Research Article

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Identities arising from higher-order Daehee polynomial bases

Abstract: Here we will derive formulas for expressing any polynomial as linear combinations of two kinds of higher-order Daehee polynomial basis. Then we will apply these formulas to certain polynomials in order to get new and interesting identities involving higher-order Daehee polynomials of the first kind and of the second kind.

Keywords: Higher-order Daehee polynomial bases, Daehee polynomial of the first kind, Daehee polynomial of the second kind, Umbral calculus

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1 Introduction

Let \( r \in \mathbb{Z}_{\geq 0} \). We recall that the Daehee polynomial of the first kind of order \( r \) \( D^{(r)}_n(x) \) and that of the second kind \( O^{(r)}_n(x) \) are given by

\[
\left( \frac{\log(1+t)}{t} \right)^r (1+t)^x = \sum_{n=0}^{\infty} D^{(r)}_n(x) \frac{t^n}{n!}, \quad (1)
\]

and

\[
\left( \frac{(1+t) \log(1+t)}{t} \right)^r (1+t)^x = \sum_{n=0}^{\infty} O^{(r)}_n(x) \frac{t^n}{n!}, \quad \text{(see [9, 12, 13])}. \quad (2)
\]

We note that \( D^{(r)}_n(x), O^{(r)}_n(x) \) are polynomials of degree \( n \) with rational coefficients for all nonnegative integers \( n \). So both \( \{D^{(r)}_0(x), D^{(r)}_1(x), \ldots, D^{(r)}_n(x)\} \) and \( \{O^{(r)}_0(x), O^{(r)}_1(x), \ldots, O^{(r)}_n(x)\} \) are bases for the \((n+1)\)-dimensional space \( \mathbb{P}_n(\mathbb{C}) = \{ p(x) \in \mathbb{C}[x] \mid \deg p(x) \leq n \} \). As is well known, the Bernoulli polynomials are defined by the generating function to be

\[
\left( \frac{t}{e^t - 1} \right) e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad \text{(see [1–27])}. \quad (3)
\]

When \( x = 0 \), \( B_n(0) = B_n \) are called the Bernoulli numbers. The Euler polynomials are given by the generating function to be

\[
\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad \text{(see [11, 16])}. \quad (4)
\]

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When $x = 0$, $E_n = E_n (0)$ are called the Euler numbers. Let $\mathcal{F}$ be the set of all formal power series in variable $t$ over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f (t) = \sum_{k=0}^{\infty} a_k t^k \left| a_k \in \mathbb{C} \right. \right\}.$$  

(5)

Let $\mathbb{P} = \mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L | p (x) \rangle$ denotes the action of the linear functional $L$ on the polynomial $p (x)$, and it is well known that the vector space operations on $\mathbb{P}^*$ are defined by $\langle L + M | p (x) \rangle = \langle L | p (x) \rangle + \langle M | p (x) \rangle$, $\langle cL | p (x) \rangle = c \langle L | p (x) \rangle$, where $c$ is a complex constant. The formal power series $f (t) = \sum_{k=0}^{\infty} a_k t^k \in \mathcal{F}$ defines a linear functional on $\mathbb{P}$ by setting

$$\langle f (t) | x^n \rangle = a_n, \quad \text{for all } n \geq 0, \text{ (see [15, 22])}.$$  

(6)

By (5) and (6), we easily get

$$\left\{ t^k \left| x^n \right\} = n! \delta_{n,k}, \quad (n, k \geq 0),$$  

(7)

where $\delta_{n,k}$ is the Kronecker’s symbol.

Let $f_L (t) = \sum_{k=0}^{\infty} \langle L | x^k \rangle \frac{t^k}{k!}$. Then, by (7), we get $\langle f_L (t) | x^n \rangle = \langle L | x^n \rangle$. Additionally, the map $L \mapsto f_L (t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of the formal power series in $t$ and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f (t)$ of $\mathcal{F}$ will be thought as both a formal power series and a linear functional. We call $\mathcal{F}$ umbral algebra. From (7), we have $\langle e^{yt} | p (x) \rangle = p (y)$. The order $o (f (t))$ of a power series $f (t) \neq 0$ is the smallest integer $k$ for which the coefficient of $t^k$ does not vanish. If $o (f (t)) = 1$, then $f (t)$ is called a delta series; if $o (f (t)) = 0$, then $f (t)$ is called an invertible series.

