A function space from a compact metrizable space to a dendrite with the hypo-graph topology

Abstract: Let \( X \) be an infinite compact metrizable space having only a finite number of isolated points and \( Y \) be a non-degenerate dendrite with a distinguished end point \( v \). For each continuous map \( f : X \to Y \), we define the hypo-graph \( \downarrow_v f = \bigcup_{x \in X} \{ x \} \times [v, f(x)] \), where \([v, f(x)]\) is the unique arc from \( v \) to \( f(x) \) in \( Y \). Then we can regard \( \downarrow_v C(X, Y) = \{ \downarrow_v f \mid f : X \to Y \text{ is continuous} \} \) as the subspace of the hyperspace \( Cld(X \times Y) \) of non-empty closed sets in \( X \times Y \) endowed with the Vietoris topology. Let \( \downarrow_v C(X, Y) \) be the closure of \( \downarrow_v C(X, Y) \) in \( Cld(X \times Y) \). In this paper, we shall prove that the pair \((\downarrow_v C(X, Y), \downarrow_v C(X, Y))\) is homeomorphic to \((Q, c_0)\), where \( Q = I^N \) is the Hilbert cube and \( c_0 = \{ (x_i)_{i \in \mathbb{N}} \in Q \mid \lim_{i \to \infty} x_i = 0 \} \).

Keywords: Function space, Hyperspace, Hypo-graph, The Vietoris topology, Dendrite, The Hilbert cube

MSC: 54B20, 54C35, 54F65, 57N20

DOI 10.1515/math-2015-0021
Received September 17, 2013; accepted December 15, 2014.

1 Introduction

In this paper, all maps are continuous, but functions are not necessarily continuous. For spaces \( X \) and \( Y \), let \( C(X, Y) \) be the set of all maps from \( X \) to \( Y \) and \( Cld(X) \) be the hyperspace of all non-empty closed sets in \( X \) endowed with the Vietoris topology. It is known that when \( X \) is compact metrizable, \( Cld(X) \) is so.

A **dendrite** is a Peano continuum\(^1\) containing no simple closed curves. Note that a non-degenerate dendrite is quite identical to a 1-dimensional compact AR\(^2\) [3, Chapter V, Corollary 13.5]. An end point of a space has an arbitrarily small open neighborhood whose boundary is a singleton. A non-degenerate dendrite contains end points [15, Chapter III, (6.1) and Chapter V, (1.1)]. Each pair of distinct points of a dendrite is connected by the unique arc [15, Chapter V, (1.2)]. So we denote the unique arc of two points \( x, y \) in a dendrite by \([x, y]\), where \([x, y]\) is the constant path if \( x = y \).

For each function \( f : X \to Y \) of \( X \) into a dendrite \( Y \) and \( v \in Y \), we define the hypo-graph \( \downarrow_v f \) of \( f \) with respect to \( v \) as follows:

\[
\downarrow_v f = \bigcup_{x \in X} \{ x \} \times [v, f(x)] \subset X \times Y.
\]

---

Hanbiao Yang: School of Mathematics and Computational Science, Wuyi University, Jiangmen 529020, China, E-mail: hongsejulebu@sina.com

Katsuro Sakai: Department of Mathematics, Faculty of Engineering, Kanagawa University, Yokohama, 221-8686, Japan, E-mail: k.sakai.top@gmail.com

*Corresponding Author: Katsuhisa Koshino: Division of Mathematics, Pure and Applied Sciences, University of Tsukuba, Tsukuba, 305-8571, Japan, E-mail: kakoshino@math.tsukuba.ac.jp

\(^1\) A continuum is a compact connected metrizable space and a Peano continuum is a locally connected continuum.

\(^2\) An AR is an absolute retract for metrizable spaces.
When $f$ is continuous, the hypo-graph $\downarrow_v f$ is a closed subset of the product space $X \times Y$. Hence we can regard

$$\downarrow_v \text{Cld}(X, Y) = \{ \downarrow_v f \mid f \in \text{C}(X, Y) \}$$

as the subspace of the hyperspace $\text{Cld}(X \times Y)$ with the relative topology. Let $\downarrow_v \text{C}(X, Y)$ be the closure of $\downarrow_v \text{C}(X, Y)$ in $\text{Cld}(X \times Y)$. In the case that $Y = I$ and $v = 0$, where $I = [0, 1]$, we can consider

$$\downarrow_0 \text{USC}(X, I) = \{ \downarrow_0 f \mid f : X \rightarrow I \text{ is upper semi-continuous} \}$$

as the subspace in $\text{Cld}(X \times I)$. Let $Q = I^N$ be the Hilbert cube and $c_0 = \{(x_i)_{i \in N} \in Q \mid \lim_{i \rightarrow \infty} x_i = 0 \}$, where $N = \{1, 2, \ldots \}$ is the set of natural numbers. Z. Yang and X. Zhou [13, 14] showed the following theorem:

**Theorem 1.1.** Let $X$ be a compact metrizable space. If the set of isolated points is not dense in $X$, then $\downarrow_0 \text{USC}(X, I) \cong \text{Cld}(X, I)$ and the pair $(\downarrow_0 \text{USC}(X, I), \downarrow_0 \text{C}(X, I))$ is homeomorphic to $(Q, c_0)$.

In the paper, we generalize this result as follows:

**Main Theorem.** Let $X$ be an infinite compact metrizable space having only a finite number of isolated points, $Y$ a non-degenerate dendrite and $v \in Y$ an end point of $Y$. Then the pair $(\downarrow_v \text{C}(X, Y), \downarrow_v \text{C}(X, Y))$ is homeomorphic to $(Q, c_0)$.

**Remark 1.2.** The space $\downarrow_v \text{C}(X, Y)$ has a cluster point in $\text{Cld}(X \times Y)$ which is not the hypo-graph of any map from $X$ to $Y$. For example, let $X = I$, $Y = [0] \times I \cup [-1, 1] \times \{1\}$ a triod and $v = (0, 0) \in Y$. Define a closed set $A$ in $X \times Y$ as follows:

$$A = I \times \{0\} \cup \{x \in [-1, 1] \times \{1\} \cup \{(x, t \sin(\pi/x), 1) \mid x \in (0, 1), t \in I \}. $$

For each $n \in \mathbb{N}$, let $f_n : X \rightarrow [-1, 1] \times \{1\} \subset Y$ be the map defined by

$$f_n(x) = \begin{cases} (\sin(\pi/x), 1) & \text{if } x \geq 1/2n, \\ (0, 1) & \text{if } x \leq 1/2n. \end{cases}$$

Then observe that

$$\downarrow_v f_n = I \times \{0\} \cup \{(x, t \sin(\pi/x), 1) \mid x \in [1/2n, 1], t \in I \}$$

and the sequence $(\downarrow_v f_n)_{n \in \mathbb{N}}$ converges to $A$ in $\text{Cld}(X \times I)$. However, the set $A$ is not the hypo-graph of any map from $X$ to $Y$.

## 2 Preliminaries

Let $X = (X, d)$ and $Y = (Y, d')$ be metric spaces, $x \in X$ and $A, B \subset X$. We denote the diameter of $A$ by $\text{diam } A = \sup \{d(x, y) \mid x, y \in A\}$, and the distance between $A$ and $B$ by $\text{dist}(A, B) = \inf \{d(x, y) \mid x \in A, y \in B\}$. For each $\epsilon > 0$, let $B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$, $\overline{B}(x, \epsilon) = \{y \in X \mid d(x, y) \leq \epsilon\}$ and $N(A, \epsilon) = \{x \in X \mid \text{dist}(\{x\}, A) < \epsilon\}$. It is said that a map $f : X \rightarrow Y$ is $\epsilon$-close to a map $g : X \rightarrow Y$ if $d'(f(x), g(x)) < \epsilon$ for all $x \in X$. Similarly, for each open cover $\mathcal{U}$ of $Y$, $f$ is $\mathcal{U}$-close to $g$ provided that for all $x \in X$, both of $f(x)$ and $g(x)$ are contained in some $U \in \mathcal{U}$. Throughout the paper, we use an admissible metric for the product space $X \times Y$ defined by

$$\rho((x, y), (x', y')) = \max\{d(x, x'), d'(y, y')\} \text{ for each } x, x' \in X \text{ and } y, y' \in Y.$$ 

When $X$ is compact, the topology of $\text{Cld}(X)$ is induced by the Hausdorff metric $d_H$ defined as follows:

$$d_H(A, B) = \inf\{r > 0 \mid A \subset N(B, r), B \subset N(A, r)\}.$$ 

The following lemma can be easily proved.
Lemma 2.1. Let $A, A', B$ and $B'$ be closed sets in a compact metric space $X = (X, d)$. Then

$$d_H(A \cup B, A' \cup B') \leq \max\{d_H(A, A'), d_H(B, B')\}.$$ 

For a metric space $X = (X, d)$, the metric $d$ is said to be convex if for each pair of points $x$ and $y$ in $X$, there exists a mid point $z \in X$ between $x$ and $y$, i.e., $d(x, z) = d(y, z) = d(x, y)/2$. As is easily observed, when the metric $d$ is convex and complete, there exists an arc from $x$ to $y$ isometric to the segment $[0, d(x, y)]$. It is well known that every Peano continuum admits a convex metric [2], [7], and hence every dendrite does so. The unique arcs in a dendrite have the following property with respect to the convex metric [5].

Lemma 2.2. Let $Y = (Y, d)$ be a dendrite with a convex metric. Then there exists a map $\gamma : Y^2 \times I \to Y$ such that for any distinct points $x, y \in Y$, the map $\gamma_{x, Y} = \gamma(x, y, \cdot) : I \ni t \mapsto \gamma(x, y, t) \in Y$ is an arc from $x$ to $y$ and the following holds:

(†) For each $x_i, y_i \in Y$, $i = 1, 2$, $d(\gamma_{x_1, y_1}(t), \gamma_{x_2, y_2}(t)) \leq \max\{d(x_1, x_2), d(y_1, y_2)\}$ for all $t \in I$.

From now on, we proceed with our argument in the following assumption:

$X = (X, d_X)$ is a compact metric space without isolated points, and $Y = (Y, d_Y)$ is a non-degenerate dendrite with a convex metric $d_Y$ and a distinguished end point $0 \in Y$.

For simplicity, we write $\downarrow C(X, Y) = \downarrow A C(X, Y)$.

3 The closure of $\downarrow C(X, Y)$ in $Cld(X \times Y)$ is an AR

This section is devoted to proving the following theorem:

Theorem 3.1. The space $\downarrow C(X, Y)$ is an AR.

For each $A \in Cld(X \times Y)$, we define a set-valued function $A : X \to Cld^*(Y)$ as follows:

$$A(x) = \{y \in Y \mid (x, y) \in A\} \in Cld^*(Y),$$

where $Cld^*(Y) = Cld(Y) \cup \{\emptyset\}$.

Lemma 3.2. The following holds:

$$\downarrow C(X, Y) = \{A \in Cld(X \times Y) \mid A(x) \neq \emptyset \text{ for all } x \in X \text{ and } y \in A(x) \Rightarrow [0, y] \subseteq A(x)\}.$$ 

Proof. For convenience sake, let $F$ be the set of the right-hand side of the above equality. Then observe that $\downarrow C(X, Y) \subseteq F$.

