On the strong metric dimension of the strong products of graphs

Abstract: Let $G$ be a connected graph. A vertex $w \in V(G)$ strongly resolves two vertices $u, v \in V(G)$ if there exists some shortest $u - w$ path containing $v$ or some shortest $v - w$ path containing $u$. A set $S$ of vertices is a strong resolving set for $G$ if every pair of vertices of $G$ is strongly resolved by some vertex of $S$. The smallest cardinality of a strong resolving set for $G$ is called the strong metric dimension of $G$. It is well known that the problem of computing this invariant is NP-hard. In this paper we study the problem of finding exact values or sharp bounds for the strong metric dimension of strong product graphs and express these in terms of invariants of the factor graphs.

Keywords: Strong metric dimension, Strong metric basis, Strong resolving set, Strong product graphs

MSC: 05C12, 05C69, 05C76

1 Introduction

The problem of uniquely recognizing the position of an intruder in a network was the principal motivation of introducing the concept of locating sets in graphs by Slater in [24]. An analogous concept was also introduced independently by Harary and Melter in [9], where the locating sets were called resolving sets. In the present work we follow the terminology of resolving sets. Given a simple connected graph $G = (V, E)$, a vertex $v \in V$ resolves two vertices $x, y \in V$ if $d_G(x, y) 
eq d_G(y, x)$, where $d_G(a, b)$ represents the length of a shortest $a - b$ path. A set $S \subseteq V$ is a resolving set for $G$ if any pair of vertices of $G$ is resolved by some element of $S$. A resolving set of minimum cardinality is called a metric basis, and its cardinality the metric dimension of $G$.

Another invariant, more restricted than the metric dimension, was presented by Sebő and Tannier in [23], and studied further in several articles. That is, a vertex $w \in V(G)$ strongly resolves two vertices $u, v \in V(G)$ if $d_G(w, u) = d_G(w, v) + d_G(v, u)$ or $d_G(w, v) = d_G(w, u) + d_G(u, v)$, i.e., there exists some shortest $w - u$ path containing $v$ or some shortest $w - v$ path containing $u$. A set $S$ of vertices in a connected graph $G$ is a strong resolving set for $G$ if every two vertices of $G$ are strongly resolved by some vertex of $S$. The smallest cardinality of a strong resolving set of $G$ is called the strong metric dimension and is denoted by $\dim_s(G)$. A strong metric basis of $G$ is a strong resolving set for $G$ of cardinality $\dim_s(G)$.

We denote two adjacent vertices $u, v$ in $G = (V, E)$ by $u \sim v$ and, in this case, we say that $uv$ is an edge of $G$, i.e., $uv \in E$. For a vertex $v \in V$, the set $N(v) = \{u \in V : u \sim v\}$ is the open neighborhood of $v$ and the set $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of $v$. The diameter of $G$ is defined as $D(G) = \max_{u, v \in V} \{d(u, v)\}$. 

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The vertex \( x \in V \) is diametral in \( G \) if there exists \( y \in V \) such that \( d_G(x, y) = D(G) \). We say that \( G \) is 2-antipodal if for each vertex \( x \in V \) there exists exactly one vertex \( y \in V \) such that \( d_G(x, y) = D(G) \).

A set \( S \) of vertices of \( G \) is a vertex cover of \( G \) if every edge of \( G \) is incident with at least one vertex of \( S \). The vertex cover number of \( G \), denoted by \( \alpha(G) \), is the smallest cardinality of a vertex cover of \( G \). We refer to an \( \alpha(G) \)-set in a graph \( G \) as a vertex cover set of cardinality \( \alpha(G) \). A vertex \( u \) of \( G \) is maximally distant from \( v \) if for every \( w \in N_G(u) \), \( d_G(v, w) \leq d_G(u, v) \). \( u \) is maximally distant from \( v \) if and only if \( u \) and \( v \) are mutually maximally distant. The boundary of \( G = (V, E) \) is defined as

\[
\partial(G) = \{ u \in V : \exists v \in V \text{ such that } u, v \text{ are mutually maximally distant} \}.
\]

Oellermann and Peters-Fransen [18] established a relationship between the boundary of a graph and strong resolving sets. This relationship was described by means of the strong resolving graph \( \text{SR}(G) \). The strong resolving graph of a graph \( G \) which has the same vertex set as \( G \) and two vertices \( u \) and \( v \) are joined by an edge in \( \text{SR}(G) \) if and only if \( u \) and \( v \) are mutually maximally distant in \( G \).