For $f (t) \in \mathcal{F}$ and $p (x) \in \mathbb{P}$, we have

$$f (t) = \sum_{k=0}^{\infty} \langle f (t) | x^k \rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \langle t^k \left| p (x) \right. \rangle \frac{x^k}{k!}.$$  

(8)

Thus, by (8), we get

$$p^{(k)} (0) = \left\langle t^k \left| p (x) \right. \rangle = \left\{ 1 \mid p^{(k)} (x) \right\}, \quad \text{where } p^{(k)} (0) = \frac{d^k p (x)}{dx^k} \bigg|_{x=0}.$$  

(9)

From (9), we have

$$t^k p (x) = p^{(k)} (x) = \frac{d^k p (x)}{dx^k}, \quad e^{yt} p (x) = p (x + y).$$  

(10)

For $f (t), g (t) \in \mathcal{F}$ with $o (f (t)) = 1, o (g (t)) = 0$, there exists a unique sequence $s_n (x)$ of polynomials such that

$$\left\{ g (t) f (t)^k \left| s_n (x) \right. \rangle = n! \delta_{n,k}, \quad \text{for } n, k \geq 0, \text{ (see [10, 22])}.\right.$$  

(11)

The sequence $s_n (x)$ is called the Sheffer sequence for $(g (t), f (t))$, which is denoted by $s_n (x) \sim (g (t), f (t))$. As is known, $s_n (x) \sim (g (t), f (t))$ if and only if

$$\frac{1}{g' (\overline{f} (t))} e^{y \overline{f} (t)} = \sum_{k=0}^{\infty} \frac{s_k (x)}{k!} \overline{f}^k, \quad \text{for all } x \in \mathbb{C}, \text{ (see [15, 22])},$$  

(12)

where $\overline{f} (t)$ is the compositional inverse of $f (t)$ with $f (\overline{f} (t)) = \overline{f} (f (t)) = t$. For $p (x) \in \mathbb{P}$, we have

$$\left\langle e^{yt} - 1 \left| p (x) \right. \rangle = \int_{0}^{y} p (u) du, \quad \left\langle e^{yt} - 1 \right| p (x) \rangle = p (y) - p (0).$$  

(13)

By (1), (2) and (12), we easily see that

$$D_n^{(r)} (x) \sim \left( \left( \frac{e^t - 1}{t} \right)^r, e^t - 1 \right), \quad \tilde{D}_n^{(r)} (x) \sim \left( \left( \frac{e^t - 1}{te^t} \right)^r, e^t - 1 \right).$$  

(14)
Let \( p(x) \in \mathbb{P}_n(\mathbb{C}) \) with \( p(x) = \sum_{k=0}^{n} c_k r_k(x) \), where \( r_k(x) \sim (g(t), f(t)) \). Then, by (11), we get
\[
\left\{ g(t) f(t)^k \right\} p(x) = \sum_{l=0}^{n} C_l \left\{ g(t) f(t)^k \right\} r_l(x) = \sum_{l=0}^{n} C_l \delta_{l,k}
\]
\[
eq k! C_k.
\]
Thus, by (15), we get
\[
C_k = \frac{1}{k!} \left\{ g(t) f(t)^k \right\} p(x), \quad (k \geq 0).
\]
If we apply this to \( B_n(x) \sim \left( \frac{e^t - 1}{t} \right) \), then we have
\[
C_0 = \left( \frac{e^t - 1}{t} \right) p(x) = \int_0^1 p(x) \, dx.
\]
\[
C_k = \frac{1}{k!} \left\{ \frac{e^t - 1}{t} \right\} p(x) = \frac{1}{k!} \left( e^t - 1 \right) \int_0^1 p(x) \, dx.
\]
\[
= \frac{1}{k!} \left\{ e^t - 1 \right\} p(k-1)(x) = \frac{1}{k!} \left( p(k-1)(1) - p(k-1)(0) \right),
\]
for \( k = 1, 2, 3, \ldots, n \).

In our previous paper, this formula was applied to the polynomial
\[
p(x) = \sum_{k=1}^{n-1} \frac{1}{k(n-k)} B_k(x) B_{n-k}(x), \quad (n \geq 2).
\]
Thereby, we obtained the following result (see [8, 11]):
\[
\sum_{k=1}^{n-1} \frac{1}{k(n-k)} B_k(x) B_{n-k}(x) = \frac{2}{n^2} \left( B_n + \frac{1}{2} \delta_{1,n} \right) + \frac{2}{n} \sum_{k=1}^{n-2} \frac{1}{n-k} \left( \begin{array}{c} n \\ k \end{array} \right) B_{n-k} B_k(x) + \frac{2}{n} H_{n-1} B_n(x).
\]
where \( H_n = \sum_{j=1}^{n} \frac{1}{j} \) are the harmonic numbers.

