Vietoris topology on spaces dominated by second countable ones

Abstract: For a given space $X$ let $C(X)$ be the family of all compact subsets of $X$. A space $X$ is dominated by a space $M$ if $X$ has an $M$-ordered compact cover, this means that there exists a family $\mathcal{F} = \{F_K : K \in C(M)\} \subset C(X)$ such that $\bigcup \mathcal{F} = X$ and $K \subseteq L$ implies that $F_K \subseteq F_L$ for any $K, L \in C(M)$. A space $X$ is strongly dominated by a space $M$ if there exists an $M$-ordered compact cover $\mathcal{F}$ such that for any compact $K \subset X$ there is $F \in \mathcal{F}$ such that $K \subset F$. Let $K(X) = C(X) \setminus \{\emptyset\}$ be the set of all nonempty compact subsets of a space $X$ endowed with the Vietoris topology. We prove that a space $X$ is strongly dominated by a space $M$ if and only if $K(X)$ is strongly dominated by $M$ and an example is given of a $\sigma$-compact space $X$ such that $K(X)$ is not Lindelöf $\Sigma$. It is established that if the weight of a scattered compact space $X$ is not less than $\aleph_0$, then the spaces $C_p(K(X))$ and $K(C_p(X))$ are not Lindelöf $\Sigma$. We show that if $X$ is the one-point compactification of a discrete space, then the hyperspace $K(X)$ is semi-Eberlein compact.

Keywords: Strong domination by second countable spaces, Hemicompact space, Lindelöf $p$-space, Lindelöf $\Sigma$-space, Vietoris topology, One-point compactification, Eberlein compact, Scattered spaces

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1 Introduction

All spaces under consideration are assumed to be Tychonoff. The space $\mathbb{R}$ is the set of real numbers with its natural topology and $\omega$ denotes its cardinality. The set of natural numbers is denoted by $\mathbb{N}$. Let $\omega$ be the first infinite ordinal number. For a space $X$ let $C_p(X)$ be the space of continuous maps from $X$ to $\mathbb{R}$ endowed with the topology of pointwise convergence. In [2] and [14] we can find properties of function spaces. For a given space $X$ let $K(X)$ be the set of all nonempty compact subsets of $X$ endowed with the Vietoris topology whose base is determined by finite families $U_1, U_2, \ldots, U_n$ of open subsets of $X$ as follows

$$\{U_1, U_2, \ldots, U_n\} = \{F \in K(X) : F \subset \bigcup\{U_i : i \leq n\} \cap U_i \neq \emptyset \forall i \leq n\}.$$ 

For a natural number $n$, let $\mathcal{F}_n(X)$ be the set of all the nonempty subsets of $X$ with at most $n$ points, endowed with the topology induced from $K(X)$. By $\mathcal{F}(X)$ we denote the hyperspace of all nonempty finite subsets of the space $X$. Note that $\mathcal{F}(X) = \bigcup \{\mathcal{F}_n(X) : n \in \mathbb{N}\} \subset K(X)$. For the information about hyperspaces of nonempty compact subsets see [6] and [7]. A space is called Lindelöf $p$ if it can be perfectly mapped onto a second countable space. A space is called Lindelöf $\Sigma$ if it is a continuous image of a Lindelöf $p$-space. In 1991 Cascales and Orihuela introduced in [4] the class of spaces dominated by a given space $M$. For a given space $X$ let $C(X)$ be the family of all compact subsets of $X$. A space $X$ is called dominated by a space $M$ if $X$ has an $M$-ordered compact cover, this
means that there exists a family $\mathcal{F} = \{ F_K : K \in \mathcal{C}(M) \} \subseteq \mathcal{C}(X)$ such that $\bigcup \mathcal{F} = X$ and $K \subseteq L$ implies that $F_K \subseteq F_L$ for any $K, L \in \mathcal{C}(M)$. In [4] it was proved that a Lindelöf $\Sigma$-space is dominated by a second countable space. A space $X$ is strongly dominated by a space $M$ if there exists an $M$-ordered compact cover $\mathcal{F}$ such that for any compact $K \subseteq X$ there is $F \in \mathcal{F}$ such that $K \subseteq F$. Strong domination was introduced by Cascales, Orihuela and Tkachuk in [5].