Applications of (strong) resolving sets in graphs have been described in several articles. The usefulness of these ideas into long range aids to navigation were described in [24]. Some applications in chemistry for representing chemical compounds [12, 13], or in problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [17] have appeared. Also, applications to navigation of robots in networks are discussed in [14]. Some interesting connections between resolving sets in graphs and the Mastermind game or coin weighing were presented in [2]. In such an article, one motivation is related to the fact that the graphs that arise in Mastermind games or coin weighing are Cartesian product graphs. These applications, and specifically the last one, suggest studying the (strong) metric dimension of other graph products.

Moreover, some applications of strong resolving sets to combinatorial searching were presented in [23]. Specifically, there were analyzed some problems on false coins arising from the connection between information theory and extremal combinatorics. Also, they dealt with a combinatorial optimization problem related to finding “connected joins” in graphs. In such a work, several results about detection of false coins are used to approximate the value of the strong metric dimension of some specific graphs, like for example the Hamming graphs.

On the other hand, studies about graph operations are being frequently presented and published in the last years. The book [8] is a very extensive reference on the theory of product graphs. The most common graph products are the Cartesian product, the strong product, the direct product, and the lexicographic product, which are called as standard products. It is therefore natural that several graphs invariants are studied for such classes of graph products, in the sense of computing or bounding their exact value, or deducing their behavior with respect to the factor graphs.

The metric dimension of product graphs is no exception. Some examples of these appear in the following published articles. The metric dimension of Cartesian product graphs, lexicographic product graphs, strong product graphs, hierarchical product graphs and corona product graphs was studied in [2], [10, 21], [20], [4] and [25], respectively. Also, the strong metric dimension of Hamming graphs and corona product graphs was studied in [15] and [16], respectively. This has motivated us to study the strong metric dimension of strong product graphs.

The second motivation of our work is related to the close relationship which exists, as we will see, between the strong metric dimension of a graph and the independence number of its strong resolving graph. The independence number of the repeated strong product of graph with itself was studied as a result of its relationship with the Shannon capacity of a graph, see [8]. As we will show in this article, the strong metric dimension of a strong product graph is related to the independence number of the strong product of the strong resolving graphs of its factors. Hence the connection and our motivation.

We recall that the Cartesian product of two graphs \( G = (V_1, E_1) \) and \( H = (V_2, E_2) \) is the graph \( G \square H = (V, E) \), such that \( V = V_1 \times V_2 \) and two vertices \( (a, b) \in V \) and \( (c, d) \in V \) are adjacent in \( G \square H \) if and only if either

- \( a = c \) and \( bd \in E_2 \), or
- \( ac \in E_1 \) and \( b = d \).

The strong product of two graphs \( G = (V_1, E_1) \) and \( H = (V_2, E_2) \) is the graph \( G \boxtimes H = (V, E) \), such that \( V = V_1 \times V_2 \) and two vertices \( (a, b) \in V \) and \( (c, d) \in V \) are adjacent in \( G \boxtimes H \) if and only if either

- \( a = c \) and \( bd \in E_2 \), or
- \(ac \in E_1\) and \(b = d\), or
- \(ac \in E_1\) and \(bd \in E_2\).

The **lexicographic product** of two graphs \(G = (V_1, E_1)\) and \(H = (V_2, E_2)\) is the graph \(G \circ H\) with the vertex set \(V = V_1 \times V_2\) and two vertices \((a, b) \in V\) and \((c, d) \in V\) are adjacent if either
- \(ac \in E_1\), or
- \(a = c\) and \(bd \in E_2\).

The **Cartesian sum** of two graphs \(G = (V_1, E_1)\) and \(H = (V_2, E_2)\), denoted by \(G \oplus H\), has as the vertex set \(V = V_1 \times V_2\) and two vertices \((a, b) \in V\) and \((c, d) \in V\) are adjacent in \(G \oplus H\) if and only if \(ac \in E_1\) or \(bd \in E_2\).

This notion of a graph product was introduced by Ore [19]. The Cartesian sum is also known as the disjunctive product [22].

Notice that the Cartesian product, the strong product and the Cartesian sum of graphs are commutative operations. This is implicitly used throughout the article. Also, the following formula on the vertex distances in strong product graphs [8] is used in several situations without specific mentioning of the result.

**Remark 1.1 ([8]).** For any graphs \(G\) and \(H\) and any two vertices \((a, b), (c, d)\) of \(G \boxtimes H\),

\[
d_{G \boxtimes H}((a, b), (c, d)) = \max\{d_G(a, c), d_H(b, d)\}.
\]

Let \(G = (V, E)\) and \(G' = (V', E')\) be two graphs. If \(V' \subseteq V\) and \(E' \subseteq E\), then \(G'\) is a subgraph of \(G\) and we denote that by \(G' \subseteq G\).