Replacing \( n \) by \( 2n \) and recalling that \( B_k = 0 \) for all odd \( k \geq 3 \), (19) can be easily modified to
\[
\sum_{k=1}^{n-1} \frac{1}{2k(2n-2k)} B_{2k}(x) B_{2n-2k}(x) + \frac{2}{2n-1} B_1(x) B_{2n-1}(x)
\]
\[
= \frac{1}{n} \sum_{k=1}^{n} \left( \frac{2n}{2k} \right) B_{2k} B_{2n-2k}(x) + \frac{1}{n} H_{2n-1} B_{2n}(x) + \frac{2}{2n-1} B_1(x) B_{2n-1}, \quad (n \geq 2).
\]
Letting \( x = 0 \) in (19), we obtain the following identity in [4, (2.16)] which is a slightly different version of the well-known Miki’s identity (see [7, 20]):
\[
\sum_{k=1}^{n-1} \frac{1}{2k(2n-2k)} B_{2k} B_{2n-2k} = \frac{1}{n} \sum_{k=1}^{n} \left( \frac{2n}{2k} \right) B_{2k} B_{2n-2k} + \frac{1}{n} H_{2n-1} B_{2n}.
\]
Put \( \overline{B}_n = \left( \frac{1 - 2^{-1-n}}{2^{-1-n}} \right) B_n = (2^{1-n} - 1) B_n = B_n \left( \frac{1}{2} \right) \) for all \( n = 0, 1, 2, \ldots \). Then we note that \( \overline{B}_1 = 0 \). Setting \( x = \frac{1}{2} \) in (19), we have
\[
\sum_{k=1}^{n-1} \frac{1}{2k(2n-2k)} B_{2k} B_{2n-2k} = \frac{1}{n} \sum_{k=1}^{n} \left( \frac{2n}{2k} \right) B_{2k} \overline{B}_{2n-2k} + \frac{1}{n} H_{2n-1} \overline{B}_{2n}, \quad (n \geq 2).
\]
which is the Faber-Pandharipande-Zagier identity. Here we emphasize that our proof of Miki’s and Faber-Pandharipande-Zagier identities depends only on the simple formulas (17) and (18) which involve the integral of
the given polynomial over the unit interval and its derivatives of all orders. However, for example, in case of Miki’s identity the other proofs are quite special and involved. Indeed, Miki’s proof uses a formula for the Fermat quotient \( \frac{a^{p-a} - a}{p} \) modulo \( p^2 \), Gessel’s proof is based on two different expressions for Stirling numbers of the second kind \( S_2(n,k) \) (see [7]), Shiratani-Yokoyama’s proof employs \( p \)-adic analysis (see [25]), and Dunne-Schubert exploits the asymptotic expansion of some special polynomials coming from the quantum field theory computations. The main purpose of this paper is to derive new and interesting identities involving higher-order Daehee polynomials of the first kind and of the second kind.

## 2 Identities arising from higher-order Daehee polynomial bases

Let \( S_2(n, k) \) be the Stirling number of the second kind \( (n, k \geq 0) \). Before we move on to the higher-order Daehee polynomials, we would like to obtain an explicit expression for the Daehee polynomial, \( D_n(x) \), of the first kind. Here, we recall that \( D_n(x) \sim (e^x - 1) \). Thus, we have

\[
D_n(x + 1) - D_n(x) = (e^x - 1) D_n(x) = n D_{n-1}(x), \quad (n \geq 1).
\]

(22)

On the other hand,

\[
\sum_{n=0}^{\infty} \frac{(D_n(x + 1) - D_n(x)) t^n}{n!} = \frac{\log(t + 1)}{t} \left( (1 + t)^{x+1} - (1 + t)^x \right)
= \log(1 + t) (1 + t)^x.
\]

(23)

By (23), we easily get

\[
D_n(x + 1) - D_n(x) = \left. \frac{d^n}{dt^n} \left( \log(1 + t) (1 + t)^x \right) \right|_{t=0}.
\]

(24)

Now, we observe that

\[
\left( \log(1 + t) (1 + t)^x \right)' = (1 + t)^{x-1} + x \log(1 + t) (1 + t)^{x-1}
\]

\[
(\log(1 + t) (1 + t)^x)^{(2)} = \{x + (x - 1)\} (1 + t)^{x-2} + x (x - 1) \log(1 + t) (1 + t)^{x-2}
\]

\[
(\log(1 + t) (1 + t)^x)^{(3)} = \{(x - 1) (x - 2) + x (x - 2) + x (x - 1)\} (1 + t)^{x-3}
+ x (x - 1) (x - 2) \log(1 + t) (1 + t)^{x-3},
\]

\[
(\log(1 + t) (1 + t)^x)^{(4)} = \{(x - 1) (x - 2) (x - 3) + x (x - 2) (x - 3) + x (x - 1) (x - 2) (x - 3) \log(1 + t) (1 + t)^{x-4}
+ x (x - 1) (x - 2) (x - 3) \log(1 + t) (1 + t)^{x-4}.
\]

Continuing this process, we have

\[
(\log(1 + t) (1 + t)^x)^{(n)} = \sum_{i=0}^{n-1} x \cdots (x - i) \cdots (x - n + 1) (1 + t)^{x-n} \quad (n \geq 1)
+ x (x - 1) \cdots (x - n + 1) \log(1 + t) (1 + t)^{x-n}.
\]

(25)

Therefore, by (22), (24) and (25), we obtain the following theorem.