First, we prove that $F$ is closed in $Cld(X \times Y)$. Let $A$ be the limit of a sequence $(A_n)_{n \in \mathbb{N}}$ in $F$. We shall show that $A(x) \neq \emptyset$ for every $x \in X$. For $n \in \mathbb{N}$, we can take $y_n \in A_n(x) \neq \emptyset$. Because of the compactness of $Y$, we can assume that $(y_n)_{n \in \mathbb{N}}$ converges to some $y \in Y$. Since $\rho_H(A_n, A) \to 0$ as $n \to \infty$ and

$$\text{dist}\{(x, y), (x, y_n)\} = \rho((x, y), (x, y_n)) \to 0 \text{ as } n \to \infty.,$$

it follows that $(x, y) \in A$. Hence $A(x) \neq \emptyset$. To show that $[0, y] \subseteq A(x)$ for each $y \in A(x)$, take any $z \in [0, y]$. Using the map $\gamma : Y^2 \times I \to Y$ as in Lemma 2.2, we can write $z = \gamma(0, y, t)$ for some $t \in I$. Since $(x, y) \in A$, we can choose $(x_n, y_n) \in A_n, n \in \mathbb{N}$, so that $(x_n, y_n) \to (x, y)$ as $n \to \infty$. Let $z_n = \gamma(0, y_n, t)$ for each $n \in \mathbb{N}$. According to Lemma 2.2, $d_Y(z_n, z) \leq d_Y(y, y_n)$. Since $y_n \to y$ as $n \to \infty$, we have $z_n \to z$ as $n \to \infty$. Then
$z_n \in [0, y_n] \subset A_n(x_n)$, so $(x_n, z_n) \in A_n$ for every $n \in \mathbb{N}$. Because $(x_n, z_n) \to (x, z)$ as $n \to \infty$, it follows that $(x, z) \in A$, so $z \in A(x)$. Thus we have $[0, y] \subset A(x)$. Consequently, $A \in F$, so $F$ is closed in $\text{Cld}(X \times Y)$.

Next, we will show that $\downarrow C(X \times Y)$ is dense in $F$. For each $\varepsilon > 0$ and $A \in F$, because of the compactness of $A$, $A$ has a finite number of points $(x_i, y_i), i = 1, \ldots, n$, such that $A \subset \bigcup_{i=1}^{n} B((x_i, y_i), \varepsilon/2)$, where we can take $x_i \neq x_j$ if $i \neq j$ because $X$ has no isolated points. Let $A_0 = X \times \{0\} \cup \bigcup_{i=1}^{n} \{x_i\} \times \{0, y_i\} \subset A$. Then $A \subset N(A_0, \varepsilon/2)$, which implies that $\rho_H(A_0, A) < \varepsilon/2$. Let $\delta = \min\{\varepsilon, d_X(x_i, x_j) \mid i \neq j\}/3 > 0$. Note that $B(x_i, \delta) \cap B(x_j, \delta) = \emptyset$ for every $i \neq j$. Using Urysohn maps, we can construct a map $f : X \to Y$ such that $f(X \setminus \bigcup_{i=1}^{n} B(x_i, \delta)) = \{0\}, f(B(x_i, \delta)) \subset \{0, y_i\}$ and $f(x_i) = y_i$ for each $i = 1, \ldots, n$. Then $\rho_H(\downarrow f, A_0) < \delta \leq \varepsilon/3$. It follows that

$$\rho_H(\downarrow f, A) \leq \rho_H(\downarrow f, A_0) + \rho_H(A_0, A) \leq \varepsilon/3 + \varepsilon/2 < \varepsilon.$$ 

Therefore $\downarrow C(X \times Y)$ is dense in $F$. \hfill \square

We define $r : Y \times I \to Y$ by $r(y, t) = r(y, 0, t)$ for each $y \in Y$ and $t \in I$, where $y$ is the map as in Lemma 2.2. Note that $r_0(Y) = \{0\}$ and $r_1 = \text{id}_Y$. Using this map $r$, we can define the homotopy $\tau : \overline{C}(X \times Y) \times I \to \overline{C}(X \times Y)$ as follows:

$$\tau(A, t) = (\text{id}_X \times r_t)(A) = \{(x, r_t(y)) \mid (x, y) \in A\}.$$ 

Then $\tau_0(\overline{C}(X \times Y)) = X \times \{0\}$ and $\tau_1 = \text{id}^{\overline{C}(X \times Y)}$. We shall verify the uniform continuity of $\tau$. According to Lemma 2.2, the map $r$ is uniformly continuous. Hence, for any $\varepsilon > 0$, we can choose $\delta > 0$ so that for each $y, y' \in Y$ and $t, t' \in I$, if $d_Y(y, y') < \delta$ and $|t - t'| < \delta$, then $d_Y(r(y, t), r(y', t')) < \varepsilon$. Now, take $A, A' \in \overline{C}(X \times Y)$ and $t, t' \in I$ so that $\rho_H(A, A') < \delta$ and $|t - t'| < \delta$. For each $(x, z) \in \tau_t(A)$, there is a point $y \in A(x)$ such that $z = r_t(y)$. Since dist$(\{(x, y)\}, A') < \delta$, we can find $(x', y') \in A'$ such that $\rho((x, y), (x', y')) < \delta$, which means that $d_X(x, x') < \delta$ and $d_Y(y, y') < \delta$. Let $z' = r_{t'}(y') \in A'(x')$. Then $(x', z') \in \tau_{t'}(A')$ and $d_Y(z, z') = d_Y(r_{t'}(y), r_{t'}(y')) < \varepsilon$, and hence $\rho((x, z), (x', z')) = \max\{d_X(x, x'), d_Y(z, z')\} < \varepsilon$. Thus we have dist$(\{(x, z)\}, \tau_{t'}(A')) < \varepsilon$. By the same argument, we can show that dist$(\{(x', z')\}, \tau_{t'}(A)) < \varepsilon$ for each $(x', z') \in \tau_{t'}(A')$. Therefore $\rho_H(\tau_t(A), \tau_{t'}(A')) < \varepsilon$. Consequently, the map $\tau$ is uniformly continuous. Then $\tau$ is a contraction $^4$ of $\overline{C}(X \times Y)$.

We show the uniformly local path-connectedness of $\downarrow C(X \times Y)$ as follows:

**Lemma 3.3.** The space $\overline{C}(X \times Y)$ is uniformly locally path-connected with respect to $\rho_H$.

**Proof.** Take $\varepsilon > 0$ and $A, A' \in \overline{C}(X \times Y)$ with $\rho_H(A, A') < \varepsilon/2$. We define a path $h : I \to \overline{C}(X \times Y)$ from $A$ to $A \cup A'$ by $h(t) = A \cup \tau_t(A')$, where Lemma 3.2 guarantees $h(I) \subset \overline{C}(X \times Y)$. The continuity of $h$ follows from the one of $\tau$ and Lemma 2.1. In fact,

$$\rho_H(h(t), h(t')) = \rho_H(A \cup \tau_t(A'), A \cup \tau_{t'}(A')) \leq \rho_H(\tau_t(A'), \tau_{t'}(A')).$$

Moreover, $A \subset h(0), h(t') \subset A \cup A'$, and hence

$$\rho_H(h(t), h(t')) \leq \rho_H(A \cup A', A \cup A') = \rho_H(A, A') < \varepsilon/2.$$ 

It follows that diam$(h(I)) \leq \rho_H(A, A') < \varepsilon/2$. Consequently, $A$ is connected with $A \cup A'$ by an $\varepsilon/2$-path. Similarly, $A'$ is connected with $A \cup A'$ by an $\varepsilon/2$-path. Therefore $A$ and $A'$ are connected by an $\varepsilon$-path. Thus the proof is complete. \hfill \square

Now, we shall prove Theorem 3.1.

**Proof of Theorem 3.1.** By Lemma 3.3, $\downarrow C(X \times Y)$ is a Peano continuum. Then, according to the Wojdyłkowski Theorem [6, Theorem 5.3.14], we have $\text{Cld}(\overline{C}(X \times Y))$ is an AR. Identifying $A \in \text{Cld}(X \times Y)$ with $\{A\} \in \text{Cld}(\text{Cld}(X \times Y))$, we can regard $\text{Cld}(X \times Y) \subset \text{Cld}(\text{Cld}(X \times Y))$. Then the union operator

$$\bigcup : \text{Cld}(\text{Cld}(X \times Y)) \ni A \mapsto \bigcup A \in \text{Cld}(X \times Y)$$

\footnote{A homotopy $h : Z \times I \to Z$ is said to be a contraction of $Z$ if $h_1 = \text{id}_Z$ and $h(Z \times \{0\})$ is a singleton.}
is a retraction, see [6, Proposition 5.3.6]. As is easily observed due to Lemma 3.2, \( \bigcup (\text{Cld}(\downarrow \text{C}(X,Y))) = \downarrow \text{C}(X,Y) \). It follows that \( \downarrow \text{C}(X,Y) \) is a retract of the AR \( \text{Cld}(\downarrow \text{C}(X,Y)) \). Therefore \( \downarrow \text{C}(X,Y) \) is an AR.

## 4 The homotopy denseness of \( \downarrow \text{C}(X,Y) \) in \( \downarrow \text{C}(X,Y) \)

A subset \( Z \) of a space \( W \) is said to be **homotopy dense** in \( W \) if there exists a homotopy \( h : W \times I \to W \) such that \( h_0 = \text{id}_W \) and \( h_t(W) \subset Z \) for every \( t > 0 \). In this section, we will prove the following theorem:

**Theorem 4.1.** The space \( \downarrow \text{C}(X,Y) \) is homotopy dense in \( \downarrow \text{C}(X,Y) \).

For a simplicial complex \( K \) and a simplex \( \sigma \in K \), let \( K^{(0)} \) and \( \sigma^{(0)} \) be the sets of vertices of \( K \) and \( \sigma \), respectively. Moreover, denote the polyhedron of \( K \) by \( |K| \). In general setting, we can restate Lemma 3 of [11], refer to Corollary 4 of [9] and Lemma 4.2 of [5], as follows:

**Lemma 4.2.** Let \( W = (W,d) \) be a compact metric space. Then a dense subset \( Z \) of \( W \) is homotopy dense in \( W \) if it has the following property:

**(hd)** There exists \( \alpha > 0 \) such that for any locally finite countable simplicial complex \( K \), each map \( f : K^{(0)} \to Z \) extends to a map \( \overline{f} : |K| \to Z \) such that \( \text{diam} \overline{f}(\sigma) \leq \alpha \text{ diam } f(\sigma^{(0)}) \) for every \( \sigma \in K \).

**Proof of Theorem 4.1.** We only need to verify condition (hd) with respect to \( \alpha = 10 \) in Lemma 4.2. Let \( K \) be a locally finite countable simplicial complex and \( f : K^{(0)} \to \downarrow \text{C}(X,Y) \). We shall construct a map \( \overline{f} : |K| \to \downarrow \text{C}(X,Y) \) such that \( \text{diam} \overline{f}(\sigma) \leq 10 \text{ diam } f(\sigma^{(0)}) \) for every \( \sigma \in K \). For simplicity, let \( \epsilon_\sigma = \text{diam } f(\sigma^{(0)}) \geq 0 \) for each \( \sigma \in K \setminus K^{(0)} \). Let \( K_0 \) be the full subcomplex of \( K \) such that

\[ K_0^{(0)} = \{ \sigma \in K^{(0)} \mid f(\text{St}(v,K)^{(0)}) \text{ is a singleton} \} \]

where \( \text{St}(v,K) \) is the star at \( v \) in \( K \). Note that \( f(\sigma^{(0)}) \) is a singleton if \( \sigma \in K \) and \( \sigma \cap |K_0| \neq \emptyset \). We define \( K_1 = \{ \sigma \in K \mid \sigma \cap |K_0| = \emptyset \} \). For every \( v \in K_1^{(0)} \), since \( \text{diam } f(\text{St}(v,K)^{(0)}) > 0 \), we can define

\[ \epsilon_v = \min \{ \epsilon_\sigma \mid \sigma \in \text{St}(v,K), \epsilon_\sigma > 0 \} > 0. \]

Let \( f_0 : |K_0| \to \downarrow \text{C}(X,Y) \) be the map such that \( f_0(\sigma) = f(\sigma^{(0)}) \) for each \( \sigma \in K_0 \).