A compact space is called Eberlein compact if it is homeomorphic to a subspace of $C_p(Y)$ for some compact space $Y$. By $\Sigma^*(S)$ we denote the so called $\Sigma^*$-product of $S$ copies of $\mathbb{R}$ which is defined by the set of all those $x \in \mathbb{R}^S$ such that the set $\{ s \in S : |x(s)| \geq \epsilon \}$ is finite for every $\epsilon > 0$. In 1968 Amir and Lindenstrauss proved in [1] that a compact space is Eberlein compact if and only if it can be embedded into a $\Sigma^*$-product of real lines. In 2004 Kubis and Leiderman defined in [9] semi-Eberlein compact spaces. A space $X$ is called semi-Eberlein if for some set $S$ there is an embedding $X \subseteq \mathbb{R}^S$ such that $\Sigma^*(S) \cap X$ is dense in $X$. By $w(X)$ we denote the weight of the space $X$. For a given cardinal number $\kappa$, a space $X$ is called $\kappa$-compact if it is the union of $\kappa$ many compact subspaces of $X$. The rest of notation and terminology is standard and follows [6].

It is well known that if the space $X$ is either compact or locally compact, then $K(X)$ is equivalently compact or locally compact. In [16] it was proved that if $X$ is a Dieudonné complete space then $K(X)$ is also Dieudonné complete. A space $X$ is called hemicompact if it has a sequence $\{ K_n : n \in \mathbb{N} \}$ of compact subsets such that for every compact $A \subseteq X$ there exists $n \in \mathbb{N}$ for which $A \subseteq K_n$. Hemicompactness (see [12]) and Lindelöf $p$-property are inherited to $K(X)$. In this work we study hyperspaces of Eberlein (semi-Eberlein) compact, Lindelöf $\Sigma$-spaces and spaces dominated by second countable ones. We prove that there exists a $\sigma$-compact space $X$ such that $K(X)$ is not Lindelöf $\Sigma$. We show that if $X$ is strongly dominated by a space $M$ then $K(X)$ is strongly dominated by $M$. We prove that if $X$ is a scattered space and its weight is not less than $\aleph$, then $K(C_p(X))$ and $C_p(K(X))$ are not Lindelöf $\Sigma$-spaces. We show that if $X$ is the one-point compactification of a discrete space, then the hyperspace $K(X)$ is a semi-Eberlein compact space.

2 Hyperspaces of Lindelöf $\Sigma$-spaces and spaces dominated by second countable ones

Recall that a space is Lindelöf $p$ if it can be perfectly mapped onto a space with a countable base. It is well known that a space $X$ is Lindelöf $p$ if and only if the hyperspace $K(X)$ is Lindelöf $p$. In 1969 Nagami introduced in [11] the class of Lindelöf $\Sigma$-spaces. A space $X$ is Lindelöf $\Sigma$-space if it is a continuous image of a Lindelöf $p$-space. In [15] we can find properties of Lindelöf $\Sigma$-spaces. It is worth to mention that any Lindelöf $\Sigma$-space is $\sigma$-compact. Since $X$ is homeomorphic to $\mathcal{F}_\Sigma(X)$ and $\mathcal{F}_\Sigma(X)$ is closed in $K(X)$ we note that if $K(X)$ is Lindelöf $\Sigma$, then $X$ is Lindelöf $\Sigma$. For a given space $X$ and every $n \in \mathbb{N}$ define the natural continuous map $\mu_n : X^n \to \mathcal{F}_n(X)$ by the rule $\mu_n(x_1, x_2, \ldots, x_n) = \{x_1, x_2, \ldots, x_n\}$.

Proosition 2.1. If $X$ is Lindelöf $\Sigma$-space, then the hyperspace $K(X)$ has an everywhere dense Lindelöf $\Sigma$-subspace.

Proof. The hyperspace $\mathcal{F}(X)$ is an everywhere dense subspace of $K(X)$. The class of Lindelöf $\Sigma$-spaces is stable under continuous images, closed subspaces, countable products and countable unions. The hyperspace $\mathcal{F}_n(X)$ is the image of $X^n$ under continuous function $\mu_n$. The hyperspace $\mathcal{F}(X)$ is the countable union of its subspaces $\mathcal{F}_n(X)$ with $n \in \mathbb{N}$, hence $\mathcal{F}(X)$ is Lindelöf $\Sigma$. Therefore $K(X)$ has an everywhere dense Lindelöf $\Sigma$-subspace. \[\square\]

