Complexity issues for the symmetric interval eigenvalue problem

Abstract: We study the problem of computing the maximal and minimal possible eigenvalues of a symmetric matrix when the matrix entries vary within compact intervals. In particular, we focus on computational complexity of determining these extremal eigenvalues with some approximation error. Besides the classical absolute and relative approximation errors, which turn out not to be suitable for this problem, we adapt a less known one related to the relative error, and also propose a novel approximation error. We show in which error factors the problem is polynomially solvable and in which factors it becomes NP-hard.

Keywords: Interval matrix, Interval analysis, Eigenvalue, Eigenvalue bounds, NP-hardness

MSC: 15A18, 15B99, 68Q17

1 Introduction

In this paper, we address the problem of computing extremal eigenvalues of symmetric interval matrices. While there are diverse methods for calculating the eigenvalues exactly (with some computational effort), or giving rigorous lower/upper bounds, there is still a relative lack in results in computationally complexity.

Before giving the state-of-the-art and formulation of the problem, let us introduce some notation first. An interval (square) matrix \( A \) is defined as

\[
A \doteq \{ A \in \mathbb{R}^{n \times n} : \underline{A} \leq A \leq \overline{A} \},
\]

where \( \underline{A}, \overline{A} \in \mathbb{R}^{n \times n}, \underline{A} \leq \overline{A} \) are given. The center and radius of \( A \) are respectively defined as

\[
A^{\text{C}} := \frac{1}{2}(\underline{A} + \overline{A}), \quad A^{\text{R}} := \frac{1}{2}(\overline{A} - \underline{A}).
\]

The symmetric interval matrix associated to \( A \) is defined as

\[
A^{\text{S}} := \{ A \in A : A = A^{T} \}.
\]

Without loss of generality we can assume that both \( A^{\text{C}} \) and \( A^{\text{R}} \) are symmetric, otherwise intervals in \( A \) can be narrowed such that the center and radius matrices are symmetric and no symmetric matrix in \( A \) was omitted. We will also assume that \( A^{\text{S}} \neq \emptyset \).

We say that \( A \) (resp. \( A^{\text{S}} \)) has property \( \mathfrak{P} \) if each \( A \in A \) (resp. \( A \in A^{\text{S}} \)) has property \( \mathfrak{P} \). This applies to positive definiteness, stability, nonsingularity etc. An interval matrix \( A \) (resp. \( A^{\text{S}} \)) is irregular if it is not nonsingular, i.e., it contains a singular matrix.

Next, \( E \) denotes the matrix of ones, \( I \) the identity matrix (with suitable dimensions), and \( \text{diag}(z) \) the diagonal matrix with entries \( z_1, \ldots, z_n \). For the definition of interval arithmetic see, e.g., [1, 2]. For \( A \) symmetric, \( \lambda_{\min}(A) \) and \( \lambda_{\max}(A) \) denote its smallest and largest eigenvalues, respectively, and \( \rho(A) \) stands for its spectral radius.

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Many properties of interval matrices are surveyed in [3–5]. Herein, we are interested in computing the smallest and the largest possible eigenvalues in $A^S$, which are formally defined as

$$\lambda_{\min}(A^S) := \min\{\lambda_{\min}(A): A \in A^S\},$$
$$\lambda_{\max}(A^S) := \max\{\lambda_{\max}(A): A \in A^S\}. $$

Due to continuity of eigenvalues and compactness of $A^S$, these extremal eigenvalues are always attained. They can be characterized by [6] as

$$\lambda_{\min}(A^S) = \min\{\lambda_{\min}(A^c - \text{diag}(z)A^\Delta \text{diag}(z)): z \in \{\pm1\}^n\},$$
$$\lambda_{\max}(A^S) = \max\{\lambda_{\max}(A^c + \text{diag}(z)A^\Delta \text{diag}(z)): z \in \{\pm1\}^n\}. $$

A partial extension to intermediate eigenvalues was presented in [7].