In this article we are interested in the study of strong resolving sets of strong product graphs. It was shown in [18] that the problem of computing \(\dim_s(G)\) is NP-hard. This suggests obtaining closed formulaes for the strong metric dimension of specific non trivial families of graphs or bounding the value of this invariant as tight as possible.

## 2 Results

Oellermann and Peters-Fransen [18] showed that the problem of finding the strong metric dimension of a graph \(G\) can be transformed into the problem of computing the vertex cover number of \(G_{SR}\).

**Theorem 2.1 ([18]).** For any connected graph \(G\),

\[
\dim_s(G) = \alpha(G_{SR}).
\]

Recall that the largest cardinality of a set of vertices of \(G\), no two of which are adjacent, is called the **independence number** of \(G\) and is denoted by \(\beta(G)\). We refer to an \(\beta(G)\)-set in a graph \(G\) as an independent set of cardinality \(\beta(G)\). The following well-known result, due to Gallai [5], states the relationship between the independence number and the vertex cover number of a graph.

**Theorem 2.2 ([5], Gallai’s theorem).** For any graph \(G\) of order \(n\),

\[
\alpha(G) + \beta(G) = n.
\]

Thus, for any graphs \(G\) and \(H\) of order \(n_1\) and \(n_2\), respectively, by using Theorems 2.1 and 2.2, we immediately obtain that

\[
\dim_s(G \boxtimes H) = n_1 \cdot n_2 - \beta((G \boxtimes H)_{SR}).
\]

The following well known result about the neighborhood of a vertex in the strong product graph \(G \boxtimes H\), which can be found for instance in [8], is useful to present our results.
Remark 2.3 ([8]). Let $G$ and $H$ be two graphs. For every $u \in V(G)$ and $v \in V(H)$

$$N_{G \boxtimes H}(u, v) = N_G[u] \times N_H[v].$$

The next result follows directly from the above remark.

Corollary 2.4. Let $G$ and $H$ be two graphs and let $u, u' \in V(G)$ and $v, v' \in V(H)$. The following assertions hold.

(i) If $(u', v') \in N_{G \boxtimes H}(u, v)$, then $u' \in N_G[u]$ and $v' \in N_H[v]$.

(ii) If $u' \in N_G(u)$ and $v' \in N_H(v)$, then $(u', v') \in N_{G \boxtimes H}(u, v)$.

The following result about the boundary of strong product graphs was presented in [1]. Nevertheless in such a paper the authors are more interested into the cardinality of the boundary $\partial(G \boxtimes H)$ than into how the subgraph induced by boundary looks like.

Theorem 2.5 ([1]). For any graphs $G$ and $H$, $\partial(G \boxtimes H) = (\partial(G) \times V(H)) \cup (V(G) \times \partial(H))$.

In the next lemma we describe the structure of the strong resolving graph of $G \boxtimes H$.

Lemma 2.6. Let $G$ and $H$ be two connected nontrivial graphs. Let $u, x$ be two vertices of $G$ and let $v, y$ be two vertices of $H$. Then $(u, v)$ and $(x, y)$ are mutually maximally distant vertices in $G \boxtimes H$ if and only if one of the following conditions holds:

(i) $u, x$ are mutually maximally distant in $G$ and $v, y$ are mutually maximally distant in $H$;

(ii) $u, x$ are mutually maximally distant in $G$ and $v = y$;

(iii) $v, y$ are mutually maximally distant in $H$ and $u = x$;

(iv) $u, x$ are mutually maximally distant in $G$ and $d_G(u, x) > d_H(v, y)$;

(v) $v, y$ are mutually maximally distant in $H$ and $d_G(u, x) < d_H(v, y)$.

Proof. (Sufficiency) Let $(u', v') \in N_{G \boxtimes H}(u, v)$ and $(x', y') \in N_{G \boxtimes H}(x, y)$. By Corollary 2.4 we have $u' \in N_G[u], x' \in N_G[x], v' \in N_H[v]$ and $y' \in N_H[y]$.

(i) If $u, x$ are mutually maximally distant in $G$ and $v, y$ are mutually maximally distant in $H$, then

$$d_{G \boxtimes H}((u', v'), (x, y)) = \max\{d_G(u', x), d_H(v', y)\} \leq \max\{d_G(u, x), d_H(v, y)\} = d_{G \boxtimes H}((u, v), (x, y))$$

and

$$d_{G \boxtimes H}((u, v), (x', y')) = \max\{d_G(u, x'), d_H(v, y')\} \leq \max\{d_G(u, x), d_H(v, y)\} = d_{G \boxtimes H}((u, v), (x, y)).$$

Thus, $(u, v)$ and $(x, y)$ are mutually maximally distant vertices in $G \boxtimes H$.