**Theorem 2.1.** For \( n \geq 0 \), we have

\[
D_n(x) = \frac{1}{n+1} \sum_{i=0}^{n} x (x - 1) \cdots (x - i) \cdots (x - n).
\]
Now, we turn our attention to higher-order Daehee polynomials. Let

\[ p(x) = \sum_{k=0}^{n} C_k^{(r)} D_k^{(r)}(x) \in \mathbb{P}_n(\mathbb{C}). \]  

(26)

Then, by (1), we get

\[
C_k^{(r)} = \frac{1}{k!} \left( \left( \frac{e^t - 1}{t} \right)^r \left( e^t - 1 \right)^k \right) p(x) = \frac{1}{k!} \left( \left( \frac{e^t - 1}{t} \right)^{r+k} \right) t^k p(x) 
\]

(27)

Therefore, by (26) and (27), we obtain the following theorem.

**Theorem 2.2.** For \( p(x) \in \mathbb{P}_n(\mathbb{C}) \) with \( p(x) = \sum_{k=0}^{n} C_k^{(r)} D_k^{(r)}(x) \), we have

\[
C_k^{(r)} = \frac{(r + k)!}{k!} \sum_{m=0}^{n-k} \frac{S_2(m + r + k, r + k)}{(m + r + k)!} t^m p^{(k)}(x) 
\]

(28)

In particular, for \( r = 1 \), let

\[
p(x) = \sum_{k=0}^{n} C_k D_k(x), \quad (n \geq 0).
\]

Then, we see that

\[
C_k = (k + 1) \sum_{m=0}^{n-k} \frac{S_2(m + 1 + k, k + 1)}{(m + 1 + k)!} p^{(m+k)}(0).
\]

Let us assume that \( p(x) \in \mathbb{P}_n(\mathbb{C}) \) with

\[
p(x) = \sum_{k=0}^{n} \hat{C}_k^{(r)} \hat{D}_k^{(r)}(x).
\]

(29)

From (2), we can derive the following equation:

\[
\hat{C}_k^{(r)} = \frac{1}{k!} \left( \left( \frac{e^t - 1}{te^t} \right)^r \left( e^t - 1 \right)^{k} \right) p(x) = \frac{1}{k!} \left( \left( \frac{e^t - 1}{t} \right)^{r+k} \right) t^k p(x) 
\]

(29)
Let us take \( p(x) = \sum_{k=0}^{n} B_k(x) B_{n-k}(x) \in \mathbb{P}_n(\mathbb{C}) \). Then, we have
\[
p^{(1)}(x) = \sum_{k=1}^{n} k B_{k-1}(x) B_{n-k}(x) + \sum_{k=0}^{n-1} (n-k) B_{n-k-1}(x) B_k(x) \\
= 2 \sum_{k=1}^{n} k B_{k-1}(x) B_{n-k}(x).
\]

\[
p^{(2)}(x) = 2 \sum_{k=2}^{n} k (k-1) B_{k-2}(x) B_{n-k}(x) + 2 \sum_{k=1}^{n-1} (n-k) B_{k-1}(x) B_{n-k-1}(x) \\
= 2 \sum_{k=2}^{n} k (k-1) B_{k-2}(x) B_{n-k}(x) + 2 \sum_{k=2}^{n} (k-1)(n-k+1) B_{k-2}(x) B_{n-k}(x) \\
= 2 (n+1) \sum_{k=2}^{n} (k-1) B_{k-2}(x) B_{n-k}(x).
\]

Continuing this process, we have
\[
p^{(k)}(x) = 2 (n+1) n \cdots (n+3-k) \sum_{l=k}^{n} (l-k+1) B_{l-k}(x) B_{n-l}(x)
\]
(30)
\[
= 2 \frac{(n+1)!}{(n-k+2)!} \sum_{l=k}^{n} (l-k+1) B_{l-k}(x) B_{n-l}(x).
\]

Therefore, by Theorem 2.2 and (29), we obtain the following theorem.

