Generalizations of Nekrasov matrices and applications

Abstract: In this paper we present a nonsingularity result which is a generalization of Nekrasov property by using two different permutations of the index set. The main motivation comes from the following observation: matrices that are Nekrasov matrices up to the same permutations of rows and columns, are nonsingular. But, testing all the permutations of the index set for the given matrix is too expensive. So, in some cases, our new nonsingularity criterion allows us to use the results already calculated in order to conclude that the given matrix is nonsingular. Also, we present new max-norm bounds for the inverse matrix and illustrate these results by numerical examples, comparing the results to some already known bounds for Nekrasov matrices.

Keywords: Nekrasov matrices, $H$-matrices

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1 Introduction

Matrix properties based on diagonal dominance that guarantee nonsingularity of matrices are very important in the field of eigenvalue localization, in analysis of iterative methods for solving systems of linear equations and in many other problems, see [2]. In this paper, we present new nonsingularity criterion starting from the well-known Nekrasov property. Throughout the paper, by $\mathbb{C}^n$ (or $\mathbb{R}^n$) we denote complex (real) $n$ dimensional vector space, by $\mathbb{C}^{n,n}$ (or $\mathbb{R}^{n,n}$) the collection of all $n \times n$ matrices with complex (real) entries, and by $N := \{1, 2, \ldots, n\}$ the set of indices.

The class of Nekrasov matrices, the one that we are here particularly interested in, is strongly related to the well-known class of strictly diagonally dominant matrices (SDD), i.e. matrices $A = [a_{ij}] \in \mathbb{C}^{n,n}$ such that

$$|a_{ii}| > r_i(A), \text{ for all } i \in N,$$

or, in other words,

$$d(A) > r(A),$$

where

$$r_i(A) = \sum_{j \in N \setminus \{i\}} |a_{ij}|$$

is the $i$ -th deleted row sum,

$$r(A) := [r_1(A), \ldots, r_n(A)]^T$$

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is the vector of deleted row sums, and the vector of moduli of diagonal entries is denoted by

\[ d(A) := [|a_{11}|, ..., |a_{nn}|]^T. \]

The class of Nekrasov matrices, see [5, 6], is defined by

\[ |a_{ii}| > h_i(A), \quad \text{for all } i \in N, \quad (1) \]

or, using vectors as above,

\[ d(A) > h(A), \]

where the values \( h_i(A), i \in N \) are defined recursively by

\[ h_1(A) := r_1(A), \quad h_i(A) := \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=i+1}^{n} |a_{ij}|, \quad i = 2, 3, \ldots, n, \quad (2) \]

and \( h(A) := [h_1(A), ..., h_n(A)]^T \).

Comparing the deleted row sums used in definition of SDD property and recursively obtained, modified row sums used in definition of Nekrasov property (Nekrasov row sums, (2)), one can easily see that the Nekrasov condition is weaker and the class of Nekrasov matrices is wider than the SDD class. Also, notice that Nekrasov row sums are obtained from deleted row sums by placing specific weights on entries in the lower triangular part of the matrix.

Having this in mind, given a matrix \( A \), by \( A = D - L - U \) we denote the standard splitting of \( A \) into its diagonal \( D \), strictly lower \( L \) and strictly upper \( U \) triangular parts.

Let us now recall two well known lemmas.

**Lemma 1.1 ([7]).** Given any matrix \( A = [a_{ij}] \in \mathbb{C}^{n \times n}, n \geq 2, \) with \( a_{ii} \neq 0 \) for all \( i \in N \), then

\[ h_i(A) = |a_{ii}| \left( (|D| - |L|)^{-1} |U| e \right)_i, \]

where \( e \in \mathbb{C}^n \) is the vector with all components equal to 1.

**Lemma 1.2 ([8]).** A matrix \( A = [a_{ij}] \in \mathbb{C}^{n \times n}, n \geq 2 \) is a Nekrasov matrix if and only if

\[ (|D| - |L|)^{-1} |U| e < e, \]

which implies that \( I - (|D| - |L|)^{-1} |U| \) is an SDD matrix, where \( I \) is the identity matrix, and \( e \in \mathbb{C}^n \) is the vector with all components equal to 1.

