) \to 0$ as $t \to \infty$. It follows from the proof of [6, Theorem 4.1, (c) ⇒ (d)] (see also the proof of Theorem 2.2) combined with the estimate

$$\left\| \frac{\overline{D}_{mn}\delta_t}{\omega_m(t)} \right\|_n = \left\| \frac{t}{\omega_m(t)}\,\delta_t * \mu \right\|_n \le \left\| \frac{t}{\omega_m(t)}\,\delta_t \right\|_n \cdot \|\mu\|_n = \frac{t\,\omega_n(t)}{\omega_m(t)} \cdot \|\mu\|_n \to 0$$

as $t \to \infty$ that $\overline{D}_{mn}$ maps the unit ball in $M(\omega_m)$ to a relatively compact set in $M(\omega_n)$. Since E is bounded in $M(\omega_m)$, we deduce that $\overline{D}_\mu(E) = \overline{D}_{mn}(E)$ is relatively compact in $M(\omega_n)$. Finally, by [15, p. 85] this proves that $\overline{D}_\mu(E)$ is relatively compact in $B(\omega)$, so $\overline{D}_\mu$ is Montel. □

References

[1] Bonet J., Lindström M., Spaces of operators between Fréchet spaces, Math. Proc. Cambridge Philos. Soc., 1994, 115(1), 133–144

[2] Choi Y., Heath M.J., Translation-finite sets and weakly compact derivations from $\ell^1(\mathbb{Z}_+)$ to its dual, Bull. Lond. Math. Soc., 2010, 42(3), 429–440

[3] Choi Y., Heath M.J., Characterizing derivations from the disk algebra to its dual, Proc. Amer. Math. Soc., 2011, 139(3), 1073–1080

[4] Conway J.B., A Course in Functional Analysis, Grad. Texts in Math., 96, Springer, New York, 1985

[5] Dales H.G., Banach Algebras and Automatic Continuity, London Math. Soc. Monogr. Ser., 24, Oxford University Press, New York, 2000

[6] Despić M., Ghahramani F., Grabiner S., Weighted convolution algebras without bounded approximate identities, Math. Scand., 1995, 76(2), 257–272

[7] Edwards R.E., Functional Analysis, Holt, Rinehart and Winston, New York, 1965

[8] Ghahramani F., McClure J.P., Automorphisms and derivations of a Fréchet algebra of locally integrable functions, Studia Math., 1992, 103(1), 51–69

[9] Grabiner S., Homomorphisms of the algebra of locally integrable functions on the half line, J. Aust. Math. Soc., 2006, 81(2), 253–278
[10] Jewell N.P., Sinclair A.M., Epimorphisms and derivations on $L^1(0,1)$ are continuous, Bull. London Math. Soc., 1976, 8(2), 135–139
[11] Kamowitz H., Scheinberg S., Derivations and automorphisms of $L^1(0,1)$, Trans. Amer. Math. Soc., 1969, 135, 415–427
[12] Pedersen T.V., A class of weighted convolution Fréchet algebras, In: Banach Algebras 2009, Banach Center Publ., 91, Polish Academy of Sciences, Warsaw, 2010, 247–259
[13] Pedersen T.V., Compactness and weak-star continuity of derivations on weighted convolution algebras, J. Math. Anal. Appl., 2013, 397(1), 402–414
[14] Pérez Carreras P., Bonet J., Barrelled Locally Convex Spaces, North-Holland Mathematics Studies, 131, North-Holland, Amsterdam, 1987
[15] Robertson A.P., Robertson W., Topological Vector Spaces, 2nd ed., Cambridge Tracts in Mathematics and Mathematical Physics, 53, Cambridge University Press, London, 1973