Solutions of minus partial ordering equations over von Neumann regular rings

Abstract: In this paper, we mainly derive the general solutions of two systems of minus partial ordering equations over von Neumann regular rings. Meanwhile, some special cases are correspondingly presented. As applications, we give some necessary and sufficient conditions for the existence of solutions. It can be seen that some known results can be regarded as the special cases of this paper.

Keywords: von Neumann regular ring, Minus partial ordering, Linear equation

1 Introduction and preliminaries

Many partial orders such as star partial order, Löwner partial order and inverse partial order have been studied before (see [3, 4, 6, 11–14, 16, 21, 22]). An obvious advantage of the partial order, compared with the others, is that its properties strongly improve the ring properties of the base algebraic structure. In this paper, we mainly study the minus partial order which was initially defined by Hartwig [13]. This relation is on the set of regular elements. And then some properties of the order on semigroups, monoids, and rings have also been established (see [1, 2, 5, 19, 24, 26]). We aim to derive the solutions of two systems of minus partial ordering equations over von Neumann regular rings. A von Neumann regular ring \( R \) is a non-commutative but associate ring with identity 1. If there exists an \( x \in R \) such that \( axa = a \), then \( x \) is called a generalized inverse of \( a \in R \). If \( axa = a \) and \( xax = x \) both hold, then \( x \) is called a strong von Neumann inverse of \( a \). We will denote by \( a^- \) a generalized inverse of \( a \) and \( a^- \) a strong von inverse of \( a \). Furthermore, define \( R_a = 1 - aa^- \) and \( L_a = 1 - a^-a \) for convenience.

Definition 1.1. Given regular element \( a \) is said to be related to \( b \) by minus partial order, denoted by \( a \preceq b \), if

\[
(a - b)R \cap aR = \{0\} \text{ and } R(a - b) \cap Ra = \{0\},
\]

both hold.

Some well-known characterizations of the minus partial ordering are given below.

Lemma 1.2 ([6]). Let \( R \) be a von Neumann regular ring, and \( a, b \in R \). Then, the following statements are equivalent:

(a) \( a \preceq b \).

(b) There exist \( x, y \in \{a^-\} \) such that \( ax = bx \) and \( ya = yb \) both hold.

(c) \( \{b^-\} \subseteq \{a^-\} \).

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Lemma 1.4. Let \( R \) be von Neumann regular, and let \( a, b, c \in R \) be given. Then,

(a) The linear equation \( ax = c \) is solvable for \( x \in R \) if and only if \( aa^\sim c = c \). In this case, the general solution can be expressed as

\[
x = a^\sim c + (1 - a^\sim a)u,
\]

where \( u \in R \) is arbitrary. In particular, the solution is unique if and only if \( a^\sim a = 1 \) for some \( a^\sim \).

(b) The linear equation \( axb = c \) is solvable for \( x \in R \) if and only if \( aa^\sim cb^-b = c \). In this case, the general solution can be expressed in the form

\[
x = a^\sim cb^- + u - a^\sim aubb^-,
\]

where \( u \in R \) is arbitrary. In particular, the solution is unique if and only if both \( a^\sim a = 1 \) and \( bb^- = 1 \) for some \( a^\sim \) and \( b^- \).

Lemma 1.3 ([23]). Let \( R \) be von Neumann regular, and let \( a, b, c \in R \) be given. Then,

(a) The linear equation \( ax = c \) is solvable for \( x \in R \) if and only if \( aa^\sim c = c \). In this case, the general solution can be expressed as

\[
x = a^\sim c + (1 - a^\sim a)u,
\]

where \( u \in R \) is arbitrary. In particular, the solution is unique if and only if \( a^\sim a = 1 \) for some \( a^\sim \).

(b) The linear equation \( axb = c \) is solvable for \( x \in R \) if and only if \( aa^\sim cb^-b = c \). In this case, the general solution can be expressed in the form

\[
x = a^\sim cb^- + u - a^\sim aubb^-,
\]

where \( u \in R \) is arbitrary. In particular, the solution is unique if and only if both \( a^\sim a = 1 \) and \( bb^- = 1 \) for some \( a^\sim \) and \( b^- \).

2 The general solutions of the system (1)

In this section, we consider the solutions of the system (1) over strong von Neumann regular ring. Applying Lemma 1.2 to the system (1), we get the following equivalent system of equations

\[
a_1x \leq c_1, xb_2 \leq c_2, a_3xb_3 \leq c_3. \tag{1}
\]

and

\[
a_1xb_1 \leq c_1, a_2xb_2 \leq c_2. \tag{2}
\]

Now we give some well-known results, which will play key roles in solving (1) and (2).