Although both SDD and Nekrasov class are related to the idea of diagonal dominance, there is an important difference between them. The class of SDD matrices is closed under similarity permutation transformations, while Nekrasov class is not. Permutations do not affect the values of deleted row sums, but they do change the values of Nekrasov row sums, for, in calculating recursively defined Nekrasov sums, the order is crucial.

Since SDD matrices are invariant under permutation of indices, while the condition (1) is not, one easily obtains a wider class.

Namely, given a permutation matrix, \( P \), a matrix \( A = [a_{ij}] \in \mathbb{C}^{n \times n} \) is called \( P \)-Nekrasov, if \( P^T AP \) is a Nekrasov matrix, i.e., if

\[ |(P^T AP)_{ii}| > h_i(P^T AP), \quad \text{for all } i \in N, \]

or, in other words,

\[ d(P^T AP) > h(P^T AP). \]

The union of all \( P \)-Nekrasov classes by permutation matrices \( P \) is known as Gudkov class, see [4].

All matrix classes defined and investigated in this paper, belong to the class of (nonsingular) \( H \)-matrices. A matrix \( A = [a_{ij}] \in \mathbb{C}^{n \times n} \) is called an \( H \)-matrix if its comparison matrix \( \langle A \rangle = [m_{ij}] \) defined by

\[ \langle A \rangle = [m_{ij}] \in \mathbb{C}^{n \times n}, \quad m_{ij} = \begin{cases} |a_{ij}|, & i = j \\ -|a_{ij}|, & i \neq j. \end{cases} \]
is an M-matrix, i.e., $(A)^{-1} \geq 0$.

Among many, one of well-known properties of H-matrices (see [1]) is the following:

- For any nonsingular H-matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $|A^{-1}| \leq (A)^{-1}$ holds.

\section{\{P_1, P_2\}-Nekrasov property}

We start with the following motivating question. Given a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, and given two permutation matrices, $P_1, P_2 \in \mathbb{C}^{n,n}$, let us suppose that $A$ is neither $P_1$-Nekrasov nor $P_2$-Nekrasov matrix. We want to define a new condition involving permuted sums, such that a matrix satisfying this condition is nonsingular.

In order to do so, suppose that for the given matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$; $n \geq 2$ and two given permutation matrices $P_1$ and $P_2$,

$$d(A) > \min \left\{ h^{P_1}(A), h^{P_2}(A) \right\},$$

where

$$h^{P_k}(A) = P_k h(P_k^T A P_k), \; k = 1, 2.$$

We call such a matrix \{P_1, P_2\}-Nekrasov matrix.

The following example shows that it can happen that a matrix is neither $P_1$-Nekrasov nor $P_2$-Nekrasov, but it is \{P_1, P_2\}-Nekrasov. Consider the matrix:

$$A = \begin{pmatrix} 5 & 1 & 1 & 1 \\ 0 & 5 & 1 & 4.3 \\ 5 & 1 & 5 & 0 \\ 1 & 1 & 1 & 5 \end{pmatrix}.$$  

For identical and counteridentical permutations, i.e., for $P_1 = I$ and

$$P_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

it is easy to see that, as $h^{P_1}(A) = [3, 5.3, 4.06, 2.47]^T$ and $h^{P_2}(A) = [2.556, 3.78, 6, 3]^T$, $A$ is neither $P_1$-Nekrasov nor $P_2$-Nekrasov, but it is \{P_1, P_2\}-Nekrasov.

For matrices satisfying \{P_1, P_2\}-Nekrasov property we prove the following results of the same type as Lemma 1.1 and Lemma 1.2.

\textbf{Lemma 2.1.} Given any matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$, with $a_{ii} \neq 0$ for all $i \in N$, and given a permutation matrix, $P \in \mathbb{C}^{n,n}$, then

$$h_i^P(A) = |a_{ii}| \left( p \left( |\tilde{D}| - |\tilde{L}| \right)^{-1} |\tilde{U}| e \right)_i,$$

where $e \in \mathbb{C}^n$ is the vector with all components equal to 1 and $(\tilde{D})$ is diagonal, $(\tilde{L})$ strictly lower and $(\tilde{U})$ strictly upper triangular part of the matrix $P^T A P$, i.e., $P^T A P = \tilde{D} - \tilde{L} - \tilde{U}$ is the standard splitting of the matrix $P^T A P$.