(ii) If $u, x$ are mutually maximally distant in $G$ and $v = y$, then

$$d_{G \boxtimes H}((u', v'), (x, y)) = \max\{d_G(u', x), d_H(v', y)\} = d_G(u', x) \leq d_G(u, x) = d_{G \boxtimes H}((u, v), (x, y))$$

and

$$d_{G \boxtimes H}((u, v), (x', y')) = \max\{d_G(u, x'), d_H(v, y')\} = d_G(u, x') \leq d_G(u, x) = d_{G \boxtimes H}((u, v), (x, y)).$$

Thus, $(u, v)$ and $(x, y)$ are mutually maximally distant vertices in $G \boxtimes H$.

(iii) According to the commutativity of the strong product of graphs, the result follows directly from (ii).

(iv) If $u, x$ are mutually maximally distant in $G$ and $d_G(u, x) > d_H(v, y)$, then

$$d_{G \boxtimes H}((u', v'), (x, y)) = \max\{d_G(u', x), d_H(v', y)\}$$

$$\leq \max\{d_G(u, x), d_H(v, y) + 1\}$$

$$= \max\{d_G(u, x), d_H(v, y)\}$$

$$= d_{G \boxtimes H}((u, v), (x, y))$$
Thus, \((u, v)\) and \((x, y)\) are mutually maximally distant vertices in \(G \boxtimes H\).

(v) According to the commutativity of the strongly product of graphs, the result follows directly from (iv).

(Necessity) Let \((u, v)\) and \((x, y)\) be two mutually maximally distant vertices in \(G \boxtimes H\). Let \(u' \in N_G(u), x' \in N_G(x), v' \in N_H(v)\) and \(y' \in N_H(y)\). Notice that, by Corollary 2.4 \((u', v') \in N_G \boxcirc H(u, v)\) and \((x', y') \in N_G \boxcirc H(x, y)\). So, we have that

\[
d_{G \boxtimes H}((u, v), (x, y)) = \max \{d_G(u, x'), d_H(v, y')\}
\]

and

\[
d_{G \boxtimes H}((u, v), (x, y)) = \max \{d_G(u, x), d_H(v, y)\}
\]

We differentiate two cases.

Case 1. \(d_G(u, x) \geq d_H(v, y)\). Hence, \(d_{G \boxtimes H}((u, v), (x, y)) = \max \{d_G(u, x), d_H(v, y)\} = d_G(u, x)\). Thus,

\[
d_G(u, x) \geq \max \{d_G(u', x), d_H(v', y)\}
\]

and

\[
d_G(u, x) \geq \max \{d_G(u, x'), d_H(v, y')\}.
\]

So, we obtain four inequalities:

\[
\begin{align*}
d_G(u, x) & \geq d_G(u', x), \quad (2) \\
d_G(u, x) & \geq d_H(v', y), \quad (3) \\
d_G(u, x) & \geq d_G(u, x'), \quad (4) \\
d_G(u, x) & \geq d_H(v, y'). \quad (5)
\end{align*}
\]

From (2) and (4) we have, that \(u\) and \(x\) are mutually maximally distant in \(G\). If \(v\) and \(y\) are mutually maximally distant in \(H\), then (i) holds and, if \(v = y\), then (ii) holds. Suppose that there exists a vertex \(v'' \in N_H(v)\) such that \(d_H(v'', y) > d_H(v, y)\) or there exists a vertex \(y'' \in N_H(y)\) such that \(d_H(v, y'') > d_H(v, y)\). In such a case,

\[
d_H(v'', y) \geq d_H(v, y) + 1
\]

or

\[
d_H(v, y'') \geq d_H(v, y) + 1.
\]

Since \(v'' \in N_H(v)\), for any \(u'' \in N_G(u)\) we have \((u'', v'') \in N_G \boxcirc H(u, v)\) and following the above procedure, taking \((u'', v'')\) instead of \((u', v')\) we obtain two inequalities equivalent to (3) and (5). Thus,

\[
d_G(u, x) \geq d_H(v'', y) > d_H(v, y) \quad (8)
\]

and

\[
d_G(u, x) \geq d_H(v, y'') > d_H(v, y). \quad (9)
\]

So, \(u, x\) are mutually maximally distant in \(G\) and \(d_G(u, x) > d_H(v, y)\). Hence, (iv) is satisfied.