Theorem 2.1. Let \( R \) be von Neumann regular ring, and \( a_1, b_2, a_3, b_3, c_1, c_2, c_3 \in R \) be given. Define \( e = R_{a_1}c_1, d = c_2L_{c_2}, g = R_{a_3}c_3, f = c_3L_{c_3}, h = R_{b_2}b_2, k = a_3L_{a_1}, i = R_{b_3}b_3, j = b_3L_{c_3}, m = R_{c_1}c_1, q_2 = k^-q_3h^-q_4 = k^-q_3h^- \). Then the following conditions are equivalent:

(a) The system (1) is consistent.

(b) \( a_1a_1^-c_1L_e = c_1L_e, R_dcb_2^-b_2 = R_dcb_2, a_3a_3^-c_3L_g = c_3L_g, R_f c_3b_3^-b_3 = R_f c_3 \)

and

\[
c_1y_1c_1b_2 = a_1c_2y_2c_2. \tag{7}
\]
The following two forms are the general solution of (1)

\[ x = a_1 c_1 L_e u_1 (u_2 c_1 L_e u_1) u_2 c_1 + L_a_1 c_2 u_5 (u_6 R_d c_2 u_5) R_e b_2 \]
\[ + L_a_1 q_2 R_b_2 + L_a_1 (u_1 - k - k u_1 h h) R_b_2. \]  
(8)

\[ x = L_{m u_3} (u_4 R_{L_{c_1}} c_1 a_1 L_{m u_3}) u_4 R_{L_{c_1}} + L_a_1 L_{R_{c_2}} u_7 (u_8 R_n b_2 c_2 L_{R_{c_2}} u_7) u_8 R_n \]
\[ + L_a_1 q_4 R_b_2 + L_a_1 (u_1 - k - k u_1 h h) R_b_2. \]  
(9)

where

\[ q_1 = c_3 L_e u_9 (u_10 R_f c_3 L_e u_9) u_10 R_f c_3 - a_3 a_1 c_1 L_e u_1 (u_2 c_1 L_e u_1) u_2 c_1 b_3 \]
\[ - k c_2 u_5 (u_6 R_d c_2 u_5) u_6 R_d c_2 b_3. \]
\[ q_3 = a_3 L_i u_11 (u_2 R_f b_3 c_3 a_3 L_i u_11) u_12 R_f b_3 - a_3 L_{m u_3} (u_4 R_{L_{c_1}} c_1 a_1 L_{m u_3}) u_4 R_{L_{c_1}} b_3 \]
\[ - k L_{R_{c_2}} u_7 (u_8 R_n b_2 c_2 L_{R_{c_2}} u_7) u_8 R_n b_3. \]

and \( u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}, u_{11}, u_{12}, u_{13} \in R \) are arbitrary.

Proof. (a) \( \Rightarrow \) (b): It is obvious that system (1) has solutions such that (7) holds. It follows from Lemma 1.4 that (6) holds.

(b) \( \Rightarrow \) (a): Suppose that (6) and (7) hold, substituting (8) into (3), (4) and (5) respectively, we get

\[ a_1 x = c_1 L_e u_1 (u_2 c_1 L_e u_1) u_2 c_1 = c_1 y c_1, \]
\[ x b_2 = a_1 c_1 L_e u_1 (u_2 c_1 L_e u_1) u_2 c_1 b_2 + L_a_1 c_2 u_5 (u_6 R_d c_2 u_5) u_6 R_d c_2 \]
\[ = c_2 y c_2, \]
\[ a_3 x b_3 = a_3 a_1 c_1 L_e u_1 (u_2 c_1 L_e u_1) u_2 c_1 b_3 \]
\[ + a_3 L_a_1 c_2 u_5 (u_6 R_d c_2 u_5) u_6 R_d c_2 b_3 + a_3 L_a_1 q_2 R_b_2 b_3 \]
\[ = c_3 y c_3, \]
\[ (a_1 x) c_1 (a_1 x) = c_1 L_e u_1 (u_2 c_1 L_e u_1) u_2 c_1 b_1 \]
\[ = c_1 L_e u_1 (u_2 c_1 L_e u_1) u_2 c_1 = a_1 x, \]
\[ (x b_2) c_2 (x b_2) = c_2 u_5 (u_6 R_d c_2 u_5) u_6 R_d c_2 \]
\[ = c_2 u_5 (u_6 R_d c_2 u_5) u_6 R_d c_2 = x b_2, \]
\[ (a_3 x b_3) c_3 (a_3 x b_3) = c_3 L_e u_9 (u_10 R_f c_3 L_e u_9) u_10 R_f c_3 = a_3 x b_3. \]