Since \( K \) is locally finite and \( X \) has no isolated points, we can choose finite sets \( A_v \subset X \) and \( \delta_v > 0, v \in K_1^{(0)} \), so that

1. \( \rho_H (f(v)|_{A_v}, f(v)) < \epsilon_v \),
2. \( B(a, \delta_v) \cap B(a', \delta_{v'}) = \emptyset \) if \( v \neq v' \in K_1^{(0)}, v \) and \( v' \) are contained in some \( \sigma \in K \), \( a \in A_v \), and \( a' \in A_{v'} \),
3. \( B(a, \delta) \cap B(a', \delta) = \emptyset \) if \( a \neq a' \in A_v \) and \( v \in K_1^{(0)} \),

where \( f(v)|_{A_v} = \bigcup_{a \in A_v} \{ a \} \times [0, f(v)(a)] \). First, we will construct a map \( f_1 : |K_1| \to \downarrow \text{C}(X,Y) \) such that \( \rho_H (f_1(v), f(v)) < \epsilon_v \) for each \( v \in K_1^{(0)} \), and \( \text{diam } f_1(\sigma) < 7 \epsilon_\sigma \) for each \( \sigma \in K_1 \). For every \( v \in K_1^{(0)} \), we define \( f_1(v) \in \text{Cld}(X \times X) \) as follows:

\[ f_1(v)(x) = \begin{cases} r(f(v)(x) \times \{ \delta_v - \text{dist}(\{x\}, A_v) \}/\delta_v) & \text{if } \text{dist}(\{x\}, A_v) \leq \delta_v, \\ \emptyset & \text{if } \text{dist}(\{x\}, A_v) > \delta_v. \end{cases} \]

Recall that \( r : Y \times I \to Y \) is the map defined by \( r(y,t) = r(0, y, t) \), where \( r \) is as in Lemma 2.2. Because \( f(v) \in \downarrow \text{C}(X,Y) \), as is easily observed, \( f_1(v) \in \downarrow \text{C}(X,Y) \). Since \( f(v)|_{A_v} \subset f_1(v) \subset f(v) \), it follows that

\[ \rho_H (f(v), f_1(v)) \leq \rho_H (f(v)|_{A_v}, f(v)) < \epsilon_v. \]

Denote the barycenter of \( \sigma \in K_1 \) by \( \bar{\sigma} \). For \( \sigma \in K_1 \), let

\[ f_1(\bar{\sigma}) = \bigcup_{v \in \sigma^{(0)}} f_1(v) \in \downarrow \text{C}(X,Y). \]
For each \( z \in \sigma \), there exist faces \( \sigma_0 < \sigma_1 < \cdots < \sigma_n < \sigma \) of \( \sigma \) such that \( z = \sum_{i=0}^{n} t_i \hat{\sigma}_i \), where \( \sum_{i=0}^{n} t_i = 1 \) and \( t_i \geq 0 \). Then we can define

\[
    f_1(z) = \bigcup_{i=0}^{n} \mathcal{P} \left( f_1(\hat{\sigma}_i), \sum_{j=i}^{n} t_j \right) \in \mathcal{C}(X, Y).
\]

For each \( \sigma \in K_1 \) and \( v \in \sigma^{(0)} \), the continuity of \( f_1|_{\text{St}(v, \text{Sd} K)|\cap \sigma} \) follows from the ones of both the map \( \mathcal{P} \) and the union operator \( |\text{Cld}(\text{Cld}(X \times Y))| \), where \( \text{Sd} K \) is the barycentric subdivision of \( K \). Indeed, for each \( z = \sum_{i=0}^{n} t_i \hat{\sigma}_i \in |\text{St}(v, \text{Sd} K)| \cap \sigma \), where \( v = \sigma_0 < \sigma_1 < \cdots < \sigma_n < \sigma \), \( \sum_{i=0}^{n} t_i = 1 \) and \( t_i \geq 0 \), the map

\[
    |\text{St}(v, \text{Sd} K)| \cap \sigma \ni z \mapsto \left\{ \mathcal{P} \left( f_1(\hat{\sigma}_i), \sum_{j=i}^{n} t_j \right) \right\}_{i=0}^{n} \in \text{Cld}(\text{Cld}(X \times Y))
\]

is continuous because \( \mathcal{P} \) is so. Moreover, the union operator \( \bigcup : \text{Cld}(\text{Cld}(X \times Y)) \to \text{Cld}(X \times Y) \) is also continuous. The map \( f_1|_{\text{St}(v, \text{Sd} K)|\cap \sigma} \) is the composition of them. Since \( K_1 \) is locally fine, it follows that \( f_1 \) is continuous. Thus we have a map \( f_1 : |K_1| \to \mathcal{C}(X, Y) \).

We shall show that \( \text{diam} f_1(\sigma) < 7\varepsilon_\sigma \) for any \( \sigma \in K_1 \). For each \( \sigma \in K_1 \), take any \( v \in \sigma^{(0)} \) and any \( z \in |\text{St}(v, \text{Sd} K)| \cap \sigma \). Then we can write \( z = \sum_{i=0}^{n} t_i \hat{\sigma}_i \in |\text{St}(v, \text{Sd} K)| \cap \sigma \), where \( v = \sigma_0 < \sigma_1 < \cdots < \sigma_n < \sigma \), \( \sum_{i=0}^{n} t_i = 1 \) and \( t_i \geq 0 \). By the definition of \( f_1 \), we have

\[
    f_1(v) = \mathcal{P} (f_1(\hat{\sigma}_0), 1) \subset \bigcup_{i=0}^{n} \mathcal{P} \left( f_1(\hat{\sigma}_i), \sum_{j=i}^{n} t_j \right) = f_1(z) \subset f_1(\hat{\sigma}) = \bigcup_{v' \in \sigma^{(0)}} f(v').
\]

Then it follows that

\[
    \rho_H (f_1(z), f_1(v)) \leq \rho_H \left( f_1(v), \bigcup_{v' \in \sigma^{(0)}} f(v') \right) \leq \rho_H (f_1(v), f(v)) + \rho_H \left( f(v), \bigcup_{v' \in \sigma^{(0)}} f(v') \right) \\
    \leq \rho_H (f_1(v), f(v)) + \max \{ \rho_H (f(v), f(v')) | v' \in \sigma^{(0)} \} \\
    \leq \rho_H (f_1(v), f(v)) + \text{diam } f(\sigma^{(0)}) \leq \varepsilon_v + \varepsilon_\sigma \leq 2\varepsilon_\sigma.
\]

For each \( z, z' \in \sigma \in K_1 \), we can choose vertices \( v, v' \in \sigma^{(0)} \) such that \( z \in |\text{St}(v, \text{Sd} K)| \) and \( z' \in |\text{St}(v', \text{Sd} K)| \). Then we have

\[
    \rho_H (f_1(z), f_1(z')) \leq \rho_H (f_1(z), f_1(v)) + \rho_H (f_1(v), f(v)) + \rho_H (f(v), f(v')) \\
    + \rho_H (f(v'), f_1(v')) + \rho_H (f_1(v'), f_1(z')) \\
    < 2\varepsilon_\sigma + \varepsilon_v + \varepsilon_\sigma + \varepsilon_v' + 2\varepsilon_\sigma \leq 7\varepsilon_\sigma.
\]

Consequently, \( \text{diam} f_1(\sigma) < 7\varepsilon_\sigma \) for each \( \sigma \in K_1 \).

Next, we construct a map \( f_* : |K| \cup K^{(0)} \times I \to \mathcal{C}(X, Y) \), where \( |K| \) is identified with \( |K| \times \{0\} \subset |K| \times I \). Let \( f_*|_{K^{(0)}} = f_0 \) and \( f_*|_{K|I} = f_1 \). For each \( z \in |K| \setminus |K^{0} \cup K_1| \), there exists \( \sigma_0 \in K^{0} \) and \( \sigma_1 \in K_1 \) such that \( z \) is contained in the join of \( \sigma_0 \) and \( \sigma_1 \), and hence \( z \) can be uniquely written as follows: \( z = tz_0 + (1-t)z_1 \) for some \( z_0 \in \sigma_0, z_1 \in \sigma_1 \) and \( t \in I \). Then we can define

\[
    f_*(z) = \mathcal{P} (f_0(z_0), t) \cup f_1(z_1) \in \mathcal{C}(X, Y).
\]

Observe that \( f_*(z_0) = f_0(z_0) \) and \( f_*(z_1) = f_1(z_1) \). For each \( (v, t) \in K^{(0)} \times I \), we define

\[
    f_* (v, t) = \begin{cases} 
    \mathcal{P} (f(v), t) \cup f_1 (v) \text{ if } v \in K^{(0)}_1, \\
    f(v) \text{ if } v \notin K^{(0)}_1.
    \end{cases}
\]

Then for each \( v \in K^{(0)}_1, f_* (v, 0) = f_1 (v) \) and \( f_* (v, 1) = f(v) \).

Thirdly, we can obtain a map \( g : |K| \to |K| \cup K^{(0)} \times I \) so that \( g(v) = (v, 1) \) for each \( v \in K^{(0)} \) and \( g(\sigma) = \sigma \cup \sigma^{(0)} \times I \) for each \( \sigma \in K \setminus K^{(0)} \). In fact, let \( v \in K^{(0)} \) and \( z = \sum_{i=0}^{n} t_i \hat{\sigma}_i \in |\text{St}(v, \text{Sd} K)| \), where \( \sigma_0 < \sigma_1 < \cdots < \sigma_n \in K, \sum_{i=0}^{n} t_i = 1 \) and \( t_i \geq 0 \). We define

\[
    g(z) = \begin{cases} 
    (1 - 2t_0)z + 2t_0v \text{ if } t_0 \leq 1/2, \\
    (v, 2t_0 - 1) \text{ if } t_0 \geq 1/2.
    \end{cases}
\]
Now, the desired map $\overline{f} : |K| \to \downarrow C(X, Y)$ can be defined by $\overline{f} = f \circ g$. As is easily observed, $\overline{f}_{|K^{(0)}} = f$. We will show that $\text{diam} \overline{f}(\sigma) \leq 10\varepsilon_\sigma$ for every $\sigma \in K$. When $\sigma \in K_0$, we have $\text{diam} \overline{f}(\sigma) = \text{diam} f(\sigma^{(0)}) = 0$. For each $\sigma \in K_1$, since $\overline{f}(\sigma) = f_1(\sigma) \cup f_3(\sigma^{(0)} \times I)$, it follows that
\[
\text{diam} \overline{f}(\sigma) \leq \text{diam} f_1(\sigma) + \text{diam } f_3(\sigma^{(0)} \times I)
\leq \text{diam } f_1(\sigma) + \text{diam } f(\sigma^{(0)}) + 2\max \{\rho_H(f_1(v), f(v)) \mid v \in \sigma^{(0)}\}
\leq 7\varepsilon_\sigma + \varepsilon_\sigma + 2\varepsilon_\sigma = 10\varepsilon_\sigma.
\]

When $\sigma \in K \setminus (K_0 \cup K_1)$, we can take $\sigma_0 \in K_0$ and $\sigma_1 \in K_1$ so that $\sigma$ is the join of $\sigma_0$ and $\sigma_1$. Since $\sigma \in \text{St}(v_0, K)$ for any $v_0 \in \sigma_0^{(0)} \subset K_0^{(0)}$, $f(\sigma^{(0)})$ is a singleton. For each $z = tz_0 + (1-t)z_1 \in \sigma$, where $z_0 \in \sigma_0, z_1 \in \sigma_1$ and $0 \leq t \leq 1$, choose $v \in \sigma_1^{(0)}$ such that $z_1 \in [\text{St}(v, \text{D} K)]$. Then $f(\sigma^{(0)}) = \{f(v)\}$, $f_1(v) \subset f_1(z_1) \subset f(v)$ and $f_3(z) = \overline{f}(f_0(z_0), t) \cup f_1(z_1) \subset f(v)$. Hence we get
\[
\text{dist } f_3(z, f(\sigma^{(0)})) = \rho_H(f_3(z), f(v)) \leq \rho_H(f_1(v), f(v)) < \varepsilon_\sigma.
\]