\textbf{Proof.} By definition we have

$$h_i^P(A) = \left( P h(P^T A P) \right)_i.$$

Notice that there exists unique index $j \in N$ for which

$$h_j^P(A) = h_j(P^T A P).$$

It is the very index $j \in N$ for which $(P)_{i,j} = 1$ holds.
From Lemma 1.1, we obtain
\[
h_i^j(P^T AP) = ((P^T AP)_{jj}) \left( (|D_j| - |L_j|)^{-1} |U_i| \right) = |a_{ij}| \left( (P(|D_i| - |L_i|)^{-1} |U_i| e) \right).
\]
and, as \( e = P^T e \),
\[
h_i^j(A) = |a_{ij}| \left( (P(|D_i| - |L_i|)^{-1} |U_i| P^T e) \right).
\]
From Lemma 2.1, we see that, for two given permutation matrices \( P_1, P_2 \in \mathbb{C}^{n,n} \), it holds
\[
h_i^{P_k}(A) = |a_{ij}| \left( (P_k(|D_k| - |L_k|)^{-1} |U_k| e) \right), \ k = 1, 2,
\]
where \( P_k^T AP_k = D_k - L_k - U_k \) is the standard splitting of matrices \( P_k^T AP_k, \ k = 1, 2 \).

Let us now construct a special matrix, \( C \in \mathbb{C}^{n,n} \), for the given \( \{P_1, P_2\}\)-Nekrasov matrix, \( A = [a_{ij}] \in \mathbb{C}^{n,n}, \ n \geq 2 \), as follows.
\[
C = \begin{bmatrix} C(1) \\ C(2) \\ \vdots \\ C(n) \end{bmatrix} \in \mathbb{C}^{n,n}
\]
with
\[
C(i) = e_i^T P_{k_1}(|D_{k_1}| - |L_{k_1}|)^{-1} |U_{k_1}| P_{k_1}^T,
\]
where \( e_i \) is the standard basis vector, whose components are equal to zero, all except the \( i \)-th component, which is equal to 1, and, for each index \( i \), the corresponding index \( k_i \in \{1, 2\} \) is chosen in such way that
\[
\min \left\{ h_i^{P_1}(A), h_i^{P_2}(A) \right\} = h_i^{P_{k_i}}(A).
\]
In other words, we construct the matrix \( C \) in the following way. We choose each row to be the corresponding row from either \( P_1(|D_1| - |L_1|)^{-1} |U_1| P_1^T \) or \( P_2(|D_2| - |L_2|)^{-1} |U_2| P_2^T \), depending on comparison of \( h_i^{P_1}(A), h_i^{P_2}(A) \), i.e., we choose the row from the very matrix where minimum of these two sums is obtained.

**Lemma 2.2.** If a matrix \( A = [a_{ij}] \in \mathbb{C}^{n,n}, \ n \geq 2 \), is a \( \{P_1, P_2\}\)-Nekrasov matrix, then the matrix \( I - C \) is an SDD matrix, where \( I \) is the identity matrix and \( C \) defined as in (4).