A space $X$ is called $\omega$-monolithic if for any countable subset $A \subseteq X$ the network weight of $\overline{A}$ is countable. If $X$ is an Eberlein compact space then $C_p(X)$ is Lindelöf $\Sigma$. A compact space $X$ is called Gulko compact if $C_p(X)$ is Lindelöf $\Sigma$, hence an Eberlein compact space is Gulko compact (see IV.2.5 in [2]). A Gulko compact space is $\omega$-monolithic. It is well known that $K(X)$ is $\sigma$-compact if and only if $K(X)$ is hemicompact and if only if $X$ is hemicompact (see [12]). The function space $C_p(X)$ is $\sigma$-compact if and only if $X$ is finite. From this fact it follows that $K(C_p(X))$ is $\sigma$-compact if and only if $X$ is finite.
Proposition 2.2. If $X$ is an Eberlein compact space, then $K(C_p(X))$ contains a dense $\sigma$-compact subspace. If $X$ is a Gulko compact space, then $K(C_p(X))$ contains an everywhere dense Lindelöf $\Sigma$-subspace.

Proof. If $X$ is an Eberlein compact, from Theorem IV.1.7 of [2] it follows that the function space $C_p(X)$ contains a $\sigma$-compact dense subspace. Thus it follows that $K(C_p(X))$ contains a $\sigma$-compact dense subspace (see [12]).

If $X$ is a Gulko compact space, then $C_p(X)$ is Lindelöf $\Sigma$, hence from Proposition 2.1 it follows that $K(C_p(X))$ contains an everywhere dense Lindelöf $\Sigma$-subspace.

Let $\kappa$ be an infinite cardinal number. A space $X$ is called $\kappa$-hemicompact if there exists a family $F \subset K(X)$ such that $|F| \leq \kappa$ and for any $A \in K(X)$ there exists $B \in F$ such that $A \subset B$. A space is hemicompact if and only if it is $\omega$-hemicompact. It is well known that any $\sigma$-compact space is Lindelöf $\Sigma$. If $X$ is hemicompact, then $K(X)$ is hemicompact and therefore Lindelöf $\Sigma$.

Proposition 2.3. Let $X$ be a compact space such that $|X| \geq 2$, $a \in X$, $\kappa$ an infinite cardinal number and the $\sigma$-product $\sigma(X^\kappa) = \{(x_\alpha) \in X^\kappa : |\{\alpha < \kappa : x_\alpha \neq a\}| < \infty\}$. Then:

i) $\sigma(X^\kappa)$ is $\sigma$-compact and it is not $\kappa$-hemicompact;

ii) if $\kappa \geq \omega$, then the hyperspace $K(\sigma(X^\kappa))$ is not Lindelöf $\Sigma$.

Proof. i) For any natural number $n$ the space $\sigma_n(X^\kappa) = \{(x_\alpha) \in X^\kappa : |\{\alpha < \kappa : x_\alpha \neq a\}| \leq n\}$ is compact. Then $\sigma(X^\kappa) = \cup j \sigma_n(X^\kappa) : n \in \mathbb{N}\}$ is a $\sigma$-compact space.

The $\sigma$-product $\sigma(\mathbb{D})$, where $\mathbb{D} = \{0, 1\}$, is homeomorphic to a closed subspace of $\sigma(X^\kappa)$. From Theorem 3.9 of [5] it follows that $\sigma(\mathbb{D})$ is not $\kappa$-hemicompact. Then the $\sigma$-product $\sigma(X^\kappa)$ is not $\kappa$-hemicompact, because $\kappa$-hemicompactness is hereditary to closed subspaces.

ii) From i) it follows that $\sigma(X^\kappa)$ is not $\kappa$-hemicompact. Thus from Theorem 2.1 of [12] it follows that the hyperspace $K(\sigma(X^\kappa))$ is not $\kappa$-compact. Since $\kappa \geq \omega$ the hyperspace $K(\sigma(X^\kappa))$ is not $\omega$-compact. Since any Lindelöf $\Sigma$-space is $\omega$-compact, we conclude that the hyperspace $K(\sigma(X^\kappa))$ is not Lindelöf $\Sigma$.

If $A(\omega)$ is the one-point compactification of a countable discrete space, then it is metrizable compact, thus its hyperspace $K(A(\omega)))$ is also metrizable compact, hence it is an Eberlein compact space. In contrast to this fact we have the next proposition. Let $\omega_1$ be the first non countable ordinal number.

Proposition 2.4. Let $A(\omega_1) = Y \cup \{\infty\}$ be the one-point compactification of the discrete space $Y = \{x_\alpha : \alpha < \omega_1\}$. Then the hyperspace $K(A(\omega_1)))$ is not $\omega$-monolithic and, therefore, it is not a Gulko compact space.

