On the use of semi-closed sets and functions in convex analysis

Abstract: The main aim of this short note is to show that the subdifferentiability and duality results established by Laghdir (2005), Laghdir and Benabbou (2007), and Alimohammady et al. (2011), stated in Fréchet spaces, are consequences of the corresponding known results using Moreau–Rockafellar type conditions.

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1 Introduction

Laghdir [2] introduced the notion of semi-closed sets and functions on separated topological vector spaces in order to get subdifferentiability results for a broader class of convex functions. Then Laghdir and Benabbou [3] extended this approach to duality theory. Later on, Alimohammady et al. [1], using the notions of semi-interior of a set and semi-closed function, declare that their “results extend and improve many known theorems of convex analysis and variational analysis as well as some results in functional analysis” (see [1, Rem. 2.13]).

The main aim of this note is to show that the subdifferentiability and duality results established in the above mentioned papers are consequences of the corresponding known results using Moreau–Rockafellar type conditions. Moreover, we show, providing a counterexample, that a result in [1] is false.

In the next section we introduce the notions and notations we deal with, as well as some quite simple results which will ease the comparison of the results in [1–3] and similar result using classical interiority and continuity conditions.

2 Notions, notations and results

According to Alimohammady et al. (see [1] and the references therein), a subset $A$ of a topological space $(X, \tau)$ is semi-open if there exists $U \in \tau$ such that $U \subset A \subset \text{cl} U$; this is equivalent to $A \subset \text{cl}(\text{int}(A))$ (and is also equivalent to $\text{cl} A = \text{cl}(\text{int}(A))$). Correspondingly, $A \subset (X, \tau)$ is semi-closed if $X \setminus A$ is semi-open; this is equivalent to each of the following assertions: there exists a closed set $F \subset X$ such that $\text{int} F \subset A \subset F$, $A \supset \text{int}(\text{cl}(A))$, $\text{int} A = \text{cl}(\text{int}(A))$. Moreover, the semi-closure of $A \subset (X, \tau)$ is the set $\text{s-cl} A := \cap \{ B \mid A \subset B, B \text{ is semi-closed} \}$, and the semi-interior of $A$ is the set $\text{s-int} A := \cup \{ B \mid B \subset A, B \text{ is semi-open} \}$. The function $f : X \to \mathbb{R}$ is semi-closed if its epigraph is semi-closed in $X \times \mathbb{R}$, where $\text{epi} f := \{(x, t) \in X \times \mathbb{R} \mid f(x) \leq t \}$.
In the sequel we assume that $X$ is a real topological vector space (tvs for short) whose topological dual is denoted by $X^*$. Note that Laghdir [2] introduced the notion of semi-closed set (function) for subsets of (functions defined on) such spaces.

A subset $A$ of $X$ is cs-closed if any convergent series of the form $\sum_{n=1}^{\infty} \lambda_n x_n$ with $(\lambda_n)_{n \geq 1} \subset \mathbb{R}_+$, $\sum_{n=1}^{\infty} \lambda_n = 1$, $(x_n)_{n \geq 1} \subset A$, has the sum in $A$; $A$ is lower cs-closed if there exists a Fréchet space $Y$ and a cs-closed set $C \subset X \times Y$ such that $A = \text{Pr}_X(C)$. A set $A \subset X$ is (lower) ideally convex if in the preceding definition the sequence $(x_n)_{n \geq 1} \subset A$ is required to be bounded. Of course, any (lower) cs-closed set is (lower) ideally convex, and any (lower) ideally convex set is a convex. A function $f : X \to \mathbb{R}$ is cs-closed (lower cs-closed, ideally convex, lower ideally convex) if $\text{epi } f$ is so. Clearly any lower ideally convex function is convex. Recall that the core (or algebraic interior) of $A \subset X$ is the set core $A := \{ a \in X \mid \forall x \in X, \exists \delta > 0, \forall t \in [0, \delta] : a + tx \in A \}$.

Let us recall other notions and notations, used in the sequel, related to functions $f : X \to \mathbb{R}$. The domain of $f$ is the set $\text{dom } f := \{ x \in X \mid f(x) < +\infty \}$; $f$ is proper if $\text{dom } f \neq \emptyset$ and $f(x) \neq -\infty$ for every $x \in X$. The conjugate of $f$ is the function $f^* : X^* \to \mathbb{R}$ defined by $f^*(x^*) := \sup \{ x^* - f \}$, and the subdifferential of $f$ at $x$ with $f(x) \in \mathbb{R}$ is the set $\partial f(x) := \{ x^* \in X^* \mid \langle x', x \rangle \leq f(x') - f(x) \forall x' \in X \}$, where $\langle x, x^* \rangle := x^*(x)$, is semi-closed, and $\partial f(x) := \emptyset$ if $f(x) \notin \mathbb{R}$. Moreover, $\text{epi } f$ is the lsc function convex hull of $f$, where lsc means lower semicontinuous. Having $A \subset X$, the indicator function of $A$ is the function $i_A : X \to \mathbb{R}$ defined by $i_A(x) := 0$ if $x \in A$ and $i_A(x) := +\infty$ if $x \in X \setminus A$; $N_A(x) := \partial i_A(x)$.