Due to intractability of the above formulae for larger dimensions, there were developed various approximation methods. Enclosure methods for the eigenvalue set, which yield lower bounds on $\lambda_{\min}(A^S)$ and upper bounds on $\lambda_{\max}(A^S)$, and serve as sufficient conditions for Schur or Hurwitz stability, were discussed in [5, 8–13], among others. Iterative refinements were proposed in [14, 15]. Inner estimation methods give upper bounds on $\lambda_{\min}(A^S)$ and lower bounds on $\lambda_{\max}(A^S)$ and provide as with a certificate of instability of interval matrices; recent works comprise, e.g., [7, 16, 17]. The related topic of finding verified intervals of eigenvalues for real matrices was studied in e.g. [18, 19].

In our paper, we will employ the following simple bounds stated, e.g., in [5, 10].

**Theorem 1.1.** We have

$$\{\lambda_{\min}(A^c): A \in A^S\} \subseteq \{\lambda_{\min}(A^c) - \rho(A^\Delta), \lambda_{\min}(A^c) + \rho(A^\Delta)\},$$
$$\{\lambda_{\max}(A^c): A \in A^S\} \subseteq \{\lambda_{\max}(A^c) - \rho(A^\Delta), \lambda_{\max}(A^c) + \rho(A^\Delta)\}. $$

This paper is focused on complexity of computing and approximation of the extremal eigenvalues. Let us review some known results. Consider the class of symmetric interval matrices with $A^c \in \mathbb{Q}^{n \times n}$ symmetric positive definite and entrywise nonnegative, and $A^\Delta = E$. On this class, it is NP-hard or co-NP-hard to check whether

- a given $\lambda \in \mathbb{Q}$ is an eigenvalue of some $A \in A^S$; see [3, 4],
- $\lambda_{\max}(A^S) \in (\underline{q}, \overline{q})$ for a given open interval $(\underline{q}, \overline{q})$; see [3, 4],
- $A^S$ is positive definite [3, 4, 20],
- $A^S$ is positive semi-definite [3, 4, 21].

Analogously, it is also co-NP-hard to check Hurwitz or Schur stability of $A^S$; see [3, 4]. Consequently, computing the spectral radius (i.e., the maximal 2-norm) of $A^S$, defined as

$$\max\{\rho(A): A \in A^S\} = \max\{-\lambda_{\min}(A^S), \lambda_{\max}(A^S)\}, $$

is NP-hard as well.

Surprisingly, checking whether $v \in \mathbb{Q}^n$ is an eigenvector of some $A \in A^S$ is a polynomial time problem [22].

The aim of this paper is to inspect complexity of computing an approximation of $\lambda_{\min}(A^S)$ and $\lambda_{\max}(A^S)$. The table below displays the main results. Besides the traditional absolute and relative approximation error, we investigate also other appropriate quantities called inverse relative and relative mid+rad errors, properly defined in subsequent sections. The table shows for which approximation factor the problem is NP-hard and for which factor is becomes polynomial. We use the symbol $\infty$ in case there is no finite approximation factor with polynomial complexity.

<table>
<thead>
<tr>
<th>NP-hard with factor polynomial with factor</th>
<th>mid+rad error</th>
<th>abs. error</th>
<th>rel. error</th>
<th>inverse rel. error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/(16 \cdot n^3)$</td>
<td>any</td>
<td>$&lt; 1$</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$1/2$</td>
<td>$\infty$</td>
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The table below gives analogous results for the specific case of approximating $\lambda_{\max}(A^S)$ when $A^c$ is positive semi-definite.
Assumption.
Throughout the paper, we consider a computational model, in which the exact eigenvalues of rational symmetric matrices are polynomially computable. Even though this assumption is not satisfied for standard computational models, in which the eigenvalues are computable in polynomial time with a given precision, the results developed in this paper apply to standard models with a slight modification involving the given precision.