**Proof.** Let us suppose that \( A \) is a \( \{P_1, P_2\}\)-Nekrasov matrix. Then,
\[
d(A) > \min \left\{ h^{P_1}(A), h^{P_2}(A) \right\}.
\]
For the \( i \)-th component we have
\[
|a_{ij}| > \min \left\{ h_i^{P_1}(A), h_i^{P_2}(A) \right\},
\]
where, from Lemma 2.1,
\[
h_i^{P_k}(A) = |a_{ij}| \left( (P_k(|D_k| - |L_k|)^{-1} |U_k| P_k^T e) \right).
\]
This implies
\[
|a_{ij}| > \min \left\{ |a_{ij}| \left( (P_1(|D_1| - |L_1|)^{-1} |U_1| P_1^T e) \right), \ |a_{ij}| \left( (P_2(|D_2| - |L_2|)^{-1} |U_2| P_2^T e) \right) \right\}.
\]
As it follows from (3) that \( a_{ij} \neq 0 \), therefore
\[
1 > \min \left\{ \left( (P_1(|D_1| - |L_1|)^{-1} |U_1| P_1^T e) \right), \ \left( (P_2(|D_2| - |L_2|)^{-1} |U_2| P_2^T e) \right) \right\}.
\]
This means that the matrix
\[
B := I - C
\]
as all row sums positive. Notice that \( (|D_k| - |L_k|) \) is a nonsingular \( M \)-matrix, so, all the off-diagonal entries of the matrix \( B \) are nonpositive, and \( B \) is an SDD matrix.
Theorem 2.3. Given a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$ and two arbitrary permutation matrices in $\mathbb{C}^{n,n}$, $P_1, P_2$, such that

$$d(A) > \min \left\{ h^{P_1}(A), h^{P_2}(A) \right\},$$

where

$$h^{P_k}(A) = P_k h(P_k^T AP_k), \quad k = 1, 2,$$

then $A$ is nonsingular.

Proof. Let us suppose that $A$ is singular. Then, there exists a nonzero vector $x$ such that $Ax = 0$. Let $\{P_1, P_2\}$ be the given set of two permutation matrices. For $k = 1, 2$, we have

$$P_k^T AP_k P_k^T x = 0,$$

which can be expressed as

$$D_k P_k^T x = L_k P_k^T x + U_k P_k^T x, \tag{6}$$

where $(D_k)$ is diagonal, $(-L_k)$ strictly lower and $(-U_k)$ strictly upper triangular part of the matrix $P_k^T AP_k$.

After using the triangular inequality and rearranging, (6) becomes

$$(|D_k| - |L_k|)|P_k^T x| \leq |U_k||P_k^T x|.$$

Since (3) implies that all diagonal entries of the matrix $A$ are nonzero, then $|D_k| - |L_k|$ is a nonsingular $M$-matrix, and, therefore,

$$|P_k^T x| \leq (|D_k| - |L_k|)^{-1}|U_k||P_k^T x|.$$

Since $|P_k^T x| = |P_k^T||x| = |P_k^T||x|$, multiplying the last inequality from the left hand side with $P_k$, we obtain

$$|x| \leq \left( P_k(|D_k| - |L_k|)^{-1}|U_k||P_k^T x| \right), \quad k = 1, 2.$$

From the above, one derives the inequality

$$|x_i| \leq \left( P_{k_i}(|D_{k_i}| - |L_{k_i}|)^{-1}|U_{k_i}||P_{k_i}^T x| \right), \quad i \in N,$$

which still holds if in each row $i$ we choose the corresponding $k_i$ as in (5). Then, the coefficient matrix in the right hand side turns to the matrix $C$ defined in (4).

Therefore,

$$|I - C||x| \leq 0. \tag{7}$$

As the matrix on the left hand side of inequality (7) has all row sums positive, which, together with the fact that all its off-diagonal entries are nonpositive, implies (see [1]) that it is a nonsingular $M$-matrix, then, from (7) it follows that $|x| \leq 0$ for nonzero vector $x$. This contradiction completes the proof.

Theorem 2.4. Given an arbitrary set of two permutation matrices $\{P_1, P_2\}$, every $\{P_1, P_2\}$-Nekrasov matrix is an $H$-matrix.

Proof. Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$, $n \geq 2$ be a $\{P_1, P_2\}$-Nekrasov matrix. Let $(A) = D - B$ be the standard splitting of the matrix $(A)$ into diagonal and off-diagonal part. The matrix $D$ is nonsingular from the $\{P_1, P_2\}$-Nekrasov condition. Let us first prove that $\rho(D^{-1}B) < 1$. Suppose the opposite - that $\rho(D^{-1}B) \geq 1$, meaning that there is an eigenvalue $\lambda \in \sigma(D^{-1}B)$ such that $|\lambda| \geq 1$. This implies

$$\det(\lambda I - A) = 0,$$

and

$$\det(D^{-1}) \det(\lambda D - B) = 0,$$

which, as matrix $D$ is nonsingular, implies

$$\det(\lambda D - B) = 0.$$
In other words, the matrix $F := \lambda D - B$ is singular. But, then, for each $i \in N$,

$$|f_{ii}| = |\lambda||a_{ii}| \geq |a_{ii}| > \min \left\{ h_i^{p_1}(A), h_i^{p_2}(A) \right\} \geq \min \left\{ h_i^{p_1}(F), h_i^{p_2}(F) \right\}.$$  