Case 2. \(d_G(u, x) < d_H(v, y)\). By using analogous procedure we can prove that \(v, y\) are mutually maximally distant in \(H\) and \(u = x\) or \(d_G(u, x) < d_H(v, y)\), showing that (iii) and (v) hold. Therefore, the result follows. \(\square\)

Notice that Lemma 2.6 leads to the following relationship.
Theorem 2.7. For any connected graphs $G$ and $H$,

$$G_{SR} \boxtimes H_{SR} \subseteq (G \boxtimes H)_{SR} \subseteq G_{SR} \otimes H_{SR}.$$ 

Proof. Notice that $V(G_{SR} \boxtimes H_{SR}) = V((G \boxtimes H)_{SR}) = V(G_{SR} \otimes H_{SR}) = V_1 \times V_2$. Let $(u, v)$ and $(x, y)$ be two vertices adjacent in $G_{SR} \boxtimes H_{SR}$. So, either
- $u = x$ and $vy \in E(H_{SR})$, or
- $ux \in E(G_{SR})$ and $v = y$, or
- $ux \in E(G_{SR})$ and $vy \in E(H_{SR})$.

Hence, by using respectively the condition (iii), (ii) and (i) of Lemma 2.6 we have that $(u, v)$ and $(x, y)$ are also adjacent in $(G \boxtimes H)_{SR}$.

Now, let $(u', v')$ and $(x', y')$ be two vertices adjacent in $(G \boxtimes H)_{SR}$. From Lemma 2.6 we obtain that $u'x' \in E(G_{SR})$ or $v'y' \in E(H_{SR})$. Thus, $(u', v')$ and $(x', y')$ are also adjacent in $G_{SR} \otimes H_{SR}$.

Corollary 2.8. For any connected graphs $G$ and $H$,

$$\beta(G_{SR} \boxtimes H_{SR}) \geq \beta((G \boxtimes H)_{SR}) \geq \beta(G_{SR} \otimes H_{SR}).$$

In order to better understand what the strong resolving graph $(G \boxtimes H)_{SR}$ looks like, by using Lemma 2.6, we prepare a kind of “graphical representation” of $(G \boxtimes H)_{SR}$ which we present in Figure 1. According to the conditions (i), (ii) and (iii) of Lemma 2.6 the solid lines represents those edges of $(G \boxtimes H)_{SR}$ which always exists. Also, from the conditions (iv) and (v) of Lemma 2.6, two vertices belonging to different rounded rectangles with identically filled areas could be adjacent or not in $(G \boxtimes H)_{SR}$.

The following three known results will be useful for our purposes.

Theorem 2.9 ([11]). For any graphs $G$ and $H$,

$$\beta(G) \cdot \beta(H) \leq \beta(G \boxtimes H) \leq \beta(G \square H).$$

Theorem 2.10 (Vizing’s theorem). For any graphs $G$ and $H$,

$$\beta(G \square H) \leq \min\{\beta(G)|V(H)|, \beta(H)|V(G)|\}.$$ 

Theorem 2.11 ([6]). For any graphs $G$ and $H$,

$$\beta(G \circ H) = \beta(G) \cdot \beta(H).$$
Next we present the following lemma, from [3], about the independence number of Cartesian sum graphs.

**Lemma 2.12** ([3]). For any graphs $G$ and $H$,

$$\beta(G \oplus H) = \beta(G) \cdot \beta(H).$$

**Theorem 2.13.** Let $G$ and $H$ be two connected nontrivial graphs of order $n_1$, $n_2$, respectively. Then

$$\max\{n_2 \cdot \dim_s(G), n_1 \cdot \dim_s(H)\} \leq \dim_s(G \boxtimes H) \leq n_2 \cdot \dim_s(G) + n_1 \cdot \dim_s(H) - \dim_s(G) \cdot \dim_s(H).$$

**Proof.** By using Corollary 2.8 we have that

$$\beta(G_{SR} \boxtimes H_{SR}) \geq \beta((G \boxtimes H)_{SR}).$$

Hence, from equality (1), Theorem 2.9 and Theorem 2.10 we obtain

$$\dim_s(G \boxtimes H) = n_1 \cdot n_2 - \beta((G \boxtimes H)_{SR})$$

$$\geq n_1 \cdot n_2 - \beta(G_{SR} \boxtimes H_{SR})$$

$$\geq n_1 \cdot n_2 - \beta(G_{SR} \boxtimes H_{SR})$$

$$\geq n_1 \cdot n_2 - \min\{n_2 \cdot \beta(G_{SR}), n_1 \cdot \beta(H_{SR})\}$$

$$= \max\{n_2(n_1 - \beta(G_{SR})), n_1(n_2 - \beta(H_{SR}))\}$$

$$= \max\{n_2 \cdot \dim_s(G), n_1 \cdot \dim_s(H)\}.$$

On the other hand, from Corollary 2.8 it follows $\beta((G \boxtimes H)_{SR}) \geq \beta(G_{SR} \oplus H_{SR})$. So, by using (1) and Lemma 2.12 we have