**Theorem 2.3.** For \( n \geq 0 \), we have
\[
\sum_{k=0}^{n} B_k(x) B_{n-k}(x) = 2 \sum_{k=0}^{n} \sum_{l=m}^{n} \sum_{m=0}^{n-k} \frac{(r+k)(n+1)}{r} \binom{m+k}{m+r+k} \binom{l-m-k+1}{m+r+k} \binom{n-m+2-k}{n-m+2-k} \\
\times S_2(m+r+k,r+k) B_{l-m-k}(-r) B_{n-l}(-r) \hat{D}^{(r)}_k(x)
\]
\[
\sum_{k=0}^{n} \frac{B_k(x) B_{n-k}(x)}{k!(n-k)!},
\]

we have
\[
p^{(k)}(x) = 2k \sum_{l=k}^{n} \frac{B_{l-k}(x) B_{n-l}(x)}{(l-k)! (n-l)!}.
\]

(31)

Therefore, by Theorem 2.2 and (29), we have the following theorem.

**Theorem 2.4.** For \( n \geq 0 \), we have
\[
\sum_{k=0}^{n} \frac{B_k(x) B_{n-k}(x)}{k!(n-k)!} = \frac{r!}{(r+n)!} \sum_{k=0}^{n} \sum_{l=m}^{n} \sum_{m=0}^{n-k} 2^{m+k} \binom{r+k}{r} \binom{n+r}{m+r+k} \binom{n-m-k}{n-l} \\
\times S_2(m+r+k,r+k) B_{l-m-k}(-r) B_{n-l}(-r) \hat{D}^{(r)}_k(x)
\]
\[
= \frac{r!}{(r+n)!} \sum_{k=0}^{n} \sum_{l=m}^{n} \sum_{m=0}^{n-k} 2^{m+k} \binom{r+k}{r} \binom{n+r}{m+r+k} \binom{n-m-k}{n-l} \\
\times S_2(m+r+k,r+k) B_{l-m-k}(-r) B_{n-l}(-r) \hat{D}^{(r)}_k(x).
\]
Let us assume that
\[
p(x) = \sum_{k=1}^{n-1} \frac{1}{k(n-k)} B_k(x) B_{n-k}(x), \quad (n \geq 2).
\]

Then, it can be shown that, for any integer \(s\) \((0 \leq s \leq n)\),
\[
p^{(s)}(x) = 2(n-1)s_{s-1} (H_{n-1} - H_{n-s-1}) B_{n-s}(x)
\]
\[+ (n-1)s \sum_{l=s+1}^{n-1} \frac{1}{(l-s)(n-l)} B_{l-s}(x) B_{n-l}(x),
\]
where the second term is equal to 0 for \(s = n-1, n\), \(H_n = \sum_{j=1}^n \frac{1}{j} (n \geq 1)\) with \(H_0 = H_{-1} = 0\), and \((n)_k = n(n-1) \cdots (n-k+1)(k \geq 1)\) with \((n)_0 = (n)_{-1} = 1\). From Theorem 2.2, and (32), we have
\[
\sum_{k=1}^{n-1} \frac{B_k(x) B_{n-k}(x)}{k(n-k)} = 2(n-1) \sum_{m=0}^{n-k} S_2(m + r + k, r + k) \frac{B_{n-m-k}(x)}{(m+r+k)!} \left(2(n-1)_{m+k-1} (H_{n-1} - H_{n-m-k-1})
\times B_{n-m-k} + (n-1)_{m+k} \sum_{l=m+k+1}^{n-1} \frac{B_{l-m-k} B_{n-l}}{(l-m-k)(n-l)} \right) D_k^{(r)}(x)
\]
\[= \sum_{k=0}^{n} \left(\frac{r+k}{k!} \sum_{m=0}^{n-k} S_2(m + r + k, r + k) \frac{B_{n-m-k}(x)}{(m+r+k)!} \left(2(n-1)_{m+k-1} (H_{n-1} - H_{n-m-k-1})
\times B_{n-m-k} (-r) + (n-1)_{m+k} \sum_{l=m+k+1}^{n-1} \frac{B_{l-m-k} (-r) B_{n-l} (-r)}{(l-m-k)(n-l)} \right) \hat{D}_k^{(r)}(x)\right).
\]

Let
\[
p(x) = \sum_{k=0}^{n} E_k(x) E_{n-k}(x).
\]

Then, we have
\[
p^{(k)}(x) = \frac{2(n+1)!}{(n+2-k)!} \sum_{l=k}^{n} (l-k+1) E_{l-k}(x) E_{n-l}(x).
\]

Therefore, by Theorem 2.2, we obtain the following theorem.