The last inequality holds from the following observations

$$h_i^{p_k}(A) = |a_{ii}| \left( P_k (|D_k| - |L_k|)^{-1} |U_k| P_k^T e \right)_i, \quad i \in N, \ k = 1, 2,$$

and, on the other hand,

$$h_i^{p_k}(F) = |\lambda||a_{ii}| \left( P_k (|\lambda||D_k| - |L_k|)^{-1} |U_k| P_k^T e \right)_i
= |\lambda||a_{ii}| \left( P_k \left( |D_k| - \frac{1}{\lambda} |L_k| \right)^{-1} \frac{1}{\lambda} |U_k| P_k^T e \right)_i
= |a_{ii}| \left( P_k \left( |D_k| - \frac{1}{\lambda} |L_k| \right)^{-1} |U_k| P_k^T e \right)_i, \quad i \in N, \ k = 1, 2.$$

Note that the matrices $M_A := |D_k| - |L_k|$ and $M_F := |D_k| - \frac{1}{|\lambda|} |L_k|$ are nonsingular $M$-matrices, with $M_F - M_A = (1 - \frac{1}{|\lambda|}) |L_k| \geq 0$. Therefore, $M_A^{-1} \geq M_F^{-1} (\geq 0)$. Hence,

$$h_i^{p_k}(A) \geq h_i^{p_k}(F), \quad i \in N, \ k = 1, 2.$$  

This means that matrix $F$ is also $\{P_1, P_2\}$-Nekrasov, and therefore nonsingular. This is a contradiction with $\det(\lambda D - B) = \det F = 0$.

As such eigenvalue $\lambda \in \sigma(D^{-1}B)$ does not exist, we conclude that $\rho(D^{-1} B) < 1$, and from $(D^{-1} A)^{-1} = (I - D^{-1} B)^{-1} = \sum_{k \geq 0} (D^{-1} B)^k \geq 0$, we have $(A)^{-1} \geq 0$, therefore $A$ is an $H$-matrix. \hfill $\Box$

Instead of a set of two permutations, we can observe a set of $p$ arbitrary permutation matrices, $\Pi_n = \{P_k\}_{k=1}^p$, and define the $\Pi_n$-Nekrasov property. Namely, given a set of $p$ arbitrary permutation matrices, $\Pi_n = \{P_k\}_{k=1}^p$, a matrix $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, $n \geq 2$, is called $\Pi_n$-Nekrasov, if

$$d(A) > \min_{k=1,\ldots,p} h^{p_k}(A).$$

Same as before, we can prove the following.

**Theorem 2.5.** Given an arbitrary set of permutations $\Pi_n$, every $\Pi_n$-Nekrasov matrix is nonsingular, moreover, it is an $H$-matrix.

Although we can easily prove the analogous statements concerning three or more permutations in the same fashion as before, the question is, whether these generalizations are worth further analysis from the practical point of view.

### 3 Max-norm bounds for the inverse of $\{P_1, P_2\}$-Nekrasov matrices

In [3], new max-norm bounds are given for the inverse of Nekrasov matrices using the well-known bound for the inverse of an SDD matrix by Varah (see [9]).