$$\dim_s(G \boxtimes H) = n_1 \cdot n_2 - \beta((G \boxtimes H)_{SR})$$

$$\leq n_1 \cdot n_2 - \beta(G_{SR} \oplus H_{SR})$$

$$= n_1 \cdot n_2 - \beta(G_{SR}) \cdot \beta(H_{SR})$$

$$= n_1 \cdot n_2 - (n_1 - \dim_s(G)) \cdot (n_2 - \dim_s(H))$$

$$= n_2 \cdot \dim_s(G) + n_1 \cdot \dim_s(H) - \dim_s(G) \cdot \dim_s(H).$$

We define a $C$-graph as a graph $G$ whose vertex set can be partitioned into $\beta(G)$ cliques. Notice that there are several graphs which are $C$-graphs. For instance, we emphasize the following cases: complete graphs and cycles of even order. In order to prove the next result we also need to introduce the following notation. Given two graphs $G = (V_1, E_1)$, $H = (V_2, E_2)$ and a subset $X$ of vertices of $G \boxtimes H = (V, E)$, the *projections* of $X$ over the graphs $G$ and $H$, respectively, are the following ones

$$P_G(X) = \{u \in V_1 : (u, v) \in X, \text{ for some } v \in V_2\},$$

$$P_H(X) = \{v \in V_2 : (u, v) \in X, \text{ for some } u \in V_1\}.$$

**Lemma 2.14.** For any $C$-graph $G$ and any graph $H$,

$$\beta(G \boxtimes H) = \beta(G) \cdot \beta(H).$$

**Proof.** Let $A_1, A_2, \ldots, A_{\beta(G)}$ be a partition of $V(G)$ such that $A_i$ is a clique for every $i \in \{1, 2, \ldots, \beta(G)\}$. Let $S$ be an $\beta(G \boxtimes H)$-set and let $S_i = S \cap (A_i \times V_2)$ for $i \in \{1, 2, \ldots, \beta(G)\}$. First we will show that $P_H(S_i)$ is an independent set in $H$. If $|P_H(S_i)| = 1$, then $P_H(S_i)$ is an independent set in $H$. If $|P_H(S_i)| \geq 2$, then for any two vertices $x, y \in P_H(S_i)$ there exist $u, v \in A_i$ such that $(u, x), (v, y) \in S_i$. We suppose that $x \sim y$. If $u = v$, then $(u, x) \sim (v, y)$, which is a contradiction. Thus, $u \neq v$. Since $(u, x) \not\sim (v, y)$, we have that $x \not\sim v$, which is a contradiction with the fact that $A_i$ is a clique. Therefore, for every $i \in \{1, 2, \ldots, \beta(G)\}$ the projection $P_H(S_i)$ is an independent set in $H$ and $\beta(H) \geq |P_H(S_i)|$.

Now, if $|S_i| > |P_H(S_i)|$ for some $i \in \{1, 2, \ldots, \beta(G)\}$, then there exists a vertex $z \in P_H(S_i)$ and two different vertices $a, b \in A_i$ such that $(a, z), (b, z) \in S_i$, and this is a contradiction with the facts that $A_i$ is a clique and $S_i$ is
an independent set. Thus, \( |S_i| = |\mathcal{P}(S_i)|, i \in \{1, 2, \ldots, \beta(G)\} \), and we have the following

\[
\beta(G \boxtimes H) = |S| = \sum_{i=1}^{\beta(G)} |S_i| = \sum_{i=1}^{\beta(G)} |\mathcal{P}(S_i)| \leq \beta(G) \cdot \beta(H).
\]

Therefore, by using Theorem 2.9 we conclude the proof.

**Theorem 2.15.** Let \( G \) and \( H \) be two connected nontrivial graphs of order \( n_1, n_2 \), respectively. If \( G \) is a \( C \)-graph, then

\[
dim_s(G \boxtimes H) = n_2 \cdot \dim_s(G) + n_1 \cdot \dim_s(H) - \dim_s(G) \cdot \dim_s(H).
\]

**Proof.** By using Corollary 2.8 we have that \( \beta(G \boxtimes H) \geq \beta((G \boxtimes H)_S) \). Hence, from equality (1) and Lemma 2.14 we have

\[
dim_s(G \boxtimes H) = n_2 \cdot \dim_s(G) + n_1 \cdot \dim_s(H) - \dim_s(G) \cdot \dim_s(H).
\]

The result now follows from Theorem 2.13.

At next we give examples of graphs for which its strong resolving graphs are \( C \)-graphs. To do so we need some additional terminology and notations.