Similarly, substituting (9) into (3), (4) and (5) respectively, we have

\[ a_1 x = a_1 L_{m u_3} (u_4 R_{L_{c_1}} c_1 a_1 L_{m u_3}) u_4 R_{L_{c_1}} = c_1 y c_1, \]
\[ x b_2 = L_{m u_3} (u_4 R_{L_{c_1}} c_1 a_1 L_{m u_3}) u_4 R_{L_{c_1}} b_2 + L_a_1 L_{R_{c_2}} u_7 (u_8 R_n b_2 c_2 L_{R_{c_2}} u_7) u_8 R_n b_2 \]
\[ = c_2 y c_2, \]
\[ a_3 x b_3 = a_3 L_{m u_3} (u_4 R_{L_{c_1}} c_1 a_1 L_{m u_3}) u_4 R_{L_{c_1}} b_3 \]
\[ + a_3 L_a_1 L_{R_{c_2}} u_7 (u_8 R_n b_2 c_2 L_{R_{c_2}} u_7) u_8 R_n b_3 \]
\[ = c_3 y c_3, \]
\[ a_1 x = (a_1 x) c_1 (a_1 x), \]
\[ x b_2 = (x b_2) c_2 (x b_2), \]
\[ a_3 x b_3 = (a_3 x b_3) c_3 (a_3 x b_3). \]

So \( x \) has the forms of (8) and (9) are solutions of the system (1).

Now we prove that for an arbitrary solution \( x_0 \) of the system (1), namely \( a_1 x_0 \leq c_1, x_0 b_2 \leq c_2, a_3 x_0 b_3 \leq c_3 \) can be expressed as the forms of (8) and (9). Set

\[ u_1 = x_1 c_1 c_1^{-1}, \]
\[ u_2 = (c_1 L_e)^{-1} (c_1 L_e) x_1. \]
\[ u_5 = x_1(R_d e_2)(R_d e_2)^-, u_6 = c_2^2 c_2 x_1, \]
\[ u_9 = x_1(R_f e_3)(R_f e_3)^-, u_{10} = (c_3 L_g)^-(c_3 L_g)x_1, \]
\[ u_{13} = x_0, \quad y_0 = L_e x_1 = x_1 R_d = L_g x_1 R_f. \]

Then, (8) reduces to
\[
x = a_1^1 c_1 L_e x_1 c_1 c_1^1 ((c_1 L_e)^- c_1 L_e x_1 c_1 c_1 L_e x_1 c_1 c_1^-) f (c_1 L_e)^- c_1 L_e x_1 c_1
+ L_a_1^1 c_2 x_1 (R_d e_2)(R_d e_2)^-(c_2^2 c_2 x_1 R_d e_2 c_2^2 c_2 x_1 (R_d e_2)^-(R_d e_2)^-) f
\cdot c_2^2 c_2 x_1 R_d e_2 b_2^2 + L_a_1^1 q_6 R_b_2 + L_a_1 (x_0 - k^- k x_0 h h^-) R_b_2,
\]
\[ q_6 = k^- q_5 h^-. \]
\[ q_5 = c_3 L_g x_1 (R_f e_3)(R_f e_3)^-(c_3 L_g)^-(c_3 L_g)x_1 R_f e_3 c_3 L_g x_1 (R_f e_3)(R_f e_3)^- f
\cdot (c_3 L_g)^- (c_3 L_g)x_1 R_f e_3 - a_3 a_1^1 c_1 L_e x_1 c_1 c_1^- f
\cdot ((c_1 L_e)^- c_1 L_e x_1 c_1 c_1^1 c_1 L_e x_1 c_1 c_1^-) f c_1 L_e x_1 c_1 b_3
\cdot k^- c_2 x_1 (R_d e_2)(R_d e_2)^-(c_2^2 c_2 x_1 R_d e_2 c_2^2 c_2 x_1 (R_d e_2)^-(R_d e_2)^-) f c_2^2 c_2 x_1 R_d e_2 b_2^2 b_3.
\]
So,
\[
x = a_1^1 c_1 L_e x_1 c_1 + L_a_1^1 c_2 x_1 R_d e_2 b_2^2 + L_a_1 k^- c_3 L_g x_1 R_f e_3 h^- R_b_2
- L_a_1 k^- a_3 a_1^1 c_1 L_e x_1 c_1 b_3 h^- R_b_2 - L_a_1 k^- k^- c_2 x_1 R_d e_2 b_2^2 b_3 h^- R_b_2 + L_a_1 (x_0 - k^- k x_0 h h^-) R_b_2
= a_1^1 a_1 x_0 + x_0 b_2 b_2^2 - a_1^1 a_1 x_0 b_2 b_2^2 + a_1^1 a_1 x_0 b_2 b_2^2 + a_1^1 a_1 x_0 b_2 b_2^2
= x_0.
\]