Therefore for each $z, z' \in \sigma$,
\[
\rho_H(f_3(z), f_3(z')) \leq \text{dist}(f_3(z), f(\sigma^{(0)})) + \text{dist}(f(\sigma^{(0)}), f_3(z')) + \text{diam } f(\sigma^{(0)}) < \varepsilon_\sigma + \varepsilon_\sigma = 2\varepsilon_\sigma.
\]

Consequently, $\text{diam } f_3(\sigma) \leq 2\varepsilon_\sigma$. Since
\[
\text{diam } f_3(\sigma^{(0)} \times I) \leq \text{diam } f(\sigma^{(0)}) + \max \{\rho_H(f(v), f_1(v)) \mid v \in \sigma_1^{(0)}\} \leq \varepsilon_\sigma \leq \varepsilon_\sigma,
\]

it follows that
\[
\text{diam } \overline{f}(\sigma) \leq \text{diam } f_3(\sigma) + \text{diam } f_3(\sigma^{(0)} \times I) \leq 2\varepsilon_\sigma + \varepsilon_\sigma = 3\varepsilon_\sigma.
\]

Thus the proof is complete. \(\square\)

### 5 The space $\downarrow C(X, Y)$ is an $F_{\sigma\delta}$-set in $\downarrow C(X, Y)$

A dendrite $Y$ has an order $\leq$ defined as follows: $x \leq y$ if $x \in [0, y]$. For each $\delta, \varepsilon > 0$, let $A(\delta, \varepsilon)$ be the set which consists of $A \in \downarrow C(X, Y)$ such that the following condition is satisfied:

For all $x, x' \in X$, if $d_X(x, x') < \delta$ and $y, y' \in Y$ are maximal points of $A(x), A(x')$, respectively, then $d_Y(y, y') \leq \varepsilon$.

To prove that $\downarrow C(X, Y)$ is an $F_{\sigma\delta}$-set in $\downarrow C(X, Y)$, we need the following lemma.

**Lemma 5.1.** For each $\delta, \varepsilon > 0$, the set $A(\delta, \varepsilon)$ is closed in $\downarrow C(X, Y)$.

**Proof.** Take any sequence $\{A_n\}_{n \in \mathbb{N}}$ in $A(\delta, \varepsilon)$ that converges to $A$ in $\downarrow C(X, Y)$. To show that $A \in A(\delta, \varepsilon)$, let $(x, y), (x', y') \in A$ such that $d_X(x, x') < \delta$ and $y, y'$ are maximal in $A(x), A(x')$, respectively. Since $A_n \to A$, there exist $(x_n, y_n), (x'_n, y'_n) \in A_n$ such that $(x_n, y_n) \to (x, y)$ and $(x'_n, y'_n) \to (x', y')$, see [6, Lemma 5.3.1]. Without loss of generality, we may assume that $d_X(x_n, x'_n) < \delta$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, there exist maximal points $z_n \in A_n(x_n)$ and $z'_n \in A_n(x'_n)$ such that $z_n \geq y_n$ and $z'_n \geq y'_n$. Because $Y$ is compact, replacing $(z_n)_{n \in \mathbb{N}}$ and $(z'_n)_{n \in \mathbb{N}}$ with subsequences, we can assume that $z_n \to z \in Y$ and $z'_n \to z' \in Y$. Using Lemma 5.3.1 of [6] again, we have $z \in A(x)$ and $z' \in A(x')$. Then $y$ is contained in the arc $[0, z]$ from $0$ to $z$. Indeed, if not, we have dist$(y, [0, z]) > 0$. Since $y_n \to y$ and $z_n \to z$, we can choose $m \in \mathbb{N}$ so that $d_Y(y_m, y) < \text{dist}(y, [0, z])/2$. Note that $y_m \in [0, z_m]$. Then there exists a point $p \in [0, z]$ such that $d_Y(y, p) \leq d_Y(z, z_m) < \text{dist}(y, [0, z])/2$ by Lemma 2.2. It follows that
\[
d_Y(y, p) \leq d_Y(y, y_m) + d_Y(y_m, p) < \text{dist}(y, [0, z])/2 + \text{dist}(y, [0, z])/2 = \text{dist}(y, [0, z]).
\]

---

5 Lemma 5.1 holds without the assumption that $X$ has no isolated points.
which is a contradiction. Hence \( y \in [0, z] \). By the maximality of \( y \) in \( A(x) \), we have \( y = z \). Similarly, \( y' = z' \).

Since each \( A_n \in A(\delta, \varepsilon) \), \( dx(x_n, x_n') < \delta \) and \( z_n, z_n' \) are maximal in \( A(x_n), A(x_n') \), respectively, it follows that \( dy(z_n, z_n') \leq \varepsilon \). Recall that \( z_n \to z = y \) and \( z_n' \to z' = y' \), so \( dy(y, y') \leq \varepsilon \). Consequently, we have \( A \in A(\delta, \varepsilon) \). Thus the proof is complete. \( \square \)

Now, we show the following:

**Proposition 5.2.** The space \( \downarrow C(X, Y) \) is an \( F_{\sigma \delta} \)-set in \( \downarrow C(X, Y) \).\(^6\)

**Proof.** By virtue of Lemma 5.1, it suffices to show that

\[
\downarrow C(X, Y) = \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} A(1/m, 1/n).
\]

From the definition, we need only to prove that \( A(x) \) has a unique maximal point in \( Y \) for every \( A \in \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} A(1/m, 1/n) \) and \( x \in X \). Indeed, assume that \( A \in \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} A(1/m, 1/n) \) such that each \( A(x) \) has a unique maximal point. Define a function \( f : X \to Y \) by \( f(x) = \max A(x) \) for any \( x \in X \). Then \( f = A \).

For each \( n \in \mathbb{N} \), there is \( m \in \mathbb{N} \) such that \( A \in A(1/m, 1/n) \), so if \( dx(x, x') < 1/m \), then \( dy(f(x), f(x')) < 1/n \). Thus \( f \) is continuous, and hence \( A = f \in \downarrow C(X, Y) \).

Now we will show that \( A(x) \) has a unique maximal point in \( Y \). Let \( y, y' \in Y \) be maximal points in \( A(x) \). For each \( n \in \mathbb{N} \), we can choose \( m \in \mathbb{N} \) such that \( A \in A(1/m, 1/n) \), which implies that \( dy(y, y') < 1/n \). It follows that \( dy(y, y') = 0 \), that is, \( y = y' \). Therefore the maximal point of \( A(x) \) is unique. This completes the proof. \( \square \)

### 6 The Digging Lemma

The following lemma will play an important role for the rest of this paper.

**Lemma 6.1 (The Digging Lemma).** Suppose that \( Z = (Z, d_Z) \) is a metric space, \( \phi : Z \to \downarrow C(X, Y) \) is a map, and \( a \in X \) is a non-isolated point. Then for each map \( \varepsilon : Z \to (0, 1) \), there exist maps \( \psi : Z \to \downarrow C(X, Y) \) and \( \delta : Z \to (0, 1) \) such that for each \( z \in Z \),

(a) \( \rho_H(\phi(z), \psi(z)) \leq \varepsilon(z) \),

(b) \( \psi(z)(B(a, \delta(z))) = \emptyset \).\(^7\)

**Proof.** For each \( z \in Z \), let \( \xi(z) = \sup \{ \eta > 0 \mid \rho_H(\phi(z), \psi(z)|_{X \setminus B(a, \eta)}) < \varepsilon(z) \} \). Since \( a \) is not isolated and \( \phi(z) \in \downarrow C(X, Y) \), we have \( \xi(z) > 0 \). We shall prove \( \xi : Z \to (0, \infty) \) is a lower semi-continuous function. Fix any \( z \in Z \) and \( \eta \in (0, \xi(z)) \). From the definition of \( \xi(z) \),

(\( * \)) \( \rho_H(\phi(z), \psi(z)|_{X \setminus B(a, \xi(z) - \eta/2)}) < (n - 1)\varepsilon(z)/n \) for some \( n \in \mathbb{N} \).\(^8\)

Let \( t = \min \{ \eta/2, \varepsilon(z)/3n \} \). Since \( \phi \) and \( \varepsilon \) are continuous, there exists \( s > 0 \) such that if \( d_Z(z, z') < s \), then \( \rho_H(\phi(z), \phi(z')) < t \) and \( \varepsilon(z) - \varepsilon(z') \leq \varepsilon(z)/3n \). We shall show that for every \( z' \in Z \) with \( d_Z(z, z') < s \), \( \xi(z') \geq \xi(z) - \eta \). Take any \( (x, y) \in \phi(z') \setminus B(a, \xi(z) - \eta) \). Since \( \rho_H(\phi(z), \phi(z)) < t \), we can choose \( (x', y') \in \phi(z) \) so that \( \rho((x, y), (x', y')) < \eta/2 \). Then \( dx(x, x') < \eta/2 \), that is, \( (x', y') \in \phi(z)|_{B(a, \xi(z) - \eta/2)} \). Due to \( (\ast) \), there exists \( (x'', y'') \in \phi(z)|_{X \setminus B(a, \xi(z) - \eta/2)} \) such that \( \rho((x', y'), (x'', y'')) < (n - 1)\varepsilon(z)/n \). Since \( \rho_H(\phi(z), \phi(z')) < t \), we can find a point \( (x''', y''') \in \phi(z') \) such that \( \rho((x'', y''), (x''', y''')) < t \leq \eta/2 \), which implies that \( x''' \in X \setminus B(a, \xi(z) - \eta) \). Then it follows that

\[
\rho((x, y), (x''', y''')) \leq \rho((x, y), (x', y')) + \rho((x', y'), (x'', y'')) + \rho((x'', y''), (x''', y'''))< t + (n - 1)\varepsilon(z)/n + t \leq (2/3n + (n - 1)/n)\varepsilon(z) = \varepsilon(z) - \varepsilon(z)/3n < \varepsilon(z)
\]

---

\(^6\) Proposition 5.2 holds without the assumption that \( X \) has no isolated points.

\(^7\) The Digging Lemma 6.1 holds without the assumption that \( X \) has no isolated points.
Thus $\xi$ is lower semi-continuous.

By Theorem 2.7.6 of [10], we can obtain a map $\delta : Z \to (0, 1)$ so that $\delta(z) < \xi(z)/2$ for each $z \in Z$. Now, we can define the desired map $\psi : Z \to \mathcal{C}(X, Y)$ as follows:

$$\psi(z) = \phi(z)|_{X \setminus B(a, 2\delta(z))} \cup B(a, \delta(z)) \times \{0\}$$

$$\cup \{(x, y) \in X \times Y : \delta(z) \leq d_X(x, a) \leq 2\delta(z), y \in [0, r(\max \phi(z)(x), d_X(x, a)/\delta(z) - 1)]\}.$$

Remark that $\phi(z) \in \mathcal{C}(X, Y)$ is the hypo-graph of the map $X \ni x \mapsto \max \phi(z)(x) \in Y$. It follows from the definition that $\psi$ satisfies condition (b) immediately. Note that $\psi(z) \subset \phi(z)$ and $\phi(z)|_{X \setminus B(a, 2\delta(z))} \subset \psi(z)$. Moreover, since

$$\rho_H(\phi(z), \phi(z))|_{X \setminus B(a, 2\delta(z))} \leq \rho_H(\phi(z), \phi(z)|_{X \setminus B(a, \delta(z))}) \leq \epsilon(z),$$

for each $(x, y) \in \phi(z)|_{B(a, 2\delta(z))}$, there is $(x', y') \in \phi(z)|_{X \setminus B(a, \delta(z))} \subset \psi(z)$ such that $\rho((x, y), (x', y')) \leq \epsilon(z)$. Hence $\psi$ satisfies condition (a).