**Proposition 2.1.** Let $X$ be a topological vector space and $A \subset X$ be a convex set.

(i) Assume that $\text{int } A \neq \emptyset$. Then $A$ is semi-closed and semi-open; consequently, s-int $A =$s-cl $A = A$.

(ii) $s$-$\text{int } A \neq \emptyset$ if and only if $\text{int } A \neq \emptyset$. Moreover, $\text{int } A \subset s$-$\text{int } A$.

**Proof.** (i) It is known (see e.g. [4, Th. 1.1.2 (iv)]) that $\text{cl } A = \text{cl}(\text{int } A)$ and $\text{int } A = \text{int}(\text{cl } A)$. Hence $A$ is semi-closed and semi-open.

(ii) Assume that $s$-$\text{int } A \neq \emptyset$. Then there exists a semi-open nonempty set $B \subset A$. Since $B \subset \text{cl}(\text{int } B)$, we have that $\text{int } B \neq \emptyset$, whence $\text{int } A \neq \emptyset$.

Assume now that $\text{int } A \neq \emptyset$. By (i) we have that $\text{int } A = A$, and so $s$-$\text{int } A \neq \emptyset$.

The claimed inclusion is obvious if $\text{int } A = \emptyset$, while for $\text{int } A \neq \emptyset$ we have that $\text{int } A \subset A = s$-$\text{int } A$. \]

In [2, p. 151] the author writes: “One may ask a natural question if a lower cs-closed function (see below the definition) is semi-closed? The answer seems to be unknown”. We give a partial answer to this question in the next result.

**Proposition 2.2.** Let $X$ be a complete semi-metrizable locally convex space (for example a Fréchet space), $A \subset X$ and $f : X \to \mathbb{R}$.

(i) If $A$ is lower ideally convex and core $A \neq \emptyset$ then $A$ is semi-closed.

(ii) If $f$ is lower ideally convex and core(\text{dom } f) \neq \emptyset$ then $f$ is semi-closed.

**Proof.** (i) By [4, Cor. 1.3.8] we have that core $A = \text{int } A \neq \emptyset$. Since $A$ is convex, by Proposition 2.1 we have that $A$ is semi-closed.

(ii) Since $f$ is convex and core(\text{dom } f) \neq \emptyset$ we have that core(\text{epi } f) \neq \emptyset; indeed, if $x \in \text{core(\text{dom } f)}$ then $(x, f(x)) \in \text{core(\text{epi } f)}$ for every $\alpha > f(x)$ (see e.g. [5, Lem. 12 (i)]). By (i) we have that $\text{epi } f$ is semi-closed, and so $f$ is semi-closed.

The next condition (extracted from (C. Q1) in [3, p. 1022]) is used in [3] (see also [1]).

(H) $X$ is a Fréchet space, $g : X \to \mathbb{R} \cup \{ +\infty \}$ is convex, proper and semi-closed, $\mathbb{R}_+(\text{dom } g - x) = X$.

Note that for $g$ convex and $x \in \text{dom } g$, $g$ is continuous at $x$ if and only if $x \in \text{core(\text{dom } g)}$ and $\text{int}(\text{epi } g) \neq \emptyset$ (see e.g. [4, Cor. 2.2.10]). Because the condition $\mathbb{R}_+(\text{dom } g - x) = X$ in (H) is equivalent to $x \in \text{core(\text{dom } g)}$, the role of the other two conditions in (H) is to ensure that $\text{int}(\text{epi } g) \neq \emptyset$. In this sense we have the following result.
Proposition 2.3. Assume that (H) holds. Then \( x \in \text{int}(\text{dom } g) \) and \( g \) is continuous on \( \text{int}(\text{dom } g) \).