Similarly, set
\[
u_3 = x_2 (R_{L_{e_1}})(R_{L_{e_1}})^-, u_4 = (a_1^1 L_m)^-(a_1^1 L_m)x_2,
\]
\[ u_7 = x_2 (R_n b_2)(R_n b_2)^-, u_8 = (L_{R_{c_2}})^-(L_{R_{c_2}}) x_2,
\]
\[ u_{11} = x_2 (R_j b_3)(R_j b_3)^-, u_{12} = (a_3^1 L_i)^-(a_3^1 L_i)x_2,
\]
\[ u_{13} = x_0. \quad L_m x_2 R_{L_{c_1}} = a_1^1 a_1 x_0.
\]
\[ L_{R_{c_2}} x_2 R_n = x_0 b_2 b_2^2, \quad L_i x_2 R_j = a_3^1 a_3 x_0 b_3 b_3^2.
\]

Then, equality (9) reduces to
\[
x = L_m x_2 (R_{L_{c_1}})(R_{L_{c_1}})^-(a_1^1 L_m)^-(a_1^1 L_m)x_2 R_{L_{c_1}} c_1^1 a_1^1 L_m x_2 (R_{L_{c_1}})(R_{L_{c_1}})^- f
\cdot (a_1^1 L_m)^-(a_1^1 L_m)x_2 R_{L_{c_1}} + L_a_1 L_{R_{c_2}} x_2 (R_n b_2)(R_n b_2)^-
\cdot ((L_{R_{c_2}})^-(L_{R_{c_2}}) x_2 R_n b_2 c_2^2 L_{R_{c_2}} x_2 (R_n b_2)(R_n b_2)^-) f (L_{R_{c_2}})^-(L_{R_{c_2}})^- x_2 R_n
+ L_a_1 q_8 R_b_2 + L_a_1 (x_0 - k^- k x_0 h h^-) R_b_2,
\]
\[ q_8 = k^- q_7 h^-.
\]
\[ q_7 = a_3^1 L_i x_2 (R_j b_3)(R_j b_3)^-(a_3^1 L_i)^-(a_3^1 L_i)x_2 R_j b_3 c_3^2 a_3^1 L_i x_2 (R_j b_3)(R_j b_3)^- f
\cdot (a_3^1 L_i)^-(a_3^1 L_i)x_2 R_j b_3 - a_3^1 L_m x_2 (R_{L_{c_1}})(R_{L_{c_1}})^-
\cdot ((a_1^1 L_m)^-(a_1^1 L_m)x_2 R_{L_{c_1}} c_1^1 a_1^1 L_m x_2 (R_{L_{c_1}})(R_{L_{c_1}})^-) f
\cdot (a_1^1 L_m)^-(a_1^1 L_m)x_2 R_{L_{c_1}} b_3 - k^- L_{R_{c_2}} x_2 (R_n b_2)(R_n b_2)^-
\cdot ((L_{R_{c_2}})^-(L_{R_{c_2}}) x_2 R_n b_2 c_2^2 L_{R_{c_2}} x_2 (R_n b_2)(R_n b_2)^-) f (L_{R_{c_2}})^-(L_{R_{c_2}})^- x_2 R_n b_3.
\]

which can be further simplified as
\[
x = L_m x_2 R_{L_{c_1}} + L_a_1 L_{R_{c_2}} x_2 R_n + L_a_1 k^- a_3^1 L_i x_2 R_j b_3 h^- R_b_2
The general solutions of the system (1) can be described as (8) and (9). Thus (8) and (9) are the general solutions of (1).

**Corollary 2.2.** Consider the simplified equations $a_1x = c_1, xb_2 = c_2, a_3xb_3 = c_3$ of system (1), then the general solution can be written in the following form

\[
x = a_1^{-1}c_1u_1(u_2c_1u_1)^r u_2c_1 + L_{a_1}c_2u_5(u_6c_2u_5)^r u_6c_2b_2 + L_{a_1}c_3u_9(u_{10}c_3u_9)^r u_{10}c_3h^{-1}R_{b_2} - L_{a_1}k^{-1}a_1c_3u_1(u_2c_1u_1)^r u_2c_1b_3h^{-1}R_{b_2} - L_{a_1}k^{-1}a_2u_5(u_6c_2u_5)^r u_6c_2b_2b_3h^{-1}R_{b_2} + L_{a_1}(u_{13} - k^{-1}ku_{13}h^{-1})R_{b_2},
\]

where $u_1, u_2, u_5, u_6, u_9, u_{10}, u_{13} \in R$ are arbitrary.