**Claim.** The function $\psi$ is continuous.

For every $z \in Z$ and $\epsilon > 0$, by Lemma 2.2, there exists $\delta_1 > 0$ such that $\delta_1 < 1/2$ and

$$d_Y(y, y_1) < \delta_1 \text{ and } |t - t_1| < \delta_1 \Rightarrow d_Y(r(y, t), r(y_1, t_1)) < \epsilon.$$  

Take $\delta_2 > 0$ such that $\delta_2 \leq \delta_1/2$ and $\delta_2 \text{ diam } Y < \epsilon$. We can choose $\delta_3 > 0$ so that $\delta_3 < \delta(z)$ and

$$a, b \in [\delta(z)/2, 5\delta(z)/2] \text{ and } |a - b| < \delta_3 \Rightarrow |b/a - 1| < \delta_2.$$

Since $\phi$ and $\delta$ are continuous, there exists a neighborhood $U$ of $z$ such that for each $z' \in U$, $\rho_H(\phi(z), \phi(z')) < \min\{\epsilon, \delta(z)\delta_1/2, \delta_3/4\}$, $|1/\delta(z) - 1/\delta(z')| < 2\delta_1/9\delta(z)$ and $|\delta(z) - \delta(z')| < \delta_3/8$. We shall verify that $\rho_H(\psi(z), \psi(z')) < \epsilon$ for each $z' \in U$. Take any $(x, y) \in \psi(z)$. It is sufficient to show that $(x, y) \in \mathcal{N}(\psi(z'), \epsilon)$.

**Case I.** $d_X(x, a) \leq \delta(z)$

Then we have $y = 0$. So $(x, y) = (x, 0) \in \psi(z')$.

**Case II.** $\delta(z) < d_X(x, a) < \delta(z) + \delta_3$

Then $|d_X(x, a)/\delta(z) - 1| < \delta_2$, so

$$d_Y(0, y) \leq d_Y(0, r(\max \phi(z)(x), d_X(x, a)/\delta(z) - 1)) = (d_X(x, a)/\delta(z) - 1)d_Y(0, \max \phi(z)(x)) < \delta_2 \text{ diam } Y < \epsilon.$$

Therefore $\rho((x, y), (x, 0)) = d_Y(0, y) < \epsilon$.

**Case III.** $d_X(x, a) \geq \delta(z) + \delta_3$

Since $\rho_H(\phi(z), \phi(z')) < \min\{\epsilon, \delta(z)\delta_1/2, \delta_3/4\}$, there exists a point $(x_1, y_1) \in \phi(z')$ such that

$$\rho((x, \max \phi(z)(x)), (x_1, y_1)) < \min\{\epsilon, \delta(z)\delta_1/2, \delta_3/4\}.$$

Then we have

$$d_X(x, x_1) \leq \rho((x, \max \phi(z)(x)), (x_1, y_1)) < \min\{\epsilon, \delta(z)\delta_1/2, \delta_3/4\}.$$  

Moreover, $|\delta(z) - \delta(z')| < \delta_3/8$, and hence

$$d_X(x_1, a) \geq d_X(x, a) - d_X(x, x_1) > \delta(z) + \delta_3 - \delta_3/4 > \delta(z') - \delta_3/8 + \delta_3 - \delta_3/4 > \delta(z').$$

If $d_X(x_1, a) \geq 2\delta(z')$, we get $(x_1, y_1) \in \psi(z')$. Since $y \in [0, \max \phi(z)(x)]$, by Lemma 2.2, we can find $y_2 \in [0, y_1]$ such that $d_Y(y, y_2) \leq d_Y(\max \phi(z)(x), y_1) < \epsilon$. It follows that $(x_1, y_2) \in \psi(z')$ and

$$\rho((x, y), (x_1, y_2)) = \max\{d_X(x, x_1), d_Y(y, y_2)\} < \epsilon.$$  

Now, we need only to consider the case that $\delta(z') < d_X(x_1, a) < 2\delta(z')$. Let $y_3 = r(y_1, d_X(x_1, a)/\delta(z') - 1)$.

Then $y_3 \in [0, r(\max \phi(z')(x_1), d_X(x_1, a)/\delta(z') - 1)]$, so $(x_1, y_3) \in \psi(z')$.  

Case III-i. \( \delta(z) + \delta_3 \leq d_X(x, a) < 2\delta(z) \)

Then we have

\[
|d_X(x, a)/\delta(z) - (d_X(x_1, a)/\delta(z') - 1)| \leq |1/\delta(z) - 1/\delta(z')|d_X(x_1, a) + |d_X(x, a) - d_X(x_1, a)|/\delta(z) \\
\leq |1/\delta(z) - 1/\delta(z')|(d_X(x, x_1) + d_X(x, a)) + d_X(x, x_1)/\delta(z) \\
\leq 2\delta_1(\delta(z)/4 + 2\delta(z))/98(z) + \delta(z)\delta_1/2\delta(z) \\
= \delta_1/2 + \delta_1/2 = \delta_1.
\]

On the other hand, we get

\[
d_Y(\max \phi(z)(x), y_1) \leq \rho((x, \max \phi(z)(x)), (x_1, y_1)) < \delta(z)\delta_1/2 < \delta_1.
\]

It follows that

\[
d_Y(r(\max \phi(z)(x), d_X(x, a)/\delta(z) - 1), y_3) = d_Y(r(\max \phi(z)(x), d_X(x, a)/\delta(z) - 1), r(y_1, d_X(x_1, a)/\delta(z') - 1)) < \epsilon.
\]

Using Lemma 2.2, we can choose \( y_4 \in [0, y_3) \) so that

\[
d_Y(y, y_4) \leq d_Y(r(\max \phi(z)(x), d_X(x, a)/\delta(z) - 1), y_3) < \epsilon.
\]

Then \( (x_1, y_4) \in \psi(z') \) and \( \rho((x, y), (x_1, y_4)) = \max\{d_X(x, x_1), d_Y(y, y_4)\} < \epsilon.

Case III-ii. \( 2\delta(z) \leq d_X(x, a) < 2\delta(z) + \delta_3/2 \)

It follows that

\[
|\delta(z') - d_X(x_1, a)| \leq |2\delta(z') - 2\delta(z)| + |2\delta(z) - d_X(x, a)| + |d_X(x, a) - d_X(x_1, a)| \\
\leq \delta_3/4 + \delta_3/2 + \delta_3/4 = \delta_3.
\]

Therefore we have

\[
|1 - (d_X(x_1, a)/\delta(z') - 1)| = |2 - d_X(x_1, a)/\delta(z')| < 2\delta_2 < \delta_1.
\]

Observe that

\[
d_Y(\max \psi(z)(x), y_3) = d_Y(\max \phi(z)(x), y_3) = d_Y(r(\max \phi(z)(x), 1), r(y_1, d_X(x_1, a)/\delta(z') - 1)) < \epsilon.
\]

Due to Lemma 2.2, there exists \( y_5 \in [0, y_3) \) such that \( d_Y(y, y_5) \leq d_Y(\max \psi(z)(x), y_3) < \epsilon. \) Then \( (x_1, y_5) \in \psi(z') \) and \( \rho((x, y), (x_1, y_5)) = \max\{d_X(x, x_1), d_Y(y, y_5)\} < \epsilon.

Case III-iii. \( d_X(x, a) \geq 2\delta(z) + \delta_3/2 \)

Note that

\[
d_X(x_1, a) \geq d_X(x, a) - d_X(x, x_1) \geq 2\delta(z) + \delta_3/2 - \delta_3/4 = 2\delta(z') + \delta_3/4 + \delta_3/2 - \delta_3/4 = 2\delta(z'),
\]

which is a contradiction.

Consequently, \( (x, y) \in N(\psi(z'), \epsilon). \) Similarly, \( \psi(z') \subset N(\psi(z), \epsilon). \) Thus \( \rho_H(\psi(z), \psi(z')) < \epsilon, \) and hence \( \psi \) is continuous.

\[\Box\]

7 The disjoint cells property of \( \Downarrow C(X, Y) \)

A space \( Z \) has the disjoint cells property provided that for any maps \( f, g : Q \to Z \) of the Hilbert cube and open cover \( \mathcal{U} \) of \( Z \), there exist maps \( f', g' : Q \to Z \) such that \( f' \) and \( g' \) are \( \mathcal{U} \)-close to \( f \) and \( g \), respectively, and \( f'(Q) \cap g'(Q) = \emptyset \).

**Proposition 7.1.** The space \( \Downarrow C(X, Y) \) has the disjoint cells property.
Lemma 8.1. Let $\mathcal{C}(X, Y)$ be maps and $0 < \epsilon < \diam Y$. Since $\mathcal{C}(X, Y)$ is homotopy dense in $\mathcal{C}(X, Y)$ by Theorem 4.1, we can obtain maps $f' : Q \rightarrow \mathcal{C}(X, Y)$ that is $\epsilon$-close to $f$, and $g' : Q \rightarrow \mathcal{C}(X, Y)$ that is $\epsilon/3$-close to $g$. Take a non-isolated point $x_0 \in X$. Using the Digging Lemma 6.1, we can find a map $g'' : Q \rightarrow \mathcal{C}(X, Y)$ such that $g''$ is $\epsilon/3$-close to $g'$ and $g''(x_0) = \emptyset$ for all $z \in Q$. Define a map $g''' : Q \rightarrow \mathcal{C}(X, Y)$ as follows:

$$g'''(z) = \tau(g''(z), 1 - \epsilon/(3 \diam Y)) \cup \{x_0\} \times \overline{B}(0, \epsilon/3).$$

Then $\rho_H(g(z), g'''(z)) < \epsilon$ and $g'''(z) \notin \mathcal{C}(X, Y)$ for each $z \in Q$. Indeed, since

$$\tau(g''(z), 1 - \epsilon/(3 \diam Y)) \subset g'''(z) \subset g''(z) \cup \{x_0\} \times \overline{B}(0, \epsilon/3),$$

it follows that

$$\rho_H(g''(z), g'''(z)) \leq \rho_H(g''(z) \cup \{x_0\} \times \overline{B}(0, \epsilon/3), \tau(g''(z), 1 - \epsilon/(3 \diam Y)))$$

$$\leq \sup\{d_Y(y, r(y, 1 - \epsilon/(3 \diam Y))), \epsilon/3 \mid y \in Y\} \leq \epsilon/3.$$

Therefore $g'''$ is $\epsilon/3$-close to $g''$, so it is $\epsilon$-close to $g$. Moreover, we have

$$\tau(g''(z), 1 - \epsilon/(3 \diam Y))(x_0) = r(g''(z)(x_0) \times \{1 - \epsilon/(3 \diam Y)\}) \subset g''(z)(x_0) \cup \overline{B}(0, \epsilon/3) = g'''(z)(x_0).$$

Take $y \in g'''(z)(x_0) \setminus \tau(g''(z), 1 - \epsilon/(3 \diam Y))(x_0)$, so we can choose $\delta > 0$ so that

$$B((x_0, y), \delta) \cap \tau(g''(z), 1 - \epsilon/(3 \diam Y)) = \emptyset,$$

which implies that $g'''(z)$ is not the hypo-graph of any map because $x_0$ is a non-isolated point. Hence $g'''(z) \notin \mathcal{C}(X, Y)$. Consequently, $f'(Q) \cap g'''(Q) = \emptyset$. Thus $\mathcal{C}(X, Y)$ has the disjoint cells property.

Combining Theorem 3.1, Proposition 7.1, and Toruńczyk’s characterization of the Hilbert cube [12], see Corollary 7.8 of [6], we can immediately obtain the following:

Corollary 7.2. The space $\mathcal{C}(X, Y)$ is homeomorphic to the Hilbert cube $Q$.

Due to Proposition 5.2, $\mathcal{C}(X, Y)$ is an $F_{\sigma\delta}$-set in $\mathcal{C}(X, Y)$ in the above. Hence we conclude as follows:

Corollary 7.3. The space $\mathcal{C}(X, Y)$ is an absolute $F_{\sigma\delta}$-set.