Proof. There is $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ a countable family of infinite subsets of $Y$ such that for every disjoint finite sets $F, G \subset Y$ there are $U_k, U_n \in \mathcal{U}$ such that $U_k \cap U_n = \emptyset$, $F \subset U_k$ and $G \subset U_n$. Let us consider the following subsets of the hyperspace $K(A(\omega_1))$:

\[ A = \{[\infty], A(\omega_1)\} \cup \{K \subset A(\omega_1) : \infty \in K\}, \]

\[ B = \{[\infty, x] : x \in Y\} \subset A \text{ and} \]

\[ D = \{[\infty] \cup (U_{1k} \cup U_{2k} \cup \cdots \cup U_{nk}) : U_{ik} \in \mathcal{U}, k = 1, \ldots, n\} \subset A. \]

The hyperspace $K(\omega_1)$ is compact and the set $\{Y\}$ is open in $K(A(\omega_1))$, then the subspace $A = K(A(\omega_1)) \setminus \{Y\}$ is compact. Since the family $\mathcal{U}$ is countable, the set $D$ is also countable. The set $B$ is an infinite non countable discrete subspace of $A$, in fact for a given point $[\infty, x] \in B$, where $x \in Y$, take the open standard set $\{[x], A(\omega_1)\}$, it is easy to see that $B \cap (\{x\}, A(\omega_1)) = \{[\infty, x]\}$. The countable set $D$ is dense in $A$. The set $B$ is contained in the closure of $D$, however $B$ does not have a countable network. Then the hyperspace $K(A(\omega_1))$ is not $\omega$-monolithic and therefore it is not a Gulko compact space.

For a given continuous map $f : X \to Y$, let $K(f) : K(X) \to K(Y)$ be the induced continuous map defined by $K(f)(A) = f(A)$ for any $A \in K(X)$.

Proposition 2.5. If $X$ is a scattered compact space and its weight is $\kappa \geq \omega_1$, then the hyperspace $K(X)$ is not a Gulko compact space.
Proof. There exists a continuous map $g$ from $X$ onto $A(\kappa)$, the one-point compactification of the discrete space of cardinality $\kappa$. The map $g$ is perfect, hence the induced map $K(g) : K(X) \to K(A(\kappa))$ is surjective (see [12]). From the previous proposition it follows that $K(A(\kappa))$ is not a Gulko compact and it is a continuous image of $K(X)$, therefore $K(X)$ is not a Gulko compact space.

\[\square\]

Proposition 2.6. Let $X$ be a scattered compact space.
\begin{enumerate}[i)]
\item If $X$ has countable weight, then $C_p(K(X))$ and $K(C_p(X))$ are Lindelöf $\Sigma$-spaces.
\item If $\kappa = w(X) > \omega$, then $K(C_p(X))$ is not a Lindelöf $\Sigma$-space.
\item If $\kappa = w(X) \geq \omega_1$, then $C_p(K(X))$ is not a Lindelöf $\Sigma$-space.
\end{enumerate}

Proof. i) If $X$ is a compact scattered space of countable weight, then it is a countable metrizable compact. Thus $K(X)$ is metrizable compact and $C_p(K(X))$ is Lindelöf $\Sigma$, because $K(X)$ is an Eberlein compact. Since $X$ is countable the weight of $C_p(X)$ is countable, then $K(C_p(X))$ has also a countable base. Any space with countable weight is Lindelöf, therefore $K(C_p(X))$ is a Lindelöf $\Sigma$-space.

ii) Let $A(\kappa)$ be the one-point compactification of a discrete set of cardinality $\kappa$. Since $\sigma(D^\kappa)$ is a closed subspace of $\Sigma_\kappa(\kappa)$ and $\Sigma_\kappa(\kappa)$ is homeomorphic to $C_p(A(\kappa))$, we conclude that $C_p(A(\kappa))$ is not $\kappa$-hemicompact and therefore it is not $\kappa$-hemicompact. From Theorem 2.1 of [12] it follows that $K(C_p(A(\kappa)))$ is not $\kappa$-compact, then it is not a Lindelöf $\Sigma$-space. There exists a continuous onto map $f$ from $X$ to $A(\kappa)$. Hence $C_p(A(\kappa))$ is homeomorphic to a closed subspace of $C_p(X)$, then $K(C_p(A(\kappa)))$ is homeomorphic to a closed subspace of $K(C_p(X))$. Since a closed subspace of a Lindelöf $\Sigma$-space is Lindelöf $\Sigma$, we conclude that $K(C_p(X))$ is not a Lindelöf $\Sigma$-space.