**Proof.** It is obvious that the equalities $e = R_{a_1}c_1 = 0, d = c_2L_{b_2} = 0, g = R_{a_5}c_3 = 0$ and $f = c_3L_{b_3} = 0$ can be derived from the fact that the system of equations $a_1x = c_1, xb_2 = c_2, a_3xb_3 = c_3$ is solvable. Thus, equality (8) can be simplified to (10).

**Corollary 2.3.** Suppose $a_1, b_2, a_3, b_3, c_1, c_2, c_3$ satisfy $a_1R \subseteq c_1R, Rb_2 \subseteq Rc_2, a_3R \subseteq c_3R$ and $Rb_3 \subseteq Rc_3$, then the general solution of the system (1) is

\[
x = u_3(u_4R_{L_{c_1}}c_1^{-1}a_1u_3)^r u_4R_{L_{c_1}} + L_{a_1}L_{R_{c_2}}u_7(u_8b_2c_2^{-1}L_{R_{c_2}}u_7)^r u_8 + L_{a_1}k^{-1}a_3u_1(u_2b_3c_3^{-1}a_3u_1)^r u_2b_3h^{-1}R_{b_2} - L_{a_1}k^{-1}a_3u_3(u_4R_{L_{c_1}}c_1^{-1}a_1u_3)^r u_4R_{L_{c_1}}b_3h^{-1}R_{b_2} - L_{a_1}k^{-1}L_{R_{c_2}}u_7(u_8b_2c_2^{-1}L_{R_{c_2}}u_7)^r u_8b_3h^{-1}R_{b_2} + L_{a_1}(u_{13} - k^{-1}ku_{13}h^{-1})R_{b_2},
\]

where $u_3, u_4, u_7, u_8, u_{11}, u_{12}, u_{13} \in R$ are arbitrary.

**Proof.** It follows from $a_1R \subseteq c_1R, Rb_2 \subseteq Rc_2, a_3R \subseteq c_3R$ and $Rb_3 \subseteq Rc_3$ that $m = R_{c_1}a_1 = 0, n = b_2L_{c_2} = 0, i = R_{c_3}a_3 = 0$ and $j = b_3L_{c_3} = 0$. Then, (9) can be reduced to (11).

### 3 The general solutions of the system (2)

In this section, our goal is to consider the general solutions of the system (2) over the strong von Neumann regular ring. By Lemma 1.2, the system (2) can be turned into the following system of equations

\[
a_1xb_1 = c_1y_1c_1, (a_1xb_1)c_1^{-1}(a_1xb_1) = a_1xb_1, \tag{12}
\]

\[
a_2xb_2 = c_2y_2c_2, (a_2xb_2)c_2^{-1}(a_2xb_2) = a_2xb_2, \tag{13}
\]

where $y_1 \in R, y_2 \in R$ are unknown elements.

**Theorem 3.1.** Let $R$ be von Neumann regular ring, and $a_1, a_2, b_1, b_2, c_1, c_2 \in R$ be given. Define $s = a_2L_{a_1}, t = R_{b_1}b_2, g = R_{a_1}d_1, e = c_1L_{b_1}, f = R_{a_2} c_2, h = c_2L_{b_2}, k = R_{a_1}a_1, i = b_1L_{c_1}, m = R_{b_2} a_2, n = b_2L_{c_2}$. Then the following conditions are equivalent:

(a) The system (2) is consistent.

(b) $a_1^{-1}c_1 L_d = c_1L_d, R_{e}c_2b_2^{-1}b_2 = R_{e}c_2, a_2^{-1}c_2 L_g = c_2L_g, R_{h}c_2b_2^{-1}b_2 = R_{h}c_2 \tag{14}$

and

\[
g(a_2^{-1}c_2y_2c_2^{-1}b_2 - a_1^{-1}c_1y_0c_1b_1) = 0. \tag{15}
\]
The following two forms are the general solutions of (2):

\[
x = a_1^- c_1 L_d u_1(2 R e c_1 L_d u_1)^- u_2 R e c_1 b_1^- \\
+ L_a s^- a_2 L_g[a_2^- c_2 L_f u_5(u_6 R_h c_2 L_f u_5)^- u_6 R_h c_2 b_2^- \\
- a_1^- c_1 L_d u_1(2 R e c_1 L_d u_1)^- u_2 R e c_1 b_1^-] b_2^- \\
+ g^- g[a_2^- c_2 L_f u_5(u_6 R_h c_2 L_f u_5)^- u_6 R_h c_2 b_2^- \\
- a_1^- c_1 L_d u_1(2 R e c_1 L_d u_1)^- u_2 R e c_1 b_1^-] b_2^- R_b_1 \\
+ L_a(u_9 - s^- s u g_9 t b_2^-) - L_a s^- a_2 L_g u_9 t b_2^- \\
+ (u_9 - g^- g u g_9 t^- R_b_1 - L_a u_9 R_b_1 + L_a s^- s u g_9 t b_2^-).
\]