8 Detecting a $Z_\sigma$-set in $\mathcal{C}(X, Y)$ containing $\mathcal{C}(X, Y)$

A closed subset $Z$ of $W$ is said to be a $Z$-set in $W$ if for each open cover $\mathcal{U}$ of $W$, there exists a map $f : W \rightarrow W$ such that $f$ is $\mathcal{U}$-close to the identity $id_W$ and $f(W) \cap Z = \emptyset$. A countable union of $Z$-sets in $W$ is called a $Z_\sigma$-set.

In addition, a $Z$-embedding is an embedding whose image is a $Z$-set in the target space. We can easily prove the following:

Lemma 8.1. Let $Z$ be a $Z$-set in a homotopy dense subset $M$ of $N$. Then the closure $\overline{Z}$ of $Z$ in $N$ is a $Z$-set in $N$.

The next lemma is very useful for detecting $Z$-sets in $\mathcal{C}(X, Y)$.

Lemma 8.2. Suppose that $F = E \cup Z$ is a closed set in $\mathcal{C}(X, Y)$ such that $Z$ is a $Z$-set in $\mathcal{C}(X, Y)$, and for each $A \in E$, there exists a point $a \in X$ with $A(a) = \{0\}$. Then $F$ is a $Z$-set in $\mathcal{C}(X, Y)$.

---

8 Lemma 8.2 holds without the assumption that $X$ has no isolated points.
Proof. Let \( \epsilon : \overline{C(X,Y)} \to (0,1) \). It suffices to construct a map \( \phi : \overline{C(X,Y)} \to \overline{C(X,Y)} \) such that 
\[ \phi(\overline{C(X,Y)}) \cap F = \emptyset \] and 
\[ \rho_H(\phi(A),A) < \epsilon(A) \] for each \( A \in \overline{C(X,Y)} \). Since \( Z \) is a \( Z \)-set, there exists a map \( \psi : \overline{C(X,Y)} \to \overline{C(X,Y)} \setminus Z \) such that \( \rho_H(\psi(A),A) < \epsilon(A)/2 \) for each \( A \in \overline{C(X,Y)} \). Fix a point \( y_0 \in Y \setminus \{0\} \). We define a map \( \phi : \overline{C(X,Y)} \to \overline{C(X,Y)} \) by

\[ \phi(A) = \psi(A) \cup \tau([0,y_0],t(A)), \]

where \( t(A) = \min\{\epsilon(A), \text{dist}(\psi(A),Z)/2(\text{diam} Y) > 0 \). Obviously, \( \phi(A)(x) \neq \{0\} \) for each \( x \in X \), that is, \( \phi(A) \neq E \). Observe that

\[ \rho_H(\phi(A), \psi(A)) \leq t(A)d_Y(0,y_0) \leq t(A) \text{diam} Y \leq \text{min}\{\epsilon(A), \text{dist}(\psi(A), Z)/2\}. \]

Hence \( \phi(A) \neq Z \) and

\[ \rho_H(\phi(A), A) \leq \rho_H(\phi(A), \psi(A)) + \rho_H(\psi(A), A) < \epsilon(A)/2 + \epsilon(A)/2 = \epsilon(A). \]

The continuity of \( \phi \) follows from the ones of \( \tau, \psi \) and \( t \), and Lemma 2.1. This completes the proof. \( \square \)

**Proposition 8.3.** The space \( \overline{C(X,Y)} \) is contained in some \( Z_\sigma \)-set in \( \overline{C(X,Y)} \).

**Proof.** Take a countable dense set \( D = \{d_n \mid n \in \mathbb{N}\} \) in \( X \). For each \( n,m \in \mathbb{N} \), let

\[ F_{n,m} = \{f \in \overline{C(X,Y)} \mid d_Y(f(d_n),0) \geq 1/m\}. \]

As is easily observed, \( F_{n,m} \) is closed in \( \overline{C(X,Y)} \). For each map \( \epsilon : \overline{C(X,Y)} \to (0,1) \), by the Digging Lemma 6.1, we have \( \phi : \overline{C(X,Y)} \to \overline{C(X,Y)} \) such that \( \rho_H(\phi(f), f) < \epsilon(f) \) and \( \phi(f)(d_n) = \{0\} \) for \( f \in \overline{C(X,Y)} \).

Obviously, \( \phi(\overline{C(X,Y)}) \cap F_{n,m} = \emptyset \). Thus each \( F_{n,m} \) is a \( Z \)-set in \( \overline{C(X,Y)} \). It follows from Theorem 4.1 and Lemma 8.1 that the closure \( \overline{F_{n,m}} \) is a \( Z \)-set in \( \overline{C(X,Y)} \).

Let \( F = \bigcap_{n \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} (\overline{C(X,Y)} \setminus F_{n,m}) \). It remains to prove that the closure \( \overline{F} \) of \( F \) in \( \overline{C(X,Y)} \) is a \( Z \)-set. Observe that

\[ F = \{f \in \overline{C(X,Y)} \mid f(d_n) = \{0\} \text{ for each } n \in \mathbb{N}\} = \{0\}, \]

where \( 0 : X \to \{0\} \subseteq Y \) is the constant map. Hence \( \overline{F} = \{0\} = \{X \times \{0\}\} \). According to Lemma 8.2, \( \overline{F} \) is a \( Z \)-set in \( \overline{C(X,Y)} \). Consequently, \( \overline{C(X,Y)} \) is contained in the \( Z_\sigma \)-set \( \overline{F} \cup \bigcup_{m,n \in \mathbb{N}} \overline{F_{n,m}} \). \( \square \)

### 9 The strong \((\mathcal{M}_0, \mathcal{F}_\sigma)\)-universality of \((\overline{C(X,Y)}, \overline{C(X,Y)})\)

In this section, we shall prove the main theorem. Let \((X_1, X_2)\) be a pair of spaces with \(X_2 \subseteq X_1\) and \((C_1, C_2)\) be a pair of classes. The notation \((X_1, X_2) \in (C_1, C_2)\) means that \(X_1 \in C_1\) and \(X_2 \in C_2\), respectively. We say that \((X_1, X_2)\) is strongly \((C_1, C_2)\)-universal if the following condition holds:

**su** Let \((Z_1, Z_2) \in (C_1, C_2)\) with \(Z_2 \subseteq Z_1\), \(K\) be a closed subset of \(Z_1\), and \(f : Z_1 \to X_1\) be a map such that \(f|_K\) is a \( Z \)-embedding. Then for every open cover \(U\) of \(X_1\), there exists a \( Z \)-embedding \(g : Z_1 \to X_1\) such that \(g\) is \(U\)-close to \(f\), \(g|_K = f|_K\) and \(g^{-1}(X_2) \setminus K = Z_2 \setminus K\).

A pair \((X_1, X_2)\) with \(X_2 \subseteq X_1\) is \((C_1, C_2)\)-absorbing \footnote{We modify the definition of \([1]\) for this paper.} provided that the following conditions are satisfied:

(i) \((X_1, X_2) \in (C_1, C_2)\),
(ii) \(X_2\) is contained in a \(Z_\sigma\)-set in \(X_1\),
(iii) \((X_1, X_2)\) is strongly \((C_1, C_2)\)-universal.
Denote the class of compact metrizable spaces by \( M_0 \), and the one of separable metrizable absolute \( F_{\alpha \delta} \)-spaces by \( F_{\alpha \delta} \). According to Theorem 1.7.6 of [1], the following can be established.

**Theorem 9.1.** Let \( X_1 \) and \( Z_1 \) be topological copies of the Hilbert cube \( Q \). If pairs \((X_1, X_2)\) and \((Z_1, Z_2)\) are \((M_0, F_{\alpha \delta})\)-absorbing, then there exists a homeomorphism \( f : X_1 \to Z_1 \) such that \( f(X_2) = Z_2 \).

Let \( c_1 = \{(x_i)_{i \in \mathbb{N}} \in Q \mid \lim_{i \to \infty} x_i = 1\} \). The following fact is well known.

**Fact 9.2.** The pairs \((Q, c_0)\) and \((Q, c_1)\) are \((M_0, F_{\alpha \delta})\)-absorbing, and hence \((Q, c_0)\) is homeomorphic to \((Q, c_1)\).

We need the following lemma to verify the strong \((M_0, F_{\alpha \delta})\)-universality of \((C(X,Y), \downarrow C(X,Y))\).

**Lemma 9.3.** Let \( x_m, x_\infty \in X, \ m \in \mathbb{N} \), such that \( \{r_m = d_X(x_m, x_\infty)\}_{m \in \mathbb{N}} \) is a strictly decreasing sequence converging to 0, and let \( y_0 \in Y \setminus \{0\} \) such that \( d_Y(0, y_0) \leq 1 \). Suppose that \( g : Z \to Q \) is a map from a space \( Z \) to the Hilbert cube \( Q \) and \( \delta : Z \to (0,1) \) is a map. Then there exists a map \( \Phi : Z \to C(X, [0, y_0]) \) satisfying the following conditions:

1. \( \Phi \) is injective,
2. \( \rho_H(\Phi(z)X, \{0\}) \leq \delta(z) \) for all \( z \in Z \),
3. \( \Phi(z)(X \setminus B(x, r_2k)) = \{0\} \) for all \( z \in Z \) with \( 2^{-k} \leq \delta(z) \leq 2^{-k+1}, k \in \mathbb{N} \),
4. \( z \in g^{-1}(c_1) \) if and only if \( \Phi(z) \in \{C(X, [0, y_0]) \),
5. \( \Phi(z)(x_\infty) = [0, r(y_0, \delta(z))] \) for all \( z \in Z \).

**Proof.** For each \( k, m \in \mathbb{N} \), let \( Z_k = \{z \in Z \mid 2^{-k} \leq \delta(z) \leq 2^{-k+1}\} \) and \( S_m = \{x \in X \mid r_m \leq d_X(x, x_\infty) \leq r_{m-1}\} \), where \( r_0 = \text{diam } X \). Note that \( Z = \bigcup_{k \in \mathbb{N}} Z_k \), \( S_m = X \setminus \{x_\infty\} \), and \( S_m \cap S_m' \neq \emptyset \) if and only if \( |m - m'| \leq 1 \). We define maps \( \phi_k : Z_k \to I \) and \( \psi_m : S_m \to I \) for each \( k, m \in \mathbb{N} \) by \( \phi_k(z) = 2^{-k} \delta(z) \) and \( \psi_m(x) = d_X(x, x_\infty) - r_m)/(r_{m-1} - r_m) \). Then \( \psi_m(x_{m-1}) = 1 \) and \( \psi_m(x_m) = 0, m \geq 2 \). For each \( i, k \in \mathbb{N} \), let \( f_k^i : Z_k \to I \) be a map defined by

\[
f_k^i(z) = \begin{cases} 0 & \text{if } i = 1, \\
(1 - \phi_k(z))\delta(z) & \text{if } i = 2, \\
(1 - \phi_k(z))\delta(z)g(z)(1) & \text{if } i = 3, \\
\delta(z)(1 - \phi_k(z))g(z)((i - 1)/2) + \phi_k(z)g(z)((i - 3)/2) & \text{if } i = 2j + 1, j \geq 2.
\end{cases}
\]

Remark that \( f_k^i(z) \leq \delta(z) \) for every \( z \in Z \). We define a map \( \Phi_k : Z_k \to C(X, [0, y_0]) \), \( k \in \mathbb{N} \), as follows:

\[
\Phi_k(z) = \{x \in X \mid d_X(x, x_\infty) \geq r_{2k}\} \times \{0\} \cup \{x_\infty\} \times [0, r(y_0, \delta(z))] \\
\cup \bigcup_{i \in \mathbb{N}} \{(x, y) \in X \times Y \mid x \in S_{2k+i}, y \in [0, r(y_0, a_k^i(x, z))]\},
\]

where \( a_k^i(x, z) = \psi_{2k+i}(x)f_k^i(z) + (1 - \psi_{2k+i}(x))f_{k+1}^i(z) \). Then \( \Phi_k(z) = \Phi_{k+1}(z) \) for every \( z \in Z_k \cap Z_{k+1} \).