iii) The induced map $K(f) : K(X) \to K(A(\kappa))$ is onto. From the compactness of $K(X)$ and $K(A(\kappa))$ it follows that $C_p(K(A(\kappa)))$ is homeomorphic to a closed subspace of $C_p(K(X))$. From Proposition 2.4 it follows that $K(A(\kappa))$ is not Gulko compact, hence $C_p(K(A(\kappa)))$ is not Lindelöf $\Sigma$, therefore $C_p(K(X))$ is not a Lindelöf $\Sigma$-space.

\[\square\]

A family $\gamma$ of subsets of a topological space $X$ is called $T_0$-separating if, for every pair of points $x, y \in X$, there is $U \in \gamma$ such that the set $U \cap \{x, y\}$ contains exactly one point. A family $\gamma$ of sets of a topological space $X$ is called point finite (countable) if every point of $X$ is contained only in a finite (countable) collection of sets of $\gamma$. In [9] Kubis and Leiderman gave an internal characterization of semi-Eberlein compact spaces.

Proposition 2.7. Let $X$ be a compact space. Then $X$ is semi-Eberlein compact if and only if there exists a $T_0$-separating collection $\mathcal{U}$ consisting of open $F_\sigma$ subsets of $X$ such that $\mathcal{U} = \bigcup\{U_n : n \in \mathbb{N}\}$ and the set $\{x \in X : \forall n \in \mathbb{N}\{U \in \mathcal{U}_n : x \in U\} \text{ is finite}\}$ is dense in $X$.

It is easy to see that any Eberlein compact space is semi-Eberlein compact. We have proved in Proposition 2.4 that the hyperspace of compact sets of the one-point compactification $A(\kappa)$ of a non-countable discrete space is not Eberlein compact. We will prove that semi-Eberlein compactness of $A(\kappa)$ is preserved by its hyperspace $K(A(\kappa))$.

Proposition 2.8. Let $A(\kappa)$ be the one-point compactification of a discrete space $X$ of cardinality $\kappa$. Then the hyperspace $K(A(\kappa))$ is a semi-Eberlein compact space.

Proof. Let $A(\kappa) = \{x_\alpha : \alpha < \kappa\} \cup \{\infty\}$ be the one-point compactification of the discrete space $X = \{x_\alpha : \alpha < \kappa\}$. It is easy to see that the family $\mathcal{U} = U_1 \cup U_2$, where $U_1 = \{(A(\kappa), \{x_{a_1}\}, \{x_{a_2}\}, \ldots, \{x_{a_n}\}) : x_{a_k} \in X \ \forall k = 1, \ldots, n \text{ and } n \in \mathbb{N}\}$ and $U_2 = \{(\{x_{a_1}\}, \{x_{a_2}\}, \ldots, \{x_{a_n}\}) : x_{a_k} \in X \ \forall k = 1, \ldots, n \text{ and } n \in \mathbb{N}\}$, contains only compact open subsets of the hyperspace $K(A(\kappa))$. We are going to prove that this family is $T_0$-separating the points of $K(A(\kappa))$. It is important to mention that any compact subset of $A(\kappa)$ is finite or infinite containing $\infty$.

Let $A, B$ be two different points of the hyperspace $K(A(\kappa))$. If $\infty \notin B$, then the open set $\{\{b_1\}, \{b_2\}, \ldots, \{b_m\}\}$, where $B = \{b_1, b_2, \ldots, b_m\}$, separates the points $A$ and $B$. If $\infty \notin A$ we can separate the points $A$ and $B$ by the same way.
If \( \infty \in A \cap B \), then there exists a point \( p \neq \infty \) such that \( p \in A \setminus B \) or \( p \in B \setminus A \). In this case the open set \( \{A(\kappa), \{p\}\} \) separates the points \( A \) and \( B \).

It is easy to see that each finite subset of \( A(\kappa) \) is contained only in a finite family of elements of \( U \). Note that every compact subset of \( X \) is finite and \( X \) is a dense subset of \( A(\kappa) \). Thus the family \( U \) is point finite in the dense set \( F(X) \) of finite nonempty subsets of \( X \), hence \( U \) satisfies the conditions of Proposition 2.7. Therefore the hyperspace \( K(A(\kappa)) \) is a semi-Eberlein compact space.