2 Relative mid+rad approximation error

As observed in [23], the classical absolute and relative errors are sometimes unsatisfactory. Their pros and cons for eigenvalue problems are also discussed in [24]. For our problem, we propose a novel approximation error called the relative mid+rad approximation error. We investigate this case first as the results developed here will be subsequently utilized for some other approximations.

Definition 2.1. An algorithm computes \( \lambda \) with relative mid+rad approximation error \( \varepsilon \) if it computes \( \lambda^0 \) such that \( \lambda \in [\lambda^0 - \varepsilon(\rho(A^c) + \rho(A^\Delta)), \lambda^0 + \varepsilon(\rho(A^c) + \rho(A^\Delta))] \).

Theorem 2.2. There is a polynomial time algorithm that computes the extremal eigenvalues with relative mid+rad approximation error \( 1/2 \).

Proof. From (2), we get

\[
\lambda_{\min}(A^S) \in [\lambda_{\min}(A^c) - \rho(A^\Delta), \lambda_{\min}(A^c)] \\
\subseteq [\lambda_{\min}(A^c) - (\rho(A^c) + \rho(A^\Delta)), \lambda_{\min}(A^c)].
\]

Thus, \( \lambda^0 := \lambda_{\min}(A^c) - \frac{1}{2}(\rho(A^c) + \rho(A^\Delta)) \) approximates \( \lambda_{\min}(A^S) \) with relative mid+rad error \( 1/2 \). Similarly for \( \lambda_{\max}(A^S) \).

Theorem 2.3. As long as \( P \neq NP \) there is no polynomial time algorithm that computes the extremal eigenvalues with relative mid+rad approximation error \( 1/(16n^4) \) on a class of interval matrices with \( A^\Delta := E \).

Before we present a proof of Theorem 2.3 we need to introduce some notions and auxiliary results first. A matrix \( A \in \mathbb{R}^{n \times n} \) is called an MC matrix if it is symmetric and satisfies \( A_{ii} = n \) and \( a_{ij} \in \{0, -1\} \) for \( i \neq j \). Any MC-matrix \( A \) is positive definite and \( \lambda_{\min}(A) \geq 1 \), which follows, e.g., from the Gerschgorin inclusion theorem. Next, for a symmetric nonsingular matrix \( A \in \mathbb{R}^{n \times n} \) its symmetric radius of nonsingularity is defined as

\[
d(A) := \inf\{\varepsilon > 0; [A - \varepsilon E, A + \varepsilon E]^S \text{ is irregular}\}.
\]

We will also employ the matrix norm

\[
\|A\|_{\infty, 1} := \max\{\|Ax\|_1; \|x\|_\infty = 1\}.
\]

using the vector \( L_1 \) and \( L_\infty \)-norms. By [3], we have

\[
d(A) = \frac{1}{\|A^{-1}\|_{\infty, 1}}
\]

for any symmetric positive definite matrix \( A \). By definition of \( d(A^c) \), if \( 0 < \lambda_{\min}(A^S) \leq p \) for some \( p > 0 \) and \( A^S \) with \( A^\Delta = E \), then \( d(A^c) > 1 \). Moreover, we can derive also an upper bound \( d(A^c) \leq 1 + p \) since
Proof of Theorem 2.3. In [25], it was proved that for a given 
by

Similarly we can approximate

is a polynomial time algorithm that computes

and by rounding we arrive at

Now,

Proof. From

Consider the class of symmetric interval matrices

Suppose that there is a polynomial time algorithm that determines

Then suppose the remaining case, when

Since

we have

Denote

with mid+rad error

Theorem 2.4. Consider the class of symmetric interval matrices

with

where

and

are polynomial time algorithms that computes

Thus, the class of symmetric interval matrices

Positive semi-definiteness of

implies

and

Since

we have

which completes the proof.