(16)

\[
x = L_k u_3(u_4 R_i b_1 c_1^- a_1 L_k u_3)^- u_4 R_i \\
+ L_a s^- a_2 L_g[L_m u_7(u_8 R_8 b_2 c_2^- a_2 L_m u_3)^- u_4 R_a \\
- L_k u_3(u_4 R_i b_1 c_1^- a_1 L_k u_3)^- u_4 R_i] b_2^- \\
+ g^- g[L_m u_7(u_8 R_8 b_2 c_2^- a_2 L_m u_3)^- u_4 R_a \\
- L_k u_3(u_4 R_i b_1 c_1^- a_1 L_k u_3)^- u_4 R_i] b_2^- R_b_1 \\
+ L_a(u_9 - s^- s u g_9 t b_2^-) - L_a s^- a_2 L_g u_9 t b_2^- \\
+ (u_9 - g^- g u g_9 t^- R_b_1 - L_a u_9 R_b_1 + L_a s^- s u g_9 t b_2^-).
\]

(17)

where \( u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9 \in R \) are arbitrary.

Proof. (a) \( \Rightarrow \) (b): Let the system (5) has a solution \( x_0 \). Then \( a_1 x_0 b_1 \leq c_1, a_2 x_0 b_2 \leq c_2 \), so it satisfies \( a_1 x_0 b_1 = c_1 y_0 c_1, a_2 x_0 b_2 = c_2 y_0 c_2 \). It follows from Lemma 1.4 that (14) holds and

\[
x_0 = a_1^- c_1 L_d u_1(2 R e c_1 L_d u_1)^- u_2 R e c_1 b_1^- + u - a_1^- a_1 u b_1 b_1^-,
\]

where \( u_1, u_2 \) and \( u \) are arbitrary over \( R \). Hence by \( a_2 x_0 b_2 = c_2 y_0 c_2 \),

\[
a_2 x_0 b_2 = a_2 a_1^- c_1 L_d u_1(2 R e c_1 L_d u_1)^- u_2 R e c_1 b_1^- b_2 + a_2 u b_2 - a_2 a_1^- a_1 u b_1 b_1^- b_2 = c_2 y_0 c_2.
\]

Note that \( R s = 0, t L_t = 0 \). Thus by (3.7) and (3.3),

\[
g(a_2^- c_2 y_0 c_2 b_2^- - a_1^- c_1 y_0 c_1 b_1^-) f = R_s(c_2 y_0 c_2 - a_2 a_1^- c_1 y_0 c_1 b_1^- b_2)L_t \\
= R_s(a_2 u b_2 - a_2 a_1^- a_1 u b_1 b_1^- b_2)L_t \\
= R_s(s u b_2 + a_2 u t - s u t)L_t = 0,
\]

i.e., (15) holds.

(b) \( \Rightarrow \) (a): Suppose that (14) and (15) hold, substituting (16) into (12) and (13), respectively, we can get

\[
a_1 x b_1 = c_1 L_d u_1(2 R e c_1 L_d u_1)^- u_2 R e c_1 = c_1 y_1 c_1,
\]

\[
a_2 x b_2 = a_2 a_1^- c_1 L_d u_1(2 R e c_1 L_d u_1)^- u_2 R e c_1 b_1^- b_2 \\
+ s s^- a_2 L_g[a_2^- c_2 L_f u_5(u_6 R_h c_2 L_f u_5)^- u_6 R_h c_2 b_2^- \\
- a_1^- c_1 L_d u_1(2 R e c_1 L_d u_1)^- u_2 R e c_1 b_1^-] b_2 \\
+ a_2 g^- g[a_2^- c_2 L_f u_5(u_6 R_h c_2 L_f u_5)^- u_6 R_h c_2 b_2^- \\
- a_1^- c_1 L_d u_1(2 R e c_1 L_d u_1)^- u_2 R e c_1 b_1^-] b_2^- t \\
- s s^- a_2 L_g u_9 t b_2^- b_2 + a_2(u_9 - g^- g u g_9 t^- t) \\
= a_2 a_1^- c_1 L_d u_1(2 R e c_1 L_d u_1)^- u_2 R e c_1 b_1^- b_2 \\
+ a_2 a_1^- c_1 L_d u_1(2 R e c_1 L_d u_1)^- u_2 R e c_1 b_1^- b_2 \\
- a_1^- c_1 L_d u_1(2 R e c_1 L_d u_1)^- u_2 R e c_1 b_1^-] b_2 \\
= c_2 y_2 c_2.
\]
Similarly, substituting (17) into (12) and (13) respectively, we can obtain

\[
a_{1}x_{1}b_{1} = a_{1}L_{d}u_{1}(u_{2}R_{c}c_{1}L_{d}u_{1})_{R}u_{2}R_{c}c_{1} = (a_{1}x_{1}b_{1})c_{1}(a_{1}x_{1}b_{1}),
\]

\[
a_{2}x_{2}b_{2} = a_{2}L_{f}u_{5}(u_{6}R_{n}c_{2}L_{f}u_{5})_{R}u_{6}R_{n}c_{2} = (a_{2}x_{2}b_{2})c_{2}(a_{2}x_{2}b_{2}).
\]