**Theorem 2.5.** For \(n \geq 0\), we have
\[
\sum_{k=0}^{n} E_k(x) E_{n-k}(x) = 2 \sum_{k=0}^{n} \sum_{m=0}^{n-k} \sum_{l=m+k}^{n} \frac{(r+k)(n+1)_{l-m-k+1}}{(m+r+k)! (n-m-k+2)} D_k^{(r)}(x)
\times S_2(m + r + k, r + k) E_{l-m-k} E_{n-l} \hat{D}_k^{(r)}(x)
\]
\[= 2 \sum_{k=0}^{n} \sum_{m=0}^{n-k} \sum_{l=m+k}^{n} \frac{(r+k)(n+1)_{l-m-k+1}}{(m+r+k)! (n-m-k+2)} \times S_2(m + r + k, r + k) E_{l-m-k} (-r) E_{n-l} (-r) \hat{D}_k^{(r)}(x).
\]

From Theorem 2.2, we can derive the following identities:
\[
\sum_{k=0}^{n} \frac{E_k(x) E_{n-k}(x)}{k!(n-k)!} = \frac{r!}{(r+n)!} \sum_{k=0}^{n} \sum_{m=0}^{n-k} \sum_{l=m+k}^{n} 2^{m+k} \binom{r+k}{k} \binom{n+r}{m+r+k}.
\]
Continuing this process, we have

Theorem 2.6. For

\[ (n-m-k) S_2 (m+r+k,r+k) E_{l-m-k} E_{n-l} \left\{ D_k^{(r)} (x) \right\} \]

Therefore, by Theorem 2.2 and (38), we obtain the following theorem.

\[ r! \sum_{k=0}^{n} \left( \sum_{m=0}^{n-k} \sum_{l=m+k}^{n} \frac{2^{m+k}}{(m+r+k)} \right) S_2 (m+r+k) \frac{(r+k)}{(n+r)} \]

\[ \times \left( \frac{n-m-k}{n-l} \right) S_2 (m+r+k) E_{l-m-k} (-r) E_{n-l} (-r) \left\{ \hat{D}_k^{(r)} (x) \right\} . \]

\[ \sum_{k=1}^{n-1} \frac{1}{k (n-k)} E_k (x) E_{n-k} (x) = \sum_{k=0}^{n} \frac{(r+k)!}{k!} \sum_{m=0}^{n-k} \frac{S_2 (m+r+k)}{(m+r+k)} \]

\[ \times (2(n-1)m_{m+k-1} (H_{n-1} - H_{n-m-k-1}) E_{n-m-k} \]

\[ + (n-1) \sum_{l=m+k+1}^{n-1} \frac{1}{(l-m-k)(n-l)} E_{l-m-k} E_{n-l} \left\{ D_k^{(r)} (x) \right\} . \]

Let us assume that \( p(x) = \sum_{k=0}^{n} B_k (x) E_{n-k} (x) \in P_n (\mathbb{C}) \). Then we have

\[ p^{(1)} (x) = \sum_{k=1}^{n} k B_k-1 (x) E_{n-k} (x) + \sum_{k=0}^{n-1} (n-k) B_k (x) E_{n-k-1} (x) \]

\[ = \sum_{k=0}^{n} k B_k-1 (x) E_{n-k} (x) + \sum_{k=1}^{n} (n-k+1) B_k-1 (x) E_{n-k} (x) \]

\[ = (n+1) \sum_{k=0}^{n} B_k-1 (x) E_{n-k} (x) \]

\[ p^{(2)} (x) = \sum_{k=2}^{n} (k-1) B_{k-2} (x) E_{n-k} (x) + \sum_{k=2}^{n} (n-k+1) B_{k-2} (x) E_{n-k} (x) \]

\[ = n (n+1) \sum_{k=2}^{n} B_{k-2} (x) E_{n-k} (x) \]

Continuing this process, we have

\[ p^{(k)} (x) = (n+1) n \cdots (n+2-k) \sum_{l=k}^{n} B_{l-k} (x) E_{n-l} (x) \]

\[ = \frac{(n+1)!}{(n+1-k)!} \sum_{l=k}^{n} B_{l-k} (x) E_{n-l} (x) . \]

Therefore, by Theorem 2.2 and (38), we obtain the following theorem.

Theorem 2.6. For \( n \geq 0 \), we have

\[ \sum_{k=0}^{n} B_k (x) E_{n-k} (x) = \sum_{k=0}^{n} \left\{ \sum_{m=0}^{n-k} \sum_{l=m+k}^{n} \frac{1}{(m+r+k)} \frac{(n+1)}{(m+k)} \right\} S_2 (m+r+k) \frac{(r+k)}{(n-r)} \]

\[ \times B_{l-m-k} (x) E_{n-l} \left\{ D_k^{(r)} (x) \right\} . \]
Remark.

\[ \sum_{k=1}^{n-1} \frac{1}{k(n-k)} B_k(x) E_{n-k}(x) = \sum_{k=0}^{n} \left\{ \frac{(r+k)!}{k!} \sum_{m=0}^{n-k} \frac{S_2(m+r+k,r+k)}{(m+r+k)!} \times ((n-1)_{m+k-1} (H_{n-1} - H_{n-m-k-1}) (B_{n-m-k} + E_{n-m-k}) \right. \\
\left. + (n-1)^{m+k} \sum_{l=m+k+1}^{n-1} \frac{B_{l-m-k} E_{n-l}}{(l-m-k)(n-l)} \right \} D_k^{(r)}(x), \quad (n \geq 2). \]