Theorem 2.5. Consider the class of symmetric interval matrices

with

As long as

there is no polynomial time algorithm that computes

with relative mid+rad approximation error

Proof. Denote

and

Let

be with

By Theorem 2,

is positive semidefinite. Suppose to the contrary that there is a polynomial algorithm that computes

such that

Now,

Similarly we can approximate

which contradicts Theorem 2.3.
3 Absolute approximation error

Definition 3.1. An algorithm computes $\lambda$ with absolute approximation error $\varepsilon$ if it computes $\lambda^0$ such that $\lambda^0 \in [\lambda - \varepsilon, \lambda + \varepsilon]$. 

Theorem 3.2. For any polynomially enumerable non-negative function $\varphi(n)$, there is no polynomial time algorithm that computes the extremal eigenvalues of $A^S$ with absolute approximation error $\varphi(n)$ unless $P=NP$.

Proof. Using the notation from the proof of Theorem 2.3, we get from (4) that it is NP-hard to approximate $\lambda_{\min}(A^S)$ with absolute approximation error

$$\varepsilon_{\alpha} = \frac{1}{16n^4}(\rho(A^c) + \rho(A^\Lambda)) \geq \frac{1}{16n^3}n = \frac{1}{16n^3}.$$ 

Now, consider the interval matrix $B^S := 16n^3 \varphi(n) A^S$. If we can efficiently determine $\lambda^0$ such that $\lambda_{\min}(B^S) \in [\lambda^0 - \varphi(n), \lambda^0 + \varphi(n)]$, then

$$\lambda_{\min}(A^S) \in \left[\frac{1}{16n^3} \varphi(n) \lambda^0 - \frac{1}{16n^3}, \frac{1}{16n^3} \varphi(n) \lambda^0 + \frac{1}{16n^3}\right],$$

which is a contradiction. \hfill $\square$

The theorem says not only that we cannot approximate in polynomial time the extremal eigenvalues with a given accuracy $\varepsilon > 0$, but we cannot do it with accuracy $n$, $n^2$, $e^n$ etc.

This result also applies to symmetric interval matrices $A^S$ with $A^c$ rational positive semi-definite since any symmetric interval matrices $B^S$ can be reduced to this case simply by a transformation $A^S := B^S + \beta I$, which shifts the eigenvalues of $B^S$ by $\beta$.

4 Relative approximation error

We use the traditional definition of a relative error from [26].

Definition 4.1. An algorithm computes $\lambda$ with relative approximation error $\varepsilon$ if it computes $\lambda^0$ such that $\lambda^0 \in (1 + [-\varepsilon, \varepsilon])\lambda$.

For complexity of relative errors, we have the exact border point between polynomiality and NP-hardness. The following result is from [3, 4].

Theorem 4.2. Consider the class of symmetric interval matrices $A^S$ with $A^c$ rational positive semi-definite and $A^\Lambda = E$. As long as $P \neq NP$ there is no polynomial time algorithm that computes $\lambda_{\min}(A^S)$ with relative approximation error less than 1.

Theorem 4.3. There is a polynomial time algorithm that computes the extremal eigenvalues of $A^S$ with relative approximation error 1.

Proof. Simply put $\lambda^0 := 0$, then $\lambda^0 \in [0, 2] \lambda_{\min}(A^S)$ and $\lambda^0 \in [0, 2] \lambda_{\max}(A^S)$. \hfill $\square$

Theorem 4.4. Consider the class of symmetric interval matrices $A^S$ with $A^c$ rational positive semi-definite. On this class, there is a polynomial time algorithm that computes $\lambda_{\max}(A^S)$ with relative approximation error $1/3$.