Proof. As seen above, \( x \in \text{core}(\text{dom } g) \). Let \( \overline{g} \) be the lsc hull of \( g \). Then \( \text{epi } \overline{g} = \text{cl}(\text{epi } g) \), and so \( \overline{g} \) is a lsc convex function; moreover, because \( \overline{g} \leq g \), \( x \in \text{core}(\text{dom } g) \subset \text{core}(\text{dom } \overline{g}) \). Using for example [4, Th. 2.2.20] we obtain that \( \text{core}(\text{dom } \overline{g}) = \text{int}(\text{dom } \overline{g}) \neq \emptyset \) and \( \overline{g} \) is continuous on \( \text{int}(\text{dom } \overline{g}) \). By [4, Cor. 2.2.10] and the fact that \( g \) is semi-closed we have that \( \emptyset \neq \text{int}(\text{epi } \overline{g}) = \text{int}(\text{epi } g) \), and so, using again [4, Th. 2.2.20] we have that \( g \) is continuous on \( \text{int}(\text{dom } g) = \text{core}(\text{dom } g) \); of course, \( x \in \text{int}(\text{dom } g) \). The proof is complete.

Related to [2, Th. 3.2] we have the following result.

Proposition 2.4. Assume that \( X \) is a separated locally convex space and \( f : X \to \mathbb{R} \) is a proper function. If \( \partial f(x) \neq \emptyset \) then \( \overline{\partial} f(x) = f(x) = f(x) \) and \( \overline{\partial} f(x) = \partial f(x) \) (in particular \( f \) is lsc at \( x \)). Conversely, if \( f \) is convex and lsc at \( x \in \text{dom } f \), then \( \overline{\partial} f(x) = \partial f(x) \).

Proof. The first part is known (see, e.g., [4, Th. 2.4.1 (ii)]). Assume that \( f \) is convex and lsc at \( x \in \text{dom } f \). Then \( f \) is convex (because \( f \) is so) and \( f(x) = f(x) \). Since for \( g \leq h \) with \( g(x) = h(x) \in \mathbb{R} \) (for arbitrary \( g, h : X \to \mathbb{R} \)) one has \( \partial g(x) \subseteq \partial h(x) \), we have that \( \partial f(x) \subseteq \partial f(x) \). The converse inclusion being true by the first part, we get the conclusion.

The next result is [2, Cor. 3.2]; taking \( x = 0 \) it is [2, Th. 3.2].

Corollary 2.5. Assume that \( X \) is a Fréchet space, \( f : X \to \mathbb{R} \) is a proper convex function and \( x \in X \) is such that \( \mathbb{R}_+(\text{dom } f - x) \) is a closed linear subspace of \( X \). Then \( \partial f(x) \neq \emptyset \) if and only if \( f \) is lsc at \( x \).

Proof. Of course, as seen in Proposition 2.4, \( \partial f(x) \neq \emptyset \) implies that \( f \) is lsc at \( x \) (without any condition on \( X \) or \( f \)).

Assume now that \( f \) is lsc at \( x \). By Proposition 2.4 we have that \( \partial f(x) = \overline{\partial} f(x) \). Without loss of generality we take \( x = 0 \). By hypothesis, \( X_0 := \mathbb{R}_+ \text{dom } f \) is a closed linear subspace of \( X \), and so \( X_0 \) is a Fréchet space. Then \( \text{dom } f \subset \overline{f} \subset X_0 \). Since \( 0 \in \text{core}(\text{dom } f |_{X_0}) \), we get \( 0 \in \text{core}(\text{dom } \overline{f} |_{X_0}) \). Because \( \overline{f} |_{X_0} \) is lsc, using [4, Th. 2.2.20], we obtain that \( \overline{f} |_{X_0} \) is continuous at \( 0 \). Using now [4, Th. 2.4.12], we have that \( \overline{\partial} f(0) \neq \emptyset \) and so \( \partial f(0) \neq \emptyset \).

3 Semi-interiority and semi-closedness vs classical interiority and continuity conditions

We quote first two results from [1]. The notation is slightly different but easy to understand. Note that there are several misprints in the quoted texts.

Theorem 2.10. Suppose that \( X \) is a Fréchet space, \( C_1 \) and \( C_2 \) are two nonempty convex subsets of \( X \). Let \( s\text{-Int}(C_1) \neq \emptyset \) and \( C_2 \cap s\text{-Int}(C_1) = \emptyset \). Then there exists \( x^* \in X^* \) such that \( \sup_{x \in C_1} (x^*, x) \leq \inf_{x \in C_2} (x^*, y) \).

Theorem 2.12. Suppose that \( X \) is a Fréchet spaces, \( C \) is a semi-closed convex subset of \( X \) and \( f : X \to \mathbb{R} \cup \{+\infty\} \) is a convex function. Let \( s\text{-Int}(C) \cap \text{dom}(f) \neq \emptyset \) and \( f \) be bounded below on \( C \). Then there exists an affine function \( \alpha \leq f \) with \( \inf_{x \in C} f(x) = \inf_{x \in C} \alpha(x) \). Moreover, \( X \) is a solution of (2.11) if and only if \( 0 \in \partial f(x) + N_C(x) \).