- Varah bound for SDD matrices:
  $$||A^{-1}||_{\infty} \leq \frac{1}{\min_{i \in N} \left( |a_{ii}| - r_i(A) \right)}.$$

Now, we are going to use statements we already proved in order to obtain a max-norm bound for the inverse of a $\{P_1, P_2\}$-Nekrasov matrix.
Theorem 3.1. Suppose that, for a given set of permutation matrices \( \{P_1, P_2\} \), a matrix \( A = [a_{ij}] \in \mathbb{C}^{n \times n} \), \( n \geq 2 \), is a \( \{P_1, P_2\}\)-Nekrasov matrix. Then,

\[
||A^{-1}||_{\infty} \leq \frac{\max_{i \in N} \left( \frac{z_i(A)}{|a_{ij}|} \right)}{\min_{i \in N} \left( 1 - \min \left\{ \frac{h_{P_1}^i(A)}{|a_{ij}|}, \frac{h_{P_2}^i(A)}{|a_{ij}|} \right\} \right)},
\]

where

\[
z_i(A) := 1, \quad z_i(A) := \sum_{j=1}^{i-1} |a_{ij}| \frac{z_j(A)}{|a_{jj}|} + 1, \quad i = 2, 3, \ldots, n,
\]

the corresponding vector is \( z(A) := [z_1(A), \ldots, z_n(A)]^T \), \( z^T(A) = P z(P^T A) \), and for the given \( i \in N \) the corresponding index \( k_i \in \{1, 2\} \) is chosen in such way that

\[
\min \left\{ h_{P_1}^i(A), h_{P_2}^i(A) \right\} = h_{k_i}^i(A).
\]

Proof. Let \( A \) be a \( \{P_1, P_2\}\)-Nekrasov matrix. From Lemma 2.2, then

\[
B := I - C
\]

is an SDD matrix. Therefore, for the inverse matrix of the matrix \( B \) the Varah bound holds:

\[
||B^{-1}||_{\infty} \leq \frac{1}{\min_{i \in N} (|b_{ii}| - r_i(B))}.
\]

In the same manner as before, we obtain

\[
|b_{ii}| - r_i(B) = 1 - \min \left\{ (P_1 (|D_1| - |L_1|)^{-1} |U_1| P_1^T e)_{ii}, (P_2 (|D_2| - |L_2|)^{-1} |U_2| P_2^T e)_{ii} \right\} =
\]

\[
= 1 - \min \left\{ h_{P_1}^i(A), h_{P_2}^i(A) \right\}, \quad i \in N.
\]

It remains to find a link between matrices \( B^{-1} \) and \( A^{-1} \). It is easy to see that, for a fixed \( k \in \{1, 2\},
\[
I - P_k (|D_k| - |L_k|)^{-1} |U_k| P_k^T = P_k (|D_k| - |L_k|)^{-1} \left( (|D_k| - |L_k|) P_k^T - |U_k| P_k^T \right) =
\]

\[
= P_k (|D_k| - |L_k|)^{-1} (|D_k| - |L_k|) P_k^T = P_k (|D_k| - |L_k|)^{-1} (P_k^T(A) P_k) P_k^T =
\]

\[
= P_k (|D_k| - |L_k|)^{-1} P_k^T(A).
\]

Therefore, we have obtained that for \( k \in \{1, 2\}, \) it holds

\[
I - P_k (|D_k| - |L_k|)^{-1} |U_k| P_k^T = P_k (|D_k| - |L_k|)^{-1} P_k^T(A).
\]

Now, if we allow different values of \( k \) in different rows, keeping the same value of \( k \) in the same row on the left and the right hand side of this equality, we obtain in that way our ”mixed - rows - matrix” from (4) by choosing \( k_j \) in each row \( i \) as described in (5).

In other words, we have

\[
B = I - C = \tilde{C}(A),
\]

where we denote by \( \tilde{C} \) the matrix defined as follows.

\[
\tilde{C} = \begin{bmatrix} \tilde{C}(1) \\ \tilde{C}(2) \\ \vdots \\ \tilde{C}(n) \end{bmatrix} \in \mathbb{C}^{n \times n}
\]
with
\[ \tilde{C}(i) = e_i^T P_{k_i} \left( \lvert D_{k_i} \rvert - \lvert L_{k_i} \rvert \right)^{-1} P_{k_i}^T, \]
where \( e_i \) is the standard basis vector, whose components are equal to zero, all except the \( i^{th} \) component, which is equal to 1, and, for each index \( i \), the corresponding index \( k_i \in \{1, 2\} \) is chosen, as in (5), in such way that
\[ \min_{i \in N} \left\{ h_i^{P_1}(A), h_i^{P_2}(A) \right\} = h_i^{P_{k_i}}(A). \]