Indeed, take any \( z \in Z_k \cap Z_{k+1} \). Since \( \delta(z) = 2^{-k} \), we have \( \phi_k(z) = 1 \) and \( \phi_{k+1}(z) = 0 \). Observe that

\[
a_k^i(x, z) = \psi_{2k+1}(x)f_k^i(z) + (1 - \psi_{2k+1}(x))f_{k+1}^i(z) = 0 \quad \text{and} \\
a_k^i(x, z) = \psi_{2k+2}(x)f_k^i(z) + (1 - \psi_{2k+2}(x))f_{k+1}^i(z) = 0.
\]

It follows that

\[
\Phi_k(z)(\{x \in X \mid d_X(x, x_\infty) \geq r_{2k+2}\}) = \{0\} = \Phi_{k+1}(z)(\{x \in X \mid d_X(x, x_\infty) \geq r_{2k+2}\}).
\]

\[10 \text{ Lemma 9.3 holds without the assumption that } X \text{ has no isolated points.} \]
We see $f_2^k(z) = 0 = f_1^{k+1}(z)$, $f_2^{k+1}(z) = \delta(z)g(z)(j) = f_2^{k+1}(z)$ and $f_2^{k+2}(z) = \delta(z) = f_2^{k+1}(z)$ for all $j \geq 1$, that is, $f_i^{k+1}(z)$ for all $i \geq 1$. Therefore for each $x \in S_{2k+i+2}$, $i \geq 1$,

$$\Phi_k(z)(x) = [0, r(y_0, \alpha_i^{k+1}(x, z))] = [0, r(y_0, \alpha_i^{k+1}(x, z))] = \Phi_{k+1}(z)(x).$$

Moreover, $\Phi_k(z)(x) = [0, r(y_0, \delta(z))] = \Phi_{k+1}(z)(x)$. Thus $\Phi_k(z) = \Phi_{k+1}(z)$.

Now, we can obtain the desired map $\Phi : Z \rightarrow \text{C}(X, [0, y_0])$ defined by $\Phi(z) = \Phi_{k}(z)$ if $z \in Z_k$. It follows from the definition that $\Phi$ satisfies conditions (2), (3) and (5). So it remains to verify that conditions (1) and (4) hold.

**Condition (1)** $\Phi$ is injective.

Take $z_1, z_2 \in Z$ with $\Phi(z_1) = \Phi(z_2)$. Then

$$[0, r(y_0, \delta(z_1))] = \Phi(z_1)(x) = \Phi(z_2)(x) = [0, r(y_0, \delta(z_2))],$$

which implies that $\delta(z_1) = \delta(z_2)$. Hence both of $z_1$ and $z_2$ are contained in $Z_k$ for some $k \in \mathbb{N}$ and

$$\phi_k(z_1) = 2 - 2^k \delta(z_1) = 2 - 2^k \delta(z_1) = \phi_k(z_2).$$

Since $\psi_{2k+i}(z_2) = 0$ for all $i \in \mathbb{N}$, we have

$$[0, r(y_0, f_i^{k+1}(z_1))] = \Phi_k(z_1)(x) = \Phi_k(z_2)(x) = [0, r(y_0, f_i^{k+1}(z_2))],$$

which implies that $f_i^k(z_1) = f_i^k(z_2)$ for every $j \geq 2$. In the case $\phi_k(z_1) = 1$, for each $j \in \mathbb{N}$, we have

$$g(z_1)(j) = f_{2j+3}^k(z_1) = f_{2j+3}^k(z_2) = g(z_2)(j).$$

In the case $\phi_k(z_1) \neq 1$, we have

$$(1 - \phi_k(z_1))\delta(z_1)g(z_1)(1) = f_2^k(z_1) = f_2^k(z_2) = (1 - \phi_k(z_2))\delta(z_2)g(z_2)(1),$$

which implies that $g(z_1)(1) = g(z_2)(1)$. Assume that $g(z_1)(i) = g(z_2)(i)$ for some $i \in \mathbb{N}$. Then

$$\delta(z_1)((1 - \phi_k(z_1))g(z_1)(i) + \phi_k(z_1)g(z_1)(i)) = f_{2i+3}^k(z_1) = f_{2i+3}^k(z_2) = \delta(z_2)((1 - \phi_k(z_2))g(z_2)(i) + \phi_k(z_2)g(z_2)(i)).$$

so $g(z_1)(i + 1) = g(z_2)(i + 1)$. By induction, for all $j \in \mathbb{N}$, we get $g(z_1)(j) = g(z_2)(j)$. It follows that $g(z_1) = g(z_2)$. Since $g$ is injective, $z_1 = z_2$. Therefore $\Phi$ is injective.

**Condition (4)** $z \in g^{-1}(c_1)$ if and only if $\Phi(z) \in \text{C}(X, [0, y_0])$.

We define a function $h(z) : X \rightarrow [0, y_0] \subset Y$ for each $z \in Z_k$ and $k \in \mathbb{N}$ as follows:

$$h(z)(x) = \begin{cases} 0 & \text{if } d_X(x, x_\infty) \geq r_{2k}, \\ r(y_0, \alpha_i^k(x, z)) & \text{if } x \in S_{2k+i}, i \in \mathbb{N}, \\ r(y_0, \delta(z)) & \text{if } x = x_\infty. \end{cases}$$

Observe that $h(z) = \Phi(z)$ and $h(z)$ is continuous on $X \setminus \{x_\infty\}$. When $h(z)$ is continuous at the point $x_\infty$, $\Phi(z) = [h(z) \in \text{C}(X, [0, y_0])$. We need only to show that $z \in g^{-1}(c_1)$ if and only if $h(z)$ is continuous at $x_\infty$.

First, we shall prove the only if part. Take any $\epsilon > 0$. We may assume that $\epsilon < \delta(z)$. Since $g(z) \in c_1$, there exists $i_0 \in \mathbb{N}$ such that for every $i \geq i_0$, $g(z)(i) > 1 - \epsilon/\delta(z)$. Fix any point $x \neq x_\infty$ in the neighborhood $\{x_\infty\} \cup \bigcup_{i \geq i_0 + 3} S_{2k+i}$ of $x_\infty$ in $X$, where $z \in Z_k$. Then $x \in S_{2k+i}$ for some $i \geq 2i_0 + 3$. When $i$ is even, $f_i^k(z) = \delta(z)$. When $i$ is odd,

$$f_i^k(z) = \delta(z)((1 - \phi_k(z))g(z)(i - 1)/2 + \phi_k(z)g(z)(i - 2)/2) = \delta(z)((1 - \phi_k(z))(1 - \epsilon/\delta(z)) + \phi_k(z)(1 - \epsilon/\delta(z))) > \delta(z) - \epsilon.$$

Hence we have

$$\alpha_i^k(x, z) = \psi_{2k+i}(x)f_i^k(z) + (1 - \psi_{2k+i}(x))f_{i+1}^k(z)$$
> \psi_{2k+i}(x)\delta(z) - \epsilon + (1 - \psi_{2k+i}(x))\delta(z) = \delta(z) - \epsilon.

It follows that
\[
d_Y(h(x_\infty), h(x)) = d_Y(r(y_0, \delta(z)), r(y_0, \alpha_i^k(z))) = (\delta(z) - \alpha_i^k(z))d_Y(0, y_0)
< \delta(z) - (\delta(z) - \epsilon) = \epsilon.
\]

Consequently, \( h(z) \) is continuous.

Next, we shall show the if part. Let \( \epsilon \in (0, 1) \) and \( \epsilon' = \epsilon \phi_k(z)\delta(z) \), where \( z \in Z_k \) with \( \phi_k(z) > 0 \). Since \( h(z) \) is continuous at \( x_\infty \), we can choose \( i_0 \geq 5 \) so that for any \( x \in X \),
\[
d_X(x, x_\infty) \leq r_{2k+i_0} \Rightarrow d_Y(h(x), h(x_\infty)) < \epsilon' d_Y(0, y_0).
\]

Recall that \( \psi_m(x_m) = 0 \) for all \( m \in \mathbb{N} \). Therefore for every \( i \geq i_0 \),
\[
d_Y(r(y_0, f_{i+1}^k(z)), r(y_0, \delta(z))) = d_Y(r(y_0, \psi_{2k+i}(x_{2k+i})f_i^k(z) + (1 - \psi_{2k+i}(x_{2k+i}))f_i^k(z)), r(y_0, \delta(z))) = d_Y(h(z)(x_{2k+i}), h(z)(x_\infty)) < \epsilon' d_Y(0, y_0).
\]

Note that for all \( i \geq i_0 + 1 \),
\[
\delta(z) - f_i^k(z) = d_Y(r(y_0, f_i^k(z)), r(y_0, \delta(z)))/d_Y(0, y_0) < \epsilon'.
\]

It follows that for any \( j \geq (i_0 + 2)/2 \),
\[
g(z)(j) = (f_{j+3}^k(z)/\delta(z) - (1 - \phi_k(z))g(z)(j + 1)/\phi_k(z) \geq (f_{j+3}^k(z)/\delta(z) - (1 - \phi_k(z))/\phi_k(z)
> ((\delta(z) - \epsilon')/\delta(z) - (1 - \phi_k(z)))/\phi_k(z) = ((\delta(z) - \epsilon \phi_k(z)\delta(z))/\delta(z) - (1 - \phi_k(z))/\phi_k(z)
= 1 - \epsilon.
\]

Hence \( g(z) \in c_1 \). Thus the proof is complete. 

\[ \square \]

Proposition 9.4. The pair \( (\overline{C(X, Y)}, \overline{C(Y, X)}) \) is strongly \((\Theta_0, \mathcal{F}_{\sigma_\delta})\)-universal.

Proof. Let \((Z, C) \in (\Theta_0, \mathcal{F}_{\sigma_\delta})\), \( K \) be a closed subset of \( Z \), \( \epsilon > 0 \) and \( \Phi : Z \to \overline{C(X, Y)} \) be a map such that the restriction \( \Phi|_K \) is a \( Z \)-embedding. We shall construct a \( Z \)-embedding \( \Psi : Z \to \overline{C(X, Y)} \) so that \( \Psi \) is \( \epsilon \)-close to \( \Phi \). Let \( \Psi^{-1}(\overline{C(X, Y)} \setminus K) = C \setminus K \). Since \( \Phi(K) \) is a \( Z \)-set in \( \overline{C(X, Y)} \), we may assume that \( \Phi(K) \cap \Phi(Z \setminus K) = \emptyset \). Indeed, since \( \overline{C(X, Y)} \setminus \Phi(K) \) is homotopy dense in \( \overline{C(X, Y)} \) [Theorem 1.4.4], there is a homotopy \( F : \overline{C(X, Y)} \times I \to \overline{C(X, Y)} \) such that \( F_0 = \text{id}_{\overline{C(X, Y)}} \) and \( F_t(\overline{C(X, Y)}) \subset \overline{C(X, Y)} \setminus \Phi(K) \) for every \( t \in (0, 1] \), then for each \( \lambda > 0 \), we can choose \( t \in (0, 1] \) so that \( F_\lambda \) is \( \lambda \)-close to \( F_0 = \text{id}_{\overline{C(X, Y)}} \) for all \( s \in [0, t] \). Taking a map \( \alpha : Z \to I \) such that \( \alpha^{-1}(0) = K \) and defining \( \Phi' : Z \to \overline{C(X, Y)} \) by \( \Phi'(z) = F(\Phi(z), \alpha(z)) \), we have \( \Phi' \) is \( \lambda \)-close to \( \Phi \), \( \Phi'|_K = \Phi|_K \) and \( \Phi'(Z \setminus K) \subset \overline{C(X, Y)} \setminus \Phi(K) \). Replace \( \Phi \) with \( \Phi' \), so \( \Phi(K) \cap \Phi(Z \setminus K) = \emptyset \).