The problem (2.11) mentioned above is: minimize \( f(x) \) subject to \( x \in C \subset X \).

- Without assuming \( x^* \neq 0 \), [1, Th. 2.10] (above) is trivial. Asking \( x^* \neq 0 \), taking into account Proposition 2.1, [1, Th. 2.10] is weaker than the usual separation theorem for two nonempty convex sets (see, e.g., [4, Th. 11.13]) because the space \( X \) is required to be Fréchet, the interior of \( C_1 \) be nonempty and \( C_1 \cap C_2 = \emptyset \). A similar remark is valid for [1, Cor. 2.11].
Theorem 2.12 in [1] is false. Theorem 2.12 is true if the condition \( s \)-int \( C \cap \text{dom } f \neq \emptyset \) is replaced by the stronger condition \( \text{int } C \cap \text{dom } f \neq \emptyset \); just use (for example) [4, Cor. 2.8.5] and [4, Th. 2.9.1] (even without requiring \( X \) be a Fréchet space).

Indeed, consider \( X = \mathbb{R}, C := (-\infty, 0] \) and \( f : X \to \mathbb{R} \) defined by \( f(x) := -\sqrt{x} \) for \( x \geq 0 \), \( f(x) := +\infty \) for \( x < 0 \). Then \( s \)-int \( C = C \), and so \( s \)-int \( C \cap \text{dom } f = \{0\} \neq \emptyset \); moreover \( \gamma := \inf_{x \in C} f(x) = f(0) = 0 \).

The conclusion that there exists an affine function \( \alpha \) such that \( \alpha \leq f \) and \( \gamma = \inf_{x \in C} \alpha(x) \) is false; indeed, taking \( \alpha(x) = ax + b \) we get \( 0 = \inf_{x \in C} \alpha(x) \), and so \( a \leq 0 \) and \( b = 0 \), and \( ax \leq -\sqrt{x} \) for every \( x \geq 0 \) (equivalently \( a\sqrt{x} \leq -1 \) for every \( x \geq 0 \)), whence the contradiction \( 0 \leq -1 \). Since \( 0 \in C \) is a solution of the problem \( f(x) \) s.t. \( x \in C \), the fact that \( \partial f(0) + N_C(0) = \emptyset \neq 0 \), shows that the second conclusion of [1, Th. 2.12] is false, too.

The conclusion "there exists an affine function \( \alpha \) such that \( \alpha \leq f \) and \( \gamma = \inf_{x \in C} \alpha(x) \) is false; indeed, taking \( \alpha(x) = ax + b \) we get \( 0 = \inf_{x \in C} \alpha(x) \), and so \( a \leq 0 \) and \( b = 0 \), and \( ax \leq -\sqrt{x} \) for every \( x \geq 0 \) (equivalently \( a\sqrt{x} \leq -1 \) for every \( x \geq 0 \)), whence the contradiction \( 0 \leq -1 \). Since \( 0 \in C \) is a solution of the problem \( f(x) \) s.t. \( x \in C \), the fact that \( \partial f(0) + N_C(0) = \emptyset \neq 0 \), shows that the second conclusion of [1, Th. 2.12] is false, too.

All the duality results in [3] and [1] (stated in Fréchet spaces) are consequences of the corresponding results using Moreau–Rockafellar type (interiority) conditions.

Indeed, using Proposition 2.3 we have that condition \((C.Q_1)\) (that is, besides \((H)\) one assumes that \( f : X \to \mathbb{R} \) is a proper convex function and \( \pi \in \text{dom } f \cap \text{dom } g \) in [3] (considered in [1] as condition (2.1)) implies that \( f, g : X \to \mathbb{R} \) are proper convex functions such that \( g \) is continuous at some \( \pi \in \text{dom } f \cap \text{dom } g \), while condition \((C.Q_2)\) in [3] implies that \( f : X \to \mathbb{R}, g : Y \to \mathbb{R} \) are proper convex functions, \( h : X \to Y \cup \{+\infty\} \) is a \( Y_+\)-convex proper function such that \( g \) is continuous at \( h(\pi) \), where \( \pi \in \text{dom } f \cap h^{-1}(\text{dom } h) \) (\( \subset \text{dom } h \)); condition (2.2) in [1] is obtained from condition \((C.Q_2)\) in [3] replacing \( h \) by \( T \) with the property that \( T_{\text{dom } T} : \text{dom } T \to Y \) is linear and continuous. So, one recognizes the usual Moreau–Rockafellar type conditions.