Proof. As in the proof of Theorem 2.4 we derive

$$\lambda_{\max}(A^S) \in [\frac{1}{3}, 1](\lambda_{\max}(A^c) + \rho(A^\Lambda)).$$

Now, $\lambda^0 := \frac{2}{3}(\lambda_{\max}(A^c) + \rho(A^\Lambda))$ approximates $\lambda_{\max}(A^S)$ with relative error $\frac{1}{3}$. \hfill $\square$
**Theorem 4.5.** Consider the class of symmetric interval matrices $A^S$ with $A^c$ rational positive semi-definite and $A^\Delta = E$. As long as $P \neq NP$ there is no polynomial time algorithm that computes the $\lambda_{\max}(A^S)$ with relative approximation error $1/(32n^4)$.

**Proof.** Denote $\varepsilon := 1/(32n^4)$ and $\alpha := \rho(A^c) + \rho(A^\Delta)$. By Theorem 2.5, there is no polynomial algorithm yielding $\lambda^0$ such that

$$
\lambda_{\max}(A^S) \in [\lambda^0 - \varepsilon, \lambda^0 + \varepsilon].
$$

This inclusion is equivalent to

$$
\lambda^0 \in [\lambda_{\max}(A^S) - \varepsilon, \lambda_{\max}(A^S) + \varepsilon].
$$

Since $|\lambda_{\max}(A^S)| \leq \alpha$, we get

$$
[\lambda_{\max}(A^S) - \varepsilon, \lambda_{\max}(A^S) + \varepsilon] \supseteq (1 + [-\varepsilon, \varepsilon])\lambda_{\max}(A^S).
$$

Therefore, we cannot approximate $\lambda_{\max}(A^S)$ in polynomial time with relative approximation error $\varepsilon$. \hfill \Box

**5 Inverse relative approximation error**

Relative approximation error can be also defined by other means. In this section, we consider the alternative definition from [23], and call it an inverse relative approximation error.

**Definition 5.1.** An algorithm computes $\lambda$ with inverse relative approximation error $\varepsilon$ if it computes $\lambda^0$ such that $\lambda \in (1 + [-\varepsilon, \varepsilon])\lambda^0$.

**Theorem 5.2.** Consider the class of symmetric interval matrices $A^S$ with $A^c$ rational positive semi-definite and $A^\Delta = E$. As long as $P \neq NP$ there is no polynomial time algorithm that computes $\lambda_{\min}(A^S)$ with inverse relative approximation error $1$.

**Proof.** We use the fact [3–5] that it is co-NP-hard to check whether every matrix in $A^S$ is positive semi-definite. Now, if we can approximate $\lambda_{\min}(A^S)$ with an error $1$, then the whole interval $(1 + [-\varepsilon, \varepsilon])\lambda^0$ lies either in the non-negative half-line or in the non-positive half-line. Thus, we would know the sign of $\lambda_{\min}(A^S)$ and could decide on positive semi-definiteness of the matrices in $A^S$ in polynomial time. \hfill \Box

**Theorem 5.3.** Consider the class of symmetric interval matrices $A^S$ with $A^c$ rational positive semi-definite. On this class, there is a polynomial time algorithm that computes $\lambda_{\max}(A^S)$ with inverse relative approximation error $1/3$.

**Proof.** As in the proof of Theorem 2.4 we derive

$$
\lambda_{\max}(A^S) \in [\frac{1}{2}, 1]([\lambda_{\max}(A^c) + \rho(A^\Delta)]).
$$

Now, $\lambda^0 := \frac{2}{3}(\lambda_{\max}(A^c) + \rho(A^\Delta))$ approximates $\lambda_{\max}(A^S)$ with inverse relative error $\frac{1}{3}$. \hfill \Box

We can state also the following general result, which does not seem to have an analogy for the classical relative error.

**Theorem 5.4.** Consider the class of symmetric interval matrices $A^S$. There is a polynomial time algorithm that computes $\lambda_{\min}(A^S)$ and $\lambda_{\max}(A^S)$ with inverse relative approximation error $2$.