Define a map \( \delta : Z \to [0, 1] \) by \( \delta(z) = \min(\epsilon, \text{dist}(\Phi(z), \Phi(K)))/4 \). Observe that \( \delta(z) = 0 \) if and only if \( z \in K \). Since \( \overline{C(X, Y)} \) is homotopy dense in \( \overline{C(X, Y)} \) by Theorem 4.1, there exists a homotopy \( H : [C(X, Y)] \times I \to \overline{C(X, Y)} \) such that \( H_0 = \text{id}_{\overline{C(X, Y)}} \) and \( H_t([C(X, Y)]) \subset C(X, Y) \) for all \( t \in (0, 1] \) and \( \rho_H(H_t([A], t) \leq t \) for each \( [A] \in [C(X, Y)] \) and \( t \in I \). Let \( h : Z \to \overline{C(X, Y)} \) be a map defined by \( h(z) = H(\Phi(z), \delta(z)) \). 
Remark that \( \rho_H(h(z), \Phi(z)) = \rho_H(H(\Phi(z), \delta(z)), \Phi(z)) \leq \delta(z) \) for every \( z \in Z \), in particular, \( h(z) = \Phi(z) \) for all \( z \in K \), and \( h(Z \setminus K) \subset C(X, Y) \). Take a non-isolated point \( x_\infty \in X \). According to the Digging Lemma 6.1, we can obtain maps \( \psi : Z \setminus K \to \overline{C(X, Y)} \) and \( s : Z \setminus K \to (0, 1) \) so that for each \( z \in Z \setminus K \),
(a) \( \rho_H(h(z), \psi(z)) \leq \delta(z) \),
(b) \( \psi(z)(B(x_\infty, s(z))) = \{0\} \).
Let $Z_k = \{z \in Z \mid 2^{-k} \leq \delta(z) \leq 2^{-k+1}\} \subset Z \setminus K$ for each $k \in \mathbb{N}$. Then each $Z_k$ is compact and $Z \setminus K = \bigcup_{k \in \mathbb{N}} Z_k$. Since $x_\infty$ is a non-isolated point, there exists a point $x_1 \in X \setminus \{x_\infty\}$ such that $d_X(x_1, x_\infty) < \min\{1, s(z) \mid z \in Z_1\}$. By induction, we can choose $x_m \in X \setminus \{x_\infty\}$ for each $m \geq 2$ so that $d_X(x_m, x_\infty) < \min\{1/m, d_X(x_{m-1}, x_\infty), s(z) \mid z \in Z_m\}$. Let $s_m = d_X(x_m, x_\infty)$ for each $m \in \mathbb{N}$, so $s_m$ converges to 0 as $m$ tends to infinity. Note that for every $z \in Z_k$ and $k \in \mathbb{N}$, $\psi'(z)(B(x_\infty, s_k)) = \{0\}$. Since the pair $(Q, c_1)$ is strongly $(\mathcal{M}_0, \mathcal{F}_0)$-universal due to Fact 9.2, we can take an embedding $g : Z \to Q$ so that $g^{-1}(c_1) = C$. Choose $y_0 \in Y \setminus \{0\}$ with $d_Y(0, y_0) \leq 1$.

Using Lemma 9.3, we can obtain a map $\psi' : Z \setminus K \to \overline{\mathcal{C}(X, [0, y_0])}$ satisfying the following conditions:

1. $\psi'$ is injective,
2. $\rho_H(\psi'(z), X \times \{0\}) \leq \delta(z)$ for all $z \in Z \setminus K$,
3. $\psi'(z)(X \setminus B(x_\infty, s_{2k})) = \{0\}$ for all $z \in Z_k, k \in \mathbb{N}$,
4. $z \in C \setminus K$ if and only if $\psi'(z) \in \overline{\mathcal{C}(X, [0, y_0])}$,
5. $\psi'(z)(x_\infty) = [0, r(y_0, \delta(z))]$ for all $z \in Z \setminus K$.

Define $\psi'' : Z \setminus K \to \overline{\mathcal{C}(X, Y)}$ by $\psi''(z) = \psi(z) \cup \psi'(z)$. The continuity of $\psi''$ follows from the ones of $\psi$ and $\psi'$, and Lemma 2.1. By conditions (a) and (2), and Lemma 2.1, for each $z \in Z \setminus K$,

$$\rho_H(h(z), \psi''(z)) = \rho_H(h(z) \cup X \times \{0\}, \psi(z) \cup \psi'(z)) \leq \max\{\rho_H(h(z), \psi(z)), \rho_H(X \times \{0\}, \psi'(z))\} \leq \delta(z).$$

According to conditions (b), (3) and (4), we have $z \in C \setminus K$ if and only if $\psi''(z) \in \mathcal{C}(X, Y)$. Moreover, $\psi''$ is injective. Indeed, take any $z_1, z_2 \in Z \setminus K$ with $\psi''(z_1) = \psi''(z_2)$. Then there exist $k_1, k_2 \in \mathbb{N}$ such that $z_1 \in Z_{k_1}$ and $z_2 \in Z_{k_2}$, respectively. It follows from (b) and (5) that

$$[0, r(y_0, \delta(z_1))] = \psi'(z_1)(x_\infty) = \psi''(z_1)(x_\infty) = \psi''(z_2)(x_\infty) = \psi'(z_2)(x_\infty) = [0, r(y_0, \delta(z_2))],$$

which implies that $\delta(z_1) = \delta(z_2)$. Hence $z_1, z_2 \in Z_{k}$, where $k = k_1 = k_2$. Since $\psi(z_1)(B(x_\infty, s_k)) = \{0\} = \psi(z_2)(B(x_\infty, s_k))$ by (b), we have

$$\psi'(z_1)(x) = \psi''(z_1)(x) = \psi'(z_2)(x) = \psi'(z_2)(x) \text{ for every } x \in B(x_\infty, s_{2k}).$$

On the other hand, by (3), $\psi'(z_1)(X \setminus B(x_\infty, s_{2k})) = \{0\} = \psi'(z_2)(X \setminus B(x_\infty, s_{2k}))$. Therefore $\psi'(z_1) = \psi'(z_2)$. Due to (1), we get $z_1 = z_2$, so $\psi''$ is injective.

We can extend $\psi''$ to the desired map $\Psi : Z \to \overline{\mathcal{C}(X, Y)}$ by $\Psi|_K = \Phi|_K$. Then for each $z \in Z$,

$$\rho_H(\Phi(z), \Psi(z)) \leq \rho_H(\Phi(z), h(z)) + \rho_H(h(z), \Psi(z)) \leq 2\delta(z) \leq \min\{\epsilon, \dist(\Psi(z), \Phi(K))\}/2,$$

which means that $\Psi$ is continuous. Moreover, it follows that $\rho_H(\Phi(z), \Psi(z)) \leq \epsilon$ for all $z \in Z$, and $\Psi(z) \in \overline{\mathcal{C}(X, Y)} \setminus \Phi(K)$ for all $z \in Z \setminus K$. Since $z \in C \setminus K$ if and only if $\psi''(z) \in \mathcal{C}(X, Y)$, we have $\Psi^{-1}(\overline{\mathcal{C}(X, Y)} \setminus K) = C \setminus K$. It remains to show that $\Psi$ is a $Z$-embedding. Note that $\Psi|_K = \Phi|_K$ is a $Z$-embedding and $\Psi|_{Z \setminus K} = \psi''$ is an injection. Since $\Psi(Z \setminus K) \subset \overline{\mathcal{C}(X, Y)} \setminus \Phi(K) = \overline{\mathcal{C}(X, Y)} \setminus \Psi(K)$, $\Psi$ is an embedding. Recall that $\Psi(K) = \Phi(K)$ is a $Z$-set in $\overline{\mathcal{C}(X, Y)}$. Since $x_{2k} \in B(x_\infty, s_k) \setminus B(x_\infty, s_{2k})$ for every $k \in \mathbb{N}$, it follows from (b) and (3) that

$$\Psi(z)(x_{2k}) = \psi''(z)(x_{2k}) = \psi(z)(x_{2k}) \cup \psi'(z)(x_{2k}) = \{0\} \text{ for each } z \in Z_k.$$

Applying Lemma 8.2, $\Psi(Z) = \Psi(Z \setminus K) \cup \Psi(K)$ is a $Z$-set in $\overline{\mathcal{C}(X, Y)}$. Consequently, $\Psi$ is a $Z$-embedding. \hfill \Box

Finally, we prove the main theorem.

**Proof of Main Theorem.** Since $X$ is infinite and has only a finite number of isolated points, we can write $X = \bigoplus_{i=0}^{n} X_i$, where $X_0 \neq \emptyset$ has no isolated points and each $X_i$, $1 \leq i \leq n$, is a singleton of an isolated point. Note that every $X_i$ is open in $X$, and hence the pair $(\bigvee_i \mathcal{C}(X_i, Y), \bigvee_i \mathcal{C}(X_i, Y))$ is homeomorphic to $(\prod_{i=0}^{n} \bigvee_i \mathcal{C}(X_i, Y), \prod_{i=0}^{n} \bigvee_i \mathcal{C}(X_i, Y))$, refer to Lemma 6.8 of [5]. For each $1 \leq i \leq n$,
A function space from a compact metrizable space to a dendrite with the hypo-graph topology

\[ \downarrow_v \mathcal{C}(X_i, Y) \] is homeomorphic to \( (Y, Y) \). Combining Corollary 7.2, Proposition 5.2, Proposition 8.3 and Proposition 9.4, we can obtain that \( \downarrow_v \mathcal{C}(X_0, Y) \) is \( (\mathcal{F}_0, \mathcal{F}_0) \)-absorbing. It follows from Theorem 9.1 and Fact 9.2 that \( \downarrow_v \mathcal{C}(X_0, Y) \) is homeomorphic to \( (Q, c_0) \). On the other hand, using Theorem 9.1, we can easily show that the pair \( (Q \times Y, c_0 \times Y) \) is homeomorphic to \( (Q, c_0) \). This means that \( \prod_{i=0}^n \downarrow_v \mathcal{C}(X_i, Y) \) is homeomorphic to \( (Q, c_0) \). Thus the proof is complete.

\[ \hfill \Box \hfill \]

10 Remarks

In this section, we will give some remarks on the main theorem. Z. Yang and X. Zhou [14] proved the stronger result as follows:

**Theorem 10.1.** The pair \( (\mathcal{USC}(X, I), \downarrow \mathcal{C}(X, I)) \) is homeomorphic to \( (Q, c_0) \) if and only if the set of isolated points of \( X \) is not dense.

It is unknown whether the above theorem holds or not even if \( I \) is replaced with \( Y \) in our assumption. However, the first author shows the following theorem in his Ph.D. thesis [4] (cf. Z. Yang [13] proved the case that \( Y = I \)).

**Theorem 10.2.** The space \( \downarrow \mathcal{C}(X, Y) \) is a Baire space if and only if the set of isolated points is dense in \( X \).

The space \( c_0 \) is not a Baire space. In fact, it is a \( Z_{\alpha} \)-set in it. Immediately, the following corollary follows:

**Corollary 10.3.** If \( \downarrow \mathcal{C}(X, Y) \) is homeomorphic to \( c_0 \), then the set of isolated points is not dense in \( X \).

Acknowledgement: The authors would like to express their gratitude to the reviewer for his helpful comments and suggestions. In particular, one of them helped us to improve the main theorem.

The authors contributed equally to this work. The first author was partially supported by Wuyi University Doctor Startup Fund.

This work was done at the University of Tsukuba, when the first and the third authors were Ph.D. students of the second author.

References