Therefore, from (9),
\[ (A)^{-1} = B^{-1} \tilde{C} \]
and
\[ \|A^{-1}\|_\infty \leq \|(A)^{-1}\|_\infty = \|B^{-1} \tilde{C}\|_\infty \leq \|B^{-1}\|_\infty \|\tilde{C}\|_\infty \leq \frac{1}{\min_{i \in N} \left( 1 - \min_{i \in N} \left\{ h_i^{P_1}(A), h_i^{P_2}(A) \right\} \right)} \|\tilde{C}\|_\infty. \]

From [3] we know that \((I - |L||D|^{-1})^{-1} e = z(A)\), with recursively defined values
\[ z_1(A) := 1, \quad z_i(A) := \sum_{j=1}^{i-1} \frac{|a_{ij}| z_j(A)}{|a_{jj}|} + 1, \quad i = 2, 3, \ldots n, \]
and the corresponding vector \( z(A) := [z_1(A), ..., z_n(A)]^T \). In the same fashion as done with Nekrasov sums, we defined permuted vector as \( z^P(A) = Pz(P^T A P) \).

Having this in mind, it is easy to see that
\[ \|\tilde{C}\|_\infty = \|\tilde{C}e\|_\infty = \max_{i \in N} \left( \frac{z_i^{P_{k_i}}(A)}{|a_{ii}|} \right), \]
where, for each index \( i \), the corresponding index \( k_i \in \{1, 2\} \) is chosen in such way that
\[ \min_{i \in N} \left\{ h_i^{P_1}(A), h_i^{P_2}(A) \right\} = h_i^{P_{k_i}}(A). \]

This completes the proof. \( \square \)

**Theorem 3.2.** Suppose that, for a given set of permutation matrices \( \{P_1, P_2\} \), a matrix \( A = [a_{ij}] \in \mathbb{C}^{n,n}, \ n \geq 2, \) is a \( \{P_1, P_2\}\)-Nekrasov matrix. Then,
\[ \|A^{-1}\|_\infty \leq \max_{i \in N} \left( \frac{z_i^{P_{k_i}}(A)}{|a_{ii}|} \right) \min_{i \in N} \left( |a_{ii}| - \min_{i \in N} \left\{ h_i^{P_1}(A), h_i^{P_2}(A) \right\} \right), \]
where
\[ z_1(A) := 1, \quad z_i(A) := \sum_{j=1}^{i-1} \frac{|a_{ij}| z_j(A)}{|a_{jj}|} + 1, \quad i = 2, 3, \ldots n, \]
the corresponding vector is \( z(A) := [z_1(A), ..., z_n(A)]^T, \ z^P(A) = Pz(P^T A P), \) and for the given \( i \in N \) the corresponding index \( k_i \in \{1, 2\} \) is chosen in such way that
\[ \min_{i \in N} \left\{ h_i^{P_1}(A), h_i^{P_2}(A) \right\} = h_i^{P_{k_i}}(A). \]

**Proof.** Instead of the matrix \( B \) from (9), we now observe the matrix \( B' = |D|B \). Then,
\[ |h_{ii}'| - r_i(B') = |a_{ii}| - \min_{i \in N} \left\{ h_i^{P_1}(A), h_i^{P_2}(A) \right\}, \]
\[ (A)^{-1} = (B')^{-1} |D| \tilde{C}. \]

Therefore,
\[
||A^{-1}||_\infty \leq ||(A)^{-1}||_\infty = ||(B')^{-1} |D| \tilde{C}||_\infty \leq ||B^{-1}||_\infty |||D| \tilde{C}||_\infty \leq \\
\leq \min_{i \in N} \left( |a_{ii}| - \min \left\{ h_i^{P_1}(A), h_i^{P_2}(A) \right\} \right) ||D| \tilde{C}||_\infty,
\]

where
\[
|||D| \tilde{C}||_\infty = ||D| \tilde{C} e||_\infty = \max_{i \in N} \left( z_i^{P_{k_i}}(A) \right),
\]

where, for each index \( i \), the corresponding index \( k_i \in \{1, 2\} \) is chosen in such a way that
\[
\min \left\{ h_i^{P_1}(A), h_i^{P_2}(A) \right\} = h_i^{P_{k_i}}(A).
\]