For \( p(x) \in \mathbb{P}_n(\mathbb{C}) \) with \( p(x) = \sum_{k=0}^{n} B_k(x) x^{n-k} \), we have

\[ p'(x) = \sum_{k=1}^{n} k B_{k-1}(x) x^{n-k} + \sum_{k=0}^{n-1} B_k(x) (x-k) x^{n-k-1} \]
\[ = \sum_{k=1}^{n} k B_{k-1}(x) x^{n-k} + \sum_{k=1}^{n} B_{k-1}(x) (x-k+1) x^{n-k} \]
\[ = (n+1) \sum_{k=1}^{n} B_{k-1}(x) x^{n-k}. \]

Continuing this process, we have

\[ p^{(k)}(x) = (n+1) \cdots (n+2-k) \sum_{l=k}^{n} B_{l-k}(x) x^{n-k} = \frac{(n+1)!}{(n+1-2)!} \sum_{l=k}^{n} B_{l-k}(x) x^{n-l}. \quad (39) \]

Therefore, by Theorem 2.2 and (39), we obtain the following theorem.

**Theorem 2.7.** For \( n \geq 0 \), we have

\[ \sum_{k=0}^{n} B_k(x) x^{n-k} = \sum_{k=0}^{n} \left\{ \sum_{m=0}^{n-k} \frac{(r+k)(n+1)_m}{(m+r+k)} S_2(m+r+k,r+k) B_{n-m-k} \right \} D_k^{(r)}(x) \]
\[ = \sum_{k=0}^{n} \left\{ \sum_{m=0}^{n-k} \sum_{l=m+k}^{n} \frac{(r+k)(n+1)_m}{(m+r+k)} S_2(m+r+k,r+k) B_{l-m-k} (-r)^{n-l} \right \} D_k^{(r)}(x). \]

Remark.

\[ \sum_{k=0}^{n} E_k(x) x^{n-k} = \sum_{k=0}^{n} \left\{ \sum_{m=0}^{n-k} \frac{(r+k)(n+1)_m}{(m+r+k)} S_2(m+r+k,r+k) E_{n-m-k} \right \} D_k^{(r)}(x). \]
\[ = \frac{r!}{(r+n)!} \sum_{k=0}^{n} \left\{ \sum_{m=0}^{n-k} 2^{m+k} \binom{r+k}{r} \binom{n+r}{m+r+k} S_2(m+r+k,r+k) B_{n-m-k} \right \} D_k^{(r)}(x). \]
\[ = \frac{r!}{(r+n)!} \sum_{k=0}^{n} \left\{ \sum_{m=0}^{n-k} 2^{m+k} \binom{r+k}{r} \binom{n+r}{m+r+k} S_2(m+r+k,r+k) E_{n-m-k} \right \} D_k^{(r)}(x). \]
where \( n \geq 2 \).

Let us assume that \( p(x) \in \mathbb{P}_n(\mathbb{C}) \) with \( p(x) = \sum_{k=0}^{n} C_k^{(r)} D_k^{(r)} (x) \). Then, by (1), we get

\[
C_k^{(r)} = \frac{1}{k!} \left( \frac{e^t - 1}{t} \right)^r (e^t - 1)^k \left| p(x) \right| = \frac{1}{k!} \left( \frac{e^t - 1}{t} \right)^r (e^t - 1)^k \left| p(x) \right| \tag{40}
\]

Therefore, by (40), we obtain the following theorem.

**Theorem 2.8.** Let \( p(x) \in \mathbb{P}_n(\mathbb{C}) \) with \( p(x) = \sum_{k=0}^{n} C_k^{(r)} D_k^{(r)} (x) \). Then, we have

\[
C_k^{(r)} = \frac{1}{k!} \sum_{j=0}^{k} \sum_{m=0}^{n} (-1)^{k-j} \binom{k}{j} S_2 (m + r, r) \frac{t^m}{(m+r)!} \left| p(x+j) \right| \tag{41}
\]