So \( x \) has the forms of (16) and (17) are solutions of the system of (2).

Now we show that if the system (2) is consistent, i.e., (14) and (15) hold, then its general solutions can be expressed as (16) and (17). In \((b) \Rightarrow (a)\), we have shown that \( x \) that has the forms of (16) and (17) are solutions of the system (2). So we only need to prove that for an arbitrary solution \( x_0 \) of the system (2) can be expressed as the forms of (16) and (17). Set

\[
u_{1} = x_{1}(R_{c}c_{1})(R_{c}c_{1})_{R}u_{2} = (c_{1}L_{d})_{R}(c_{1}L_{d})x_{1},
\]

\[
u_{2} = x_{1}(R_{h}c_{2})(R_{h}c_{2})_{R}u_{6} = (c_{2}L_{f})_{R}(c_{2}L_{f})x_{1},
\]

\[
u_{9} = x_{0}, \quad \nu_{0} = L_{d}x_{1}R_{e} = L_{f}x_{1}R_{h}.
\]

We also notice that

\[
g^{-}g_{x_{0}}t^{-}R_{b_{1}} - g^{-}g(a_{2}a_{2}x_{0}b_{2}b_{2} - a_{1}a_{1}x_{0}b_{1}b_{1})b_{2}t^{-}R_{b_{1}}
\]

\[
= g^{-}R_{s}(a_{2}a_{2}x_{0}b_{2}b_{2} - a_{2}a_{2}x_{0}b_{1}b_{1})t^{-}R_{b_{1}}
\]

\[
= g^{-}R_{s}(a_{2}a_{2}a_{1}b_{1}b_{1}b_{2} - a_{2}a_{2}x_{0}b_{1}b_{1}b_{2})t^{-}R_{b_{1}}
\]

\[
= -g^{-}R_{s}a_{2}L_{a_{2}x_{0}b_{1}b_{1}b_{2}}t^{-}R_{b_{1}}
\]

\[
= -g^{-}R_{s}a_{2}x_{0}b_{1}b_{1}b_{2}t^{-}R_{b_{1}}
\]

\[
= 0.
\]

Then, (14) reduces to

\[
x = a_{1}c_{1}L_{d}x_{1}(R_{e}c_{1})(R_{e}c_{1})^{-}(c_{1}L_{d})_{R}(c_{1}L_{d})x_{1}R_{e}c_{1}L_{d}x_{1}(R_{e}c_{1})(R_{e}c_{1})^{-}.
\]
\[
\begin{align*}
\cdot (c_1 L_d)^{-}(c_1 L_d)x_1 R_c c_1 b_1^- + L_{a_1} s^- a_2 L_g [a_2^- c_2 L_f x_1(R_h c_2)(R_h c_2)^- \\
\cdot ((c_2 L_f)^-) (c_2 L_f)x_1 R_c c_2 L_f x_1(R_h c_2)(R_h c_2)^-) r^- (c_2 L_f)^-(c_2 L_f)x_1 R_h c_2 b_2^- \\
- a_1^- c_1 L_d x_1(R_c c_1)(R_c c_1)^- ((c_1 L_d)^- (c_1 L_d)x_1 R_c c_1 L_d x_1(R_c c_1)(R_c c_1)^-) r^- \\
\cdot (c_1 L_d)^{-}(c_1 L_d)x_1 R_c c_1 b_1^- b_2^- + g^- g [a_2^- c_2 L_f x_1(R_h c_2)(R_h c_2)^- \\
\cdot ((c_2 L_f)^-) (c_2 L_f)x_1 R_h c_2 L_f x_1(R_h c_2)(R_h c_2)^-) r^- (c_2 L_f)^-(c_2 L_f)x_1 R_h c_2 b_2^- \\
- a_1^- c_1 L_d x_1(R_c c_1)(R_c c_1)^- ((c_1 L_d)^- (c_1 L_d)x_1 R_c c_1 L_d x_1(R_c c_1)(R_c c_1)^-) r^- \\
\cdot (c_1 L_d)^{-}(c_1 L_d)x_1 R_c c_1 b_1^- b_2^- R_b + L_{a_1} (x_0 - s^- s x_0 b_2^-) \\
\cdot L_{a_1} s^- a_2 L_g x_0 b_2^- + (x_0 - g^- g x_0 t^-) R_b \\
\cdot L_{a_1} x_0 R_b + L_{a_1} s^- s x_0 b_2^- \\
= a_1^- c_1 L_d x_1 R_c c_1 b_1^- + L_{a_1} s^- a_2 L_g [a_2^- c_2 L_f x_1 R_h c_2 b_2^- - a_1^- c_1 L_d x_1 R_c c_1 b_1^-] b_2^- \\
+ g^- g [a_2^- c_2 L_f x_1 R_h c_2 b_2^- - a_1^- c_1 L_d x_1 R_c c_1 b_1^-] b_2^- R_b \\
+ L_{a_1} (x_0 - s^- s x_0 b_2^-) - L_{a_1} s^- a_2 L_g x_0 b_2^- \\
+ (x_0 - g^- g x_0 t^-) R_b - L_{a_1} x_0 R_b + L_{a_1} s^- s x_0 b_2^- \\
= a_1^- a_1 x_0 b_1^- + L_{a_1} s^- a_2 L_g [a_2^- a_1 x_0 b_2^- - a_1^- a_1 x_0 b_1^-] b_2^- \\
+ g^- g [a_2^- a_1 x_0 b_2^- - a_1^- a_1 x_0 b_1^-] b_2^- R_b \\
+ L_{a_1} x_0 R_b + L_{a_1} s^- s x_0 b_2^- + x_0 R_b - g^- g x_0 t^- R_b \\
= a_1^- a_1 x_0 b_1^- + L_{a_1} x_0 + x_0 R_b - L_{a_1} x_0 R_b \\
= x_0.
\end{align*}
\]