Other less important remarks:

In the proof of [2, Th. 3.1] one obtained that the function \( f \) is finite at 0 and bounded above on a neighborhood of 0, but, instead of concluding that \( f \) is continuous at 0, the conclusion was only that \( \partial f(0) \neq \emptyset \).

In [2, Rem. 3.1 38)] it is written: 'Note that for a convex set \( A \) of \( X \) one has \( \mathbb{R}_+A = X \) if, and only if, 0 is in the interior of \( A \). So the condition "\( \mathbb{R}_+[\text{dom } f - \pi] = X \)" is equivalent to "\( x \) is the interior of \( \text{dom } f \)" (for \( f \) convex, which is the case throughout the paper), condition which is much older than the Attouch-Brézis condition'.

In fact the condition \( \mathbb{R}_+A = X \) for \( A \) convex is equivalent to 0 \( \in \text{int } A \), and this is far from being equivalent to 0 \( \in \text{int } A \) for \( X \) an infinite dimensional tvs, even if \( X \) is a Fréchet space (which is not assumed here). One of the most general sufficient conditions for \( A = \text{int } A \) is provided by [4, Cor. 1.3.8] (used in the proof of Proposition 2.2).

Remark 2.1 5) in [3] asserts that "In [10], Laghdir studied the subdifferentiability of a convex semi-closed function, i.e. \( \partial f(\pi) \neq \emptyset \) whenever \( \pi \in \text{dom } f, \mathbb{R}_+[\text{dom } f - \pi] = X \) and \( X \) is a Fréchet space. It was proved in [10], that this result false under the weakened condition: \( \mathbb{R}_+[\text{dom } f - \pi] \) is a closed vector subspace". (Here [10] means our reference [2].

This assertion is quite strange because it contradicts [2, Cor. 3.2], that is Corollary 2.5 above.

In Corollaries 3.5, 3.6, 3.7 of [3] one must assume that \( X \) is a Fréchet space, while in Corollary 4.5 one must assume that \( Y \) is a Fréchet space. On the other hand, in the results of Section 4 of [3] it is not needed to assume that \( X \) is a Fréchet space.

I suppose that by cone one means convex cone in [1] (otherwise the relation \( \leq_k \) defined on line 2 of page 1290 is not transitive).

In condition (2.2) of [1] one assumes "\( T : X \to Y \) is linear and bounded"; this sounds strange because \( Y \cup \{+\infty\} \) is not a (topological) vector space. Probably the authors wished to take \( T_0 \in L(X_0, Y) \) with \( X_0 \subset X \) a linear subspace and \( T : X \to Y \cup \{+\infty\} \) defined by \( T(x) := T_0(x) \) for \( x \in X_0, T(x) := +\infty \) for \( x \in X \setminus X_0 \); said differently, the quoted text probably means that \( T_{\text{dom } T} : \text{dom } T \to Y \) is a continuous linear operator. Observe also that in Th. 2.5 and Th. 2.7 of [1] T* (the adjoint?) of such an operator T is considered!

The conclusion "there exists \((x^*, y^*) \in X^* \times Y^* \) such that \( \inf_{x \in X} \{f(x) + g(T(x))\} \leq [f(x) - (x^*, u)] + [g(y + v) - (y^*, v)] \) for all \((u, v) \in X \times Y \) and \((x, y) \in \text{Epi}_K(T) \) of [1, Th. 2.2] is equivalent to the existence of \((x^* = 0 \text{ and } y^* \in K^+ \) such that \( \gamma := \inf_{x \in X} \{f(x) + g(T(x))\} \leq f(x) + g(y) - (y, y^*) + y^*(T(x)) \) for all \( x \in X \) and all \( y \in Y \), where \( y^*(+\infty) := +\infty := g(+\infty) \). Since \( \gamma = -(f + g \circ T)(0) \), the conclusion of..."
[1, Th. 2.2] is equivalent the existence of $y^* \in K^+$ such that $(f + g \circ T)^*(0) \geq (f^* \circ T)^*(0) + g^*(y^*)$.
Of course, this conclusion can be obtained from [4, Th. 2.8.10 (iii)] taking $x^* = 0$ in (2.66).

The discussion above shows that the claim of Alimohammady et al. in [1, Rem. 2.13] that “Our results extend and improve many known theorems of convex analysis and variational analysis as well as some results in functional analysis, the original forms of which can be found in [12, 7, 1, 2, 13, 9, 3, 11, 10, 8, 4–6] and the references cited therein” is exaggerated. In fact the results are particular cases of known results, one of them being even false.

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