A **cut vertex** in a graph is a vertex whose removal increases the number of connected component and a **simplicial vertex** is a vertex \( v \) such that the subgraph induced by \( N[v] \) is isomorphic to a complete graph. Also, a **block** is a maximal biconnected subgraph of the graph. Now, let \( \mathcal{G} \) be the family of sequences of connected graphs \( G_1, G_2, \ldots, G_k \), \( k \geq 2 \), such that \( G_1 \) is a complete graph \( K_{n_1}, n_1 \geq 2 \), and \( G_i, i \geq 2 \), is obtained recursively from \( G_{i-1} \) by adding a complete graph \( K_{n_i}, n_i \geq 2 \), and identifying a vertex of \( G_{i-1} \) with a vertex in \( K_{n_i} \).

From this point we will say that a connected graph \( G \) is a **generalized tree** \(^1\) if and only if there exists a sequence \{\( G_1, G_2, \ldots, G_k \)\} \( \in \mathcal{G} \) such that \( G_k = G \) for some \( k \geq 2 \). Notice that in these generalized trees every vertex is either, a cut vertex or a simplicial vertex. Also, every complete graph used to obtain the generalized tree is a block of the graph. Note that if every \( G_i \) is isomorphic to \( K_{2} \), then \( G_k \) is a tree, justifying the terminology used.

- \((K_n)_S\) is isomorphic to \( K_n \).
- For any complete \( k \)-partite graph such that at least all but one \( p_i \geq 2 \), \( i \in \{1, 2, \ldots, k\} \), \((K_{p_1}, K_{p_2}, \ldots, K_{p_k})_S\) is isomorphic to the graph \( \bigcup_{i=1}^{k} K_{p_i} \).
- If \( G \) is a generalized tree of order \( n \) and \( c \) cut vertices, then \( G_S \) is isomorphic to the graph \( K_{n-c} \cup \bigcup_{i=1}^{c} K_1 \).
- For any 2-antipodal \(^2\) graph \( G \) of order \( n \), \( G_S \) is isomorphic to the graph \( \bigcup_{i=1}^{n} K_2 \).
- For any grid graph, \((P_n \boxtimes P_r)_S\) is isomorphic to the graph \( K_2 \cup K_2 \cup \bigcup_{i=1}^{r-4} K_1 \).

By using the above examples and Theorem 2.15 we have the following corollary.

**Corollary 2.16.** Let \( G \) and \( H \) be two connected nontrivial graphs of order \( n_1 \) and \( n_2 \), respectively.

(i) \( \dim_s(K_{n_1} \boxtimes H) = n_2(n_1-1) + n_1 \cdot \dim_s(H) - (n_1-1) \dim_s(H) \).

(ii) If \( G \) is a complete \( k \)-partite graph, then

\[
\dim_s(G \boxtimes H) = n_2(n_1-k) + n_1 \cdot \dim_s(H) - (n_1-k) \dim_s(H).
\]

\(^1\) In some works those graphs are called block graphs.

\(^2\) Notice that for instance cycles of even order are 2-antipodal graphs.
If $G$ is a generalized tree with $c$ cut vertices, then
\[ \dim_s(G \boxtimes H) = n_2(n_1 - c - 1) + n_1 \cdot \dim_s(H) - (n_1 - c - 1)\dim_s(H). \]

Particularly, if $G$ is a tree with $l(G)$ leaves, then
\[ \dim_s(G \boxtimes H) = n_2(l(G) - 1) + n_1 \cdot \dim_s(H) - (l(G) - 1)\dim_s(H). \]

If $G$ is a 2-antipodal graph, then
\[ \dim_s(G \boxtimes H) = \frac{n_2 \cdot n_1}{2} + n_1 \cdot \dim_s(H) - \frac{n_1^2}{2} \cdot \dim_s(H). \]

If $G$ is a grid graph, then
\[ \dim_s(G \boxtimes H) = 3n_2 + n_1 \cdot \dim_s(H) - 3\dim_s(H). \]

Notice that Corollary 2.16 (iv) gives the value for the strong metric dimension of $C_r \boxtimes H$ for any graph $H$ and $r$ even. Next we study separately the strong product graphs $C_r \boxtimes H$ for any graph $H$ and $r$ odd. In order to prove the next result we need to introduce the following notation. We define a $C_1$-graph as a graph whose vertex set can be partitioned into $\beta(G)$ cliques and one isolated vertex. Notice that cycles of odd order are $C_1$-graphs.