A space \( X \) is called dominated by a space \( M \) if \( X \) has an \( M \)-ordered compact cover i.e. there exists a family \( F = \{F_K : K \in C(M)\} \subseteq C(X) \) such that \( \bigcup F = X \) and \( K \subseteq L \) implies that \( F_K \subseteq F_L \) for any \( K, L \in C(M) \). In [4] it was proved that any Lindelöf \( \Sigma \)-space is dominated by a second countable space. If \( X \) is dominated by a space \( M \) of weight \( w(M) = \kappa \geq \omega \), then \( X \) is \( 2^\kappa \)-compact, where \( 2^\kappa \) denotes the cardinality of the family of all subsets of a set of cardinality \( \kappa \).

A space is called strongly dominated by a space \( M \) if there exists an \( M \)-ordered compact cover \( F \) such that for any compact \( K \subseteq X \) there is \( F \in F \) such that \( K \subseteq F \). If \( X \) is strongly dominated by a space \( M \) of weight \( w(M) = \kappa \geq \omega \), then \( X \) is \( 2^\kappa \)-hemicompact.

**Proposition 2.9.** A space \( X \) is strongly dominated by a space \( M \) if and only if the hyperspace \( K(X) \) is strongly dominated by \( M \).

**Proof.** Suppose that \( X \) is strongly dominated by \( M \) and take \( F = \{F_K : K \in C(M)\} \subseteq C(X) \) an \( M \)-ordered compact cover of \( X \), such that for any compact \( K \subseteq X \) there is \( F \in F \) such that \( K \subseteq F \). Let define \( G = \{K(F_K) : K \in C(M)\} \), since for any \( K \in C(M) \) the hyperspace \( K(F_K) \) is compact we have \( G \subseteq C(K(X)) \). Take \( K, L \in C(M) \) such that \( K \subseteq L \), then \( F_K \subseteq F_L \) and therefore \( K(F_K) \subseteq K(F_L) \), hence \( G \) is \( M \)-ordered. Take a compact subset \( D \subseteq K(X) \). From properties of compact subsets of the hyperspace \( K(X) \) it follows that the set \( D = \bigcup D \subseteq X \) is compact, then by strong domination of \( X \) by \( M \) there exists a compact set \( F \in F \) such that \( D \subseteq F \). It is easy to see that \( D \subseteq K(D) \subseteq K(F) \subseteq G \). Therefore \( K(X) \) is strongly dominated by the space \( M \).

Suppose that \( K(X) \) is strongly dominated by the space \( M \). The space \( X \) is homeomorphic to the closed subspace \( F(X) \subseteq K(X) \), then from 3.3(d) of [5] it follows that \( X \) is strongly dominated by the space \( M \).

**Corollary 2.10.** If \( X \) is strongly dominated by a second countable space, then \( K(X) \) is strongly dominated by a second countable space and it is \( e \)-hemicompact.

**Proposition 2.11.** If \( f : X \rightarrow Y \) is a perfect onto map, then \( X \) is strongly dominated by a space \( M \) if and only if \( Y \) is strongly dominated by \( M \).

**Proof.** If \( X \) is strongly dominated by a space \( M \), then by Theorem 3.3(a) of [5] it follows that \( Y \) is strongly dominated by a space \( M \).

Let \( Y \) be strongly dominated by the space \( M \) and take \( F = \{F_K : K \in C(M)\} \subseteq C(Y) \), an \( M \)-ordered compact cover of \( Y \) such that for any compact \( B \subseteq Y \) there is \( F \in F \) such that \( B \subseteq F \). The map \( f \) is perfect, hence \( f^{-1}(F) \subseteq X \) is compact for any \( F \in F \), so we define the family

\[ G = \{G_K = f^{-1}(F_K) : K \in C(M)\} \subseteq C(X). \]

If we take two subsets \( K, L \in C(M) \) such that \( K \subseteq L \), then \( F_K \subseteq F_L \) and therefore \( G_K = f^{-1}(F_K) \subseteq f^{-1}(F_L) = G_L \). Take a compact set \( A \subseteq X \) and define the compact set \( B = f(A) \). From strong domination of \( Y \) by \( M \) there exists \( F_K \in F \) such that \( B \subseteq F_K \), hence \( A \subseteq f^{-1}(f(A)) = f^{-1}(B) \subseteq f^{-1}(F_K) \subseteq G \). Therefore \( X \) is strongly dominated by \( M \).

**Proposition 2.12.** If the space \( X \) is Lindelöf \( p \), then it is strongly dominated by a second countable space.