Let us consider \( p(x) \in \mathbb{P}_n(\mathbb{C}) \) with \( p(x) = \sum_{k=0}^{n} \hat{C}_k^{(r)} D_k^{(r)} (x) \). From (2), we have

\[
\hat{C}_k^{(r)} = \frac{1}{k!} \left( \frac{e^t - 1}{t e^t} \right)^r (e^t - 1)^k \left| p(x) \right| \tag{41}
\]
For example, let $p(x) = \sum_{k=0}^{n} B_k(x) B_{n-k}(x) \in \mathbb{P}_n(\mathbb{C})$. Then, we have

$$p^{(1)}(x) = \sum_{k=1}^{n} k B_{k-1}(x) B_{n-k}(x) + \sum_{k=0}^{n-1} (n - k) B_k(x) B_{n-k-1}(x)$$

$$= \sum_{k=1}^{n} k B_{k-1}(x) B_{n-k}(x) + \sum_{k=1}^{n} (n - k + 1) B_{k-1}(x) B_{n-k}(x)$$

$$= (n + 1) \sum_{k=1}^{n} B_{k-1}(x) B_{n-k}(x).$$

Continuing this process, we have

$$p^{(2)}(x) = n (n + 1) \sum_{k=2}^{n} B_{k-2}(x) B_{n-k}(x).$$

From Theorem 2.8, (41) and (42), we have

$$\sum_{k=0}^{n} B_k(x) B_{n-k}(x) = \sum_{k=0}^{n} \left\{ \frac{r!}{k!} \sum_{j=0}^{k} \sum_{m=0}^{n} (-1)^{k-j} \binom{k}{j} S_2(m + r, j) \frac{(n + 1)!}{(m + r)! (n - m + 1)!} \right\} D_k^{(r)}(x)$$

$$= \sum_{k=0}^{n} \left\{ \frac{r!}{k!} \sum_{j=0}^{k} \sum_{m=0}^{n} \frac{(-1)^{k-j} \binom{k}{j} (n + 1)}{k!(m + r)} \times S_2(m + r, j) B_{l-m}(j) B_{n-l}(j) \right\} D_k^{(r)}(x)$$

$$= \sum_{k=0}^{n} \left\{ \frac{r!}{k!} \sum_{j=0}^{k} \sum_{m=0}^{n} \frac{(-1)^{k-j} \binom{k}{j} (n + 1)}{k!(m + r)} \times S_2(m + r, j) B_{l-m}(j) B_{n-l}(j) \right\} D_k^{(r)}(x).$$

From Theorem 2.8, we can derive the following identities:

$$\sum_{k=0}^{n} \frac{1}{k! (n-k)!} B_k(x) B_{n-k}(x) = \frac{1}{n!} \sum_{k=0}^{n} \left\{ \frac{r!}{k!} \sum_{j=0}^{k} \sum_{m=0}^{n} \frac{(-1)^{k-j} \binom{k}{j} (n-l; l-m, m)}{k!(m + r)} \times S_2(m + r, j) B_{l-m}(j) B_{n-l}(j) \right\} D_k^{(r)}(x)$$

$$= \frac{1}{n!} \sum_{k=0}^{n} \left\{ \frac{r!}{k!} \sum_{j=0}^{k} \sum_{m=0}^{n} \frac{(-1)^{k-j} \binom{k}{j} (n-l; l-m, m)}{k!(m + r)} \times 2^m S_2(m + r, j) B_{l-m}(j) B_{n-l}(j) \right\} D_k^{(r)}(x)$$

$$= \frac{1}{n!} \sum_{k=0}^{n} \left\{ \frac{r!}{k!} \sum_{j=0}^{k} \sum_{m=0}^{n} \frac{(-1)^{k-j} \binom{k}{j} (n-l; l-m, m)}{k!(m + r)} \times 2^m S_2(m + r, j) B_{l-m}(j) B_{n-l}(j) \right\} D_k^{(r)}(x).$$

$$\sum_{k=1}^{n-1} \frac{1}{k (n-k)} B_k(x) B_{n-k}(x) = r! \sum_{k=0}^{n-1} \left\{ \frac{r!}{k! (m + r)!} \sum_{j=0}^{k} \sum_{m=0}^{n} \frac{(-1)^{k-j} \binom{k}{j} S_2(m + r, j)}{k!(m + r)!} \times (2 (n - 1)_m (H_m - H_{m-1}) B_{n-m}(j) \right\} D_k^{(r)}(x).$$
\begin{align*}
+ (n - 1)m \sum_{l=m+1}^{n-1} \frac{1}{(l-m)(n-l)} B_{l-m} (j) B_{n-l} (j) \right] D_k^{(r)} (x) \\
= r! \sum_{k=0}^{n} \left\{ \sum_{j=0}^{k} \sum_{m=0}^{j} \frac{(-1)^{k-j}}{j! (m+r)!} S_2 (m+r, r) \times (2(n-1)_{m-1} (H_{n-1} - H_{n-m-1}) B_{n-m} (j-r)) \\
+ (n-1)m \sum_{l=m+1}^{n-1} \frac{1}{(l-m)(n-l)} B_{l-m} (j-r) B_{n-l} (j-r) \right\} \hat{D}_k^{(r)} (x),
\end{align*}

where \( n \geq 2 \).

References