This completes the proof. \( \square \)

### 4 Numerical examples

Consider the following matrices:

\[
A_0 = \begin{pmatrix}
1 & 0.3 & 0 & 0 \\
0 & 1 & 0.6 & 0 \\
0 & 1 & 1 & 0.7 \\
3 & 0 & 0 & 1
\end{pmatrix},
\]

\[
A_1 = \begin{pmatrix}
12 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 12 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 12 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 8 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 2 & 2 \\
0 & 0 & 2 & 2 & 2 & 12 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 114 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3
\end{pmatrix},
\]

\[
A_2 = \begin{pmatrix}
7 & -2 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 8 & 4 & 1 & -2 & 0 & 0 & 0 & 0 \\
-2 & 0 & 1 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 8 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 2 & 2 & 7 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 6 & 2 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 & 2 & 8 & 0 & 0 \\
0 & -2 & 0 & 0 & 1 & 0 & 0 & 5 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & -1 & 0 & 1 & 0 & 8
\end{pmatrix},
\]

\[
A_3 = \begin{pmatrix}
-1.5 & -0.1 & 0 & -0.1 & 0 & 0 \\
-0.1 & 2 & -0.1 & -1.9 & 0 & 0 \\
0 & -0.1 & 23 & -0.1 & -1 & -0.1 \\
0 & 0 & -0.5 & 44 & 0 & 0 \\
0 & 0 & 0 & -0.1 & 44 & -0.4 \\
0 & 0 & -0.5 & 0 & -1 & 1
\end{pmatrix}.
\]

Matrix \( A_0 \) shows that finding a permutation that transforms the given matrix to Nekrasov matrix is not as easy as one may think at first glance. Namely, the rows in this matrix are ordered by descending diagonal dominance degree, from top to bottom. But, in this order, the matrix is not a Nekrasov matrix. On the other hand, if we put the last row in the second position, what we end up with is a Nekrasov matrix! So, it is a tricky task to find a permutation that turns a matrix into a Nekrasov matrix, even if we are certain that this can be done.
Matrix $A_1$ is an SDD matrix, while $A_2$ is a Nekrasov matrix. Matrix $A_3$ is neither SDD nor Nekrasov, but it does satisfy our new $(P_1, P_2)$-Nekrasov condition, where $P_1$ is the identical permutation of order 6 and $P_2$ is counteridentical permutation of order 6. In the following table, we compare the results for max-norm bounds of the inverse matrix obtained using Theorem 3.1 and Theorem 3.2 of this paper to those of Varah, for SDD matrices, and to the bounds for Nekrasov matrices presented in [3] (in Table 1 we call them Nekrasov I and Nekrasov II).

Table 1

<table>
<thead>
<tr>
<th>Bound</th>
<th>Varah</th>
<th>Nekrasov I</th>
<th>Nekrasov II</th>
<th>$(P_1, P_2)$-Nek I</th>
<th>$(P_1, P_2)$-Nek II</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>0.5</td>
<td>0.2443</td>
<td>0.3108</td>
<td>0.2132</td>
<td>0.2443</td>
</tr>
<tr>
<td>$A_2$</td>
<td>-</td>
<td>2.2282</td>
<td>2.8729</td>
<td>0.7726</td>
<td>0.5992</td>
</tr>
<tr>
<td>$A_3$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>1.114</td>
<td>1.1255</td>
</tr>
</tbody>
</table>

Exact values for the max-norm of the inverse matrix are as follows:

\[
\|A_1^{-1}\|_\infty = 0.1796, \quad \|A_2^{-1}\|_\infty = 0.3445, \quad \|A_3^{-1}\|_\infty = 1.0578.
\]

As one can see from Table 1, our bounds are better than Varah for some SDD matrices, and, in some cases, they are better than bounds for Nekrasov matrices presented in [3]. If the matrix is neither SDD nor Nekrasov, like, for example, $A_3$, the only bounds that can be applied are bounds (8) and (10).

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References