We also set
\[
\begin{align*}
u_3 &= x_2(R_h b_1)(R_h b_1)^-, u_4 = (a_1 L_k)^-(a_1 L_k)x_2, \\
u_7 &= x_2(R_h b_2)(R_h b_2)^-, u_8 = (a_2 L_m)^-(a_2 L_m)x_2, \\
u_9 &= x_0, L_k x_2 R_i = a_1^- a_1 x_0 b_1^- . L_m x_2 R_n = a_2^- a_2 x_0 b_2^- .
\end{align*}
\]

Then, (15) reduces to
\[
\begin{align*}
x = L_k x_2(R_h b_1)(R_h b_1)^- (a_1 L_k)^-(a_1 L_k)x_2 R_i b_1 c_1^- a_1 L_k x_2(R_h b_1)(R_h b_1)^- r^- \\
\cdot (a_1 L_k)^-(a_1 L_k)x_2 R_i + L_{a_1} s^- a_2 L_g [L_m x_2(R_h b_2)(R_h b_2)^-(a_2 L_m)^-(a_2 L_m) \\
\cdot x_2 R_b b_2^- c_2^- a_2 L_m x_2(R_h b_2)(R_h b_2)^-) r^- (a_2 L_m)^-(a_2 L_m)x_2 R_n \\
- L_k x_2(R_h b_1)(R_h b_1)^- (a_1 L_k)^-(a_1 L_k)x_2 R_i b_1 c_1^- a_1 L_k x_2(R_h b_1)(R_h b_1)^- r^- \\
\cdot (a_1 L_k)^-(a_1 L_k)x_2 R_i b_2^- b_2^- R_b + L_{a_1} (x_0 - s^- s x_0 b_2^-) \\
- L_{a_1} s^- a_2 L_g x_0 b_2^- + (x_0 - g^- g x_0 t^-) R_b - L_{a_1} x_0 R_b + L_{a_1} s^- s x_0 b_2^- \\
= L_k x_2 R_i + L_{a_1} s^- a_2 L_g [L_m x_2 R_n - L_k x_2 R_i] b_2^- \\
+ g^- g [L_m x_2 R_n - L_k x_2 R_i] b_2^- R_b + L_{a_1} (x_0 - s^- s x_0 b_2^-) \\
- L_{a_1} s^- a_2 L_g x_0 b_2^- + (x_0 - g^- g x_0 t^-) R_b - L_{a_1} x_0 R_b + L_{a_1} s^- s x_0 b_2^- \\
= a_1^- a_1 x_0 b_1^- + L_{a_1} x_0 + x_0 R_b - L_{a_1} x_0 R_b \\
= x_0.
\end{align*}
\]

That is, any solution of (2) can be represented by (16) and (17). Thus (16) and (17) are the general solutions of (2).
Corollary 3.2. The general solution of $a_1x_1 = c_1, a_2x_2 = c_2$