**Proof.** There exists a second countable space \( M \) and a perfect onto map \( f : X \rightarrow M \). It is clear that \( M \) is strongly dominated by \( M \). Therefore from Proposition 2.11 it follows that \( X \) is strongly dominated by the second countable space \( M \).
For a given cardinal number \( \kappa \), let \( D(\kappa) \) be the discrete space of cardinality \( \kappa \).

**Proposition 2.13.** Let \( \kappa \) be an infinite cardinal. A space \( X \) is \( \kappa \)-hemicompact space if and only if it is strongly dominated by \( D(\kappa) \).

**Proof.** Assume that \( X \) is \( \kappa \)-hemicompact. Let \( \mathcal{H} = \{ H_\alpha : \alpha < \kappa \} \) be a family of compact subsets of \( X \) which witnesses the \( \kappa \)-hemicompactness of \( X \). The hyperspace \( C(D(\kappa)) \) consists of all the finite subsets of \( D(\kappa) \). For any compact \( K \in C(D(\kappa)) \) define \( F_K = \bigcup\{ H_\alpha : \alpha \in K \} \in C(X) \). It is easy to see that for any \( L, M \in C(D(\kappa)) \) such that \( L \subseteq M \) we have \( F_L \subseteq F_M \). Let \( B \) be a compact subset of \( X \), there exists \( \alpha < \kappa \) such that \( B \subseteq H_\alpha \), then \( B \subseteq F(\alpha) \). Therefore the family \( \mathcal{F} = \{ F_K : K \in C(D(\kappa)) \} \) is a compact cover of \( X \) which witnesses that \( X \) is strongly dominated by the space \( D(\kappa) \).

Suppose that \( X \) is a space strongly dominated by \( D(\kappa) \). The cardinality of \( C(D(\kappa)) \) is \( \kappa \), therefore \( X \) is \( \kappa \)-hemicompact.

**Proposition 2.14.** If \( X \) is a hemicompact space, then \( X \) and \( \mathcal{K}(X) \) are strongly dominated by a second countable space.

**Proof.** The space \( D(\omega) \) is second countable.

In Theorem 2.1 of [5] it was proved that in the class of Dieudonné complete spaces domination by a second countable space and the Lindelöf \( \Sigma \) property are equivalent. We will show that in the class of Lindelöf \( \Sigma \)-spaces strong domination by a second countable space and the Lindelöf \( p \) property are not equivalent. The character of \( X \) at its subspace \( A \subset X \), denoted by \( \chi(A, X) \), is the minimal of the cardinalities of all outer bases of \( A \) in \( X \). A space \( X \) is called ultracomplete if \( \chi(X, cX) \leq \omega \) for some (equivalently for any) compactification \( cX \) of \( X \) (see [13]).

**Proposition 2.15.** If \( X \) is an ultracomplete space without points of local compactness and \( cX \) is a compactification of \( X \), then the remainder \( R(X) = cX \setminus X \) is Lindelöf \( \Sigma \), strongly dominated by a second countable space and it is not Lindelöf \( p \).

**Proof.** Ponomarev and Tkachuk proved in [13] that the remainder of an ultracomplete space in any of its compactification is hemicompact. Thus the remainder \( R(X) \) is hemicompact and therefore it is a Lindelöf \( \Sigma \)-space. By Proposition 2.14 the remainder \( R(X) \) is strongly dominated by a second countable space.

The space \( X \) is not Lindelöf, because any Lindelöf ultracomplete space has points of local compactness (see [13]) and \( X \) does not have points of local compactness. The space \( R(X) \) is a dense subset of \( cX \), hence \( cX \) is also a compactification of \( R(X) \) and the remainder of \( R(X) \) in \( cX \) is \( R(R(X)) = X \). It is well known that the remainder of a Lindelöf \( p \)-space in each compactification is also Lindelöf \( p \). Therefore \( R(X) \) is not Lindelöf \( p \).

**Example 2.16.** i) Let us consider \( \omega_1 \) the space of all countable ordinals with its interval topology and \( \omega_1^{\omega} \) with the product topology. Then \( \omega_1^{\omega} \) is a non Lindelöf space which is strongly dominated by a second countable one.

ii) If \( c\omega_1^{\omega} \) is a compactification of \( \omega_1^{\omega} \), then \( R(\omega_1^{\omega}) = c\omega_1^{\omega} \setminus \omega_1^{\omega} \) is Lindelöf \( \Sigma \), strongly dominated by a second countable space and is not Lindelöf \( p \).