**Proof.** Recall that $\lambda_{\max}(A^S) \in [\lambda_{\max}(A^c), \lambda_{\max}(A^c) + \rho(A^\Delta)]$. Let $\lambda^0$ be an endpoint of this interval with the greatest absolute value. Then $\lambda^0$ approximates $\lambda_{\max}(A^S)$ with inverse relative error $2$. \hfill \Box

**Theorem 5.5.** Consider the class of symmetric interval matrices $A^S$ with $A^c$ rational positive semi-definite and $A^\Delta = E$. As long as $P \neq NP$ there is no polynomial time algorithm that computes the $\lambda_{\max}(A^S)$ with inverse relative approximation error $1/(32n^4)$.
Proof. Denote \( \epsilon := 1/(32n^4) \) and \( \alpha := \rho(A^c) + \rho(A^\Delta) \). By Theorem 2.5, there is no polynomial algorithm yielding \( \lambda^0 \) such that

\[
\lambda_{\max}(A^S) \in [\lambda^0 - \epsilon \alpha, \lambda^0 + \epsilon \alpha].
\]

We can estimate \( |\lambda^0| \leq \alpha \), whence

\[
[\lambda^0 - \epsilon \alpha, \lambda^0 + \epsilon \alpha] \supseteq (1 + [-\epsilon, \epsilon]) \lambda^0.
\]

Therefore, we cannot approximate \( \lambda_{\max}(A^S) \) in polynomial time with inverse relative approximation error \( \epsilon \). \( \square \)

6 Tractable cases

Herein, we consider special cases, for which one or both extremal eigenvalues are polynomially solvable.

Theorem 6.1. If \( A^c \) is essentially non-negative, i.e., \( A_{ij}^c \geq 0 \) \( \forall i \neq j \), then \( \lambda_{\max}(A^S) = \lambda_{\max}(\overline{A}) \).

Proof. Define \( B := A + \alpha I \), where \( \alpha \) is large enough, e.g., \( \alpha = \max_{i=1,\ldots,n} \{|\alpha_{ii}|, |\bar{\alpha}_{ii}|\} \). Then for every \( B \in B^S \) we have \( |B| \leq \overline{B} \), whence \( \lambda_{\max}(B) \leq \lambda_{\max}(\overline{B}) \). Therefore, for any \( A \in A^S \) we also have \( \lambda_{\max}(A) \leq \lambda_{\max}(\overline{A}) \). \( \square \)

Theorem 6.2. If \( A^\Delta \) is diagonal, then \( \lambda_{\min}(A^S) = \lambda_{\min}(\overline{A}) \) and \( \lambda_{\max}(A^S) = \lambda_{\max}(\overline{A}) \).

Proof. It follows from (1) as for any \( z \in \{\pm 1\}^n \) we have \( \text{diag}(z)A^\Delta \text{diag}(z) = A^\Delta \). \( \square \)

Theorem 6.3. Suppose that \( \lambda_{\max}(A^c) \) is a simple eigenvalue, and an eigenvector \( x \) to \( \lambda_{\max}(A^c) \) contains no zero entry. If \( A^\Delta \) is sufficiently small, then

\[
\lambda_{\min}(A^S) = \lambda_{\min}(A^c - \text{diag}(z)A^\Delta \text{diag}(z)),
\]

\[
\lambda_{\max}(A^S) = \lambda_{\max}(A^c + \text{diag}(z)A^\Delta \text{diag}(z)),
\]

where \( z = \text{sgn}(x) \).

Proof. It follows from (1). For sufficiently small perturbations of \( A^c \), \( z = \text{sgn}(x) \) remains the optimal sign vector in (1). \( \square \)

7 Conclusion

We discussed computational complexity of determining the extremal eigenvalues of symmetric interval matrices with certain approximation factors. For the traditional absolute and relative errors, the frontier between polynomiality and NP-hardness is known quite precisely, but the results are not very meaningful. That is why we employed an alternative relative approximation error, and proposed also a new one called mid+rad approximation error. This one seems to be appropriate for our problem, and there is open research space in tightening the gap between the polynomial and NP-hard factors.

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References