**Lemma 2.17.** For any $C_1$-graph $G$ and any graph $H$,
\[ \beta(G \boxtimes H) \leq \beta(G) \beta(H) + 1. \]

**Proof.** Let $A_1, A_2, ..., A_{\beta(G)}, B$ be a partition of $V(G)$ such that $A_i$ is a clique for every $i \in \{1, 2, ..., \beta(G)\}$ and $B = \{b\}$, where $b$ is isolated vertex. Let $S$ be an $\beta(G) \boxtimes H$-set and let $S_i = S \cap (A_i \times V_2)$ and $i \in \{1, 2, ..., \beta(G)\}$. Let $S_B = S \setminus (B \times V_2)$. By using analogous procedures as in proof of Lemma 2.14 we can show that for every $i \in \{1, 2, ..., \beta(G)\}$, $P_H(S_i)$ is an independent set in $H$ and $|S_i| = |P_H(S_i)|$. Moreover, since $|B| = 1$ we have that $P_H(S_B)$ is an independent set in $H$ and $|S_B| = |P_H(S_B)|$. Thus, we obtain the following
\[ \beta(G \boxtimes H) = |S| = \sum_{i=1}^{\beta(G)} |S_i| + |S_B| = \sum_{i=1}^{\beta(G)} |P_H(S_i)| + |P_H(S_B)| \]
\[ \leq \beta(G) \cdot \beta(H) + \beta(H) = \beta(G) \beta(H) + 1. \]

**Theorem 2.18.** Let $G$ and $H$ be two connected nontrivial graphs of order $n_1, n_2$, respectively. If $G_{SR}$ is a $C_1$-graph, then
\[ \dim_s(G \boxtimes H) \geq n_1(\dim_s(H) - 1) + \dim_s(G)(n_2 - \dim_s(H) + 1). \]

**Proof.** By using Corollary 2.8 we have that $\beta(G_{SR} \boxtimes H_{SR}) \geq \beta((G \boxtimes H)_{SR})$. Hence, from equality (1) and Lemma 2.17 we have
\[ \dim_s(G \boxtimes H) = n_1 \cdot n_2 - \beta((G \boxtimes H)_{SR}) \]
\[ \geq n_1 \cdot n_2 - \beta(G_{SR} \boxtimes H_{SR}) \]
\[ \geq n_1 \cdot n_2 - \beta(G_{SR})(\beta(H_{SR}) + 1) \]
\[ = n_1 \cdot n_2 - (n_1 - \dim_s(G)) \cdot (n_2 - \dim_s(H) + 1) \]
\[ = n_1(\dim_s(H) - 1) + \dim_s(G)(n_2 - \dim_s(H) + 1). \]

Since $\dim_s(C_{2r+1}) = r + 1$, Theorems 2.13 and 2.18 lead to the following result.

**Theorem 2.19.** Let $H$ be a connected nontrivial graphs of order $n$ and $r \geq 1$. Then
\[ n(r + 1) + r(\dim_s(H) - 1) \leq \dim_s(C_{2r+1} \boxtimes H) \leq n(r + 1) + r \cdot \dim_s(H). \]
The following results on the independence number of strong products of odd cycles is obtained in [7].

**Theorem 2.20 ([7]).** For \(1 \leq r \leq t\),

\[
\beta(C_{2r+1} \boxtimes C_{2t+1}) = r \cdot t + \left\lfloor \frac{r}{2} \right\rfloor.
\]

By using the above result we obtain the following.

**Theorem 2.21.** For \(1 \leq r \leq t\),

\[
3rt + 2r + 2t + 1 - \left\lfloor \frac{r}{2} \right\rfloor \leq \dim_s(C_{2r+1} \boxtimes C_{2t+1}) \leq 3rt + 2r + 2t + 1.
\]

**Proof.** By using Theorem 2.7 we have that 
\(G_{SR} \boxtimes H_{SR} \subseteq (G \boxtimes H)_{SR}\). Thus, \(\beta(G_{SR} \boxtimes H_{SR}) \geq \beta((G \boxtimes H)_{SR})\).

Hence, from equality (1) and Theorem 2.20 we have

\[
dim_s(C_{2r+1} \boxtimes C_{2t+1}) = (2r + 1) \cdot (2t + 1) - \beta((C_{2r+1} \boxtimes C_{2t+1})_{SR})
\]

\[
\geq (2r + 1) \cdot (2t + 1) - \beta((C_{2r+1})_{SR} \boxtimes (C_{2t+1})_{SR})
\]

\[
= (2r + 1) \cdot (2t + 1) - \beta(C_{2r+1} \boxtimes C_{2t+1})
\]

\[
= (2r + 1) \cdot (2t + 1) - r \cdot t - \left\lfloor \frac{r}{2} \right\rfloor
\]

\[
= 3rt + 2r + 2t + 1 - \left\lfloor \frac{r}{2} \right\rfloor.
\]

The upper bound is a direct consequence of Theorem 2.18.

Notice that for \(r = 1\) the lower bound is equal to the upper bound in the above theorem. Thus, \(\dim_s(C_{3} \boxtimes C_{2t+1}) = 5t + 3\) for every \(t \geq 1\).

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**References**