**Proof.** i) In Proposition 3.4 of [5] it was proved that \( \omega_1 \) is strongly dominated by a second countable space. From Theorem 3.3 of [5] it follows that \( \omega_1^{\omega} \) is strongly dominated by a second countable space. The space \( \omega_1^{\omega} \) is a non compact countably compact space and, hence, it is not Lindelöf.

ii) Buhagiar and Yoshioka proved in Theorem 4.16 of [3] that \( \omega_1^{\omega} \) is an ultracomplete space. The space \( \omega_1^{\omega} \) does not have points of local compactness. From Proposition 2.15 it follows that the remainder \( R(\omega_1^{\omega}) = c\omega_1^{\omega} \setminus \omega_1^{\omega} \) is Lindelöf \( \Sigma \), strongly dominated by a second countable space and it is not Lindelöf \( p \).

**Proposition 2.17.** If \( A(\kappa) \) is the one-point compactification of a discrete space of cardinality \( \kappa \geq \omega \), then \( \mathcal{K}(C_p(A(\kappa))) \) is not dominated by a second countable and \( C_p(A(\kappa)) \) is not strongly dominated by a second countable space.
Proof. The one-point compactification $A(\kappa)$ is an Eberlein compact, then $C_p(A(\kappa))$ is Lindelöf $\Sigma$. Theorem 2.1 of [5] implies that the function space $C_p(A(\kappa))$ is a Dieudonné complete and dominated by a second countable space. The hyperspace $K(C_p(A(\kappa)))$ is also Dieudonné complete (see [16]). From Proposition 2.6 it follows that $K(C_p(A(\kappa)))$ is not Lindelöf $\Sigma$. Therefore from Theorem 2.1 of [5] it follows that $K(C_p(A(\kappa)))$ is not dominated by a second countable space. From Proposition 2.9 it follows that $C_p(A(\kappa))$ is not strongly dominated by a second countable space.

Corollary 2.18. If $X$ is a scattered compact space of weight $w(X) = \kappa \geq \mathfrak{c}$, then $K(C_p(X))$ is not dominated by a second countable space and $C_p(X)$ is not strongly dominated by a second countable space.

Proof. The hyperspace $K(C_p(A(\kappa)))$ is homeomorphic to a closed subspace of $K(C_p(X))$ (see the proof of Proposition 2.6).

Proposition 2.19. If $X$ is strongly dominated by a second countable space, then for any $n \in \mathbb{N}$ the hyperspace $F_n(X)$ is strongly dominated by a second countable space.

Proof. From Proposition 2.9 it follows that $K(X)$ is strongly dominated by a second countable space. The class of spaces strongly dominated by a second countable space is closed under closed subspaces. Therefore for any $n \in \mathbb{N}$ the hyperspace $F_n(X)$ is strongly dominated by a second countable space.

Domination by second countable spaces is stable under countable unions (see 2.1(d) of [5]). The following example shows that the same conclusion is not true in the class of spaces strongly dominated by second countable ones.

Example 2.20. Let $\kappa \geq \mathfrak{c}$. If $X$ is a compact space with more than one point, then $\sigma(X^{\kappa})$ is $\sigma$-compact and it is not $\kappa$-hemicompact (see Proposition 2.3). Therefore $\sigma(X^{\kappa})$ is a countable union of spaces strongly dominated by second countable ones, thus from Corollary 2.1 it follows that $\sigma(X^{\kappa})$ is not strongly dominated by a second countable space, because it is not $\mathfrak{c}$-hemicompact.

The second part of Proposition 2.17 and the second part of Corollary 2.18 are related to Problem 4.11 of [5]. The anonymous referee informed the authors that a complete solution of that problem was obtained by Gartidse and Mamatelashvili in [10] and independently by Guerrero Sánchez in [8].

We conclude with some questions related to this work.

Question 2.21. If $X$ is a scattered Eberlein compact space, is then $K(X)$ semi-Eberlein compact?

Question 2.22. Is it true in ZFC that the hyperspace $K(C_p(A(\omega_1)))$ is not Lindelöf $\Sigma$?

Question 2.23. Is it true in ZFC that the hyperspace $K(C_p(A(\omega_1)))$ is not dominated by a second countable space?

A space $X$ is of countable type if for any compact $F \subset X$ there exists a compact subspace $K \subset X$ of countable character such that $A \subset X$. It is worth to mention that any Čech complete space is of countable type.

Question 2.24. If $X$ is a $\sigma$-compact space of countable type, is then $X$ strongly dominated by a second countable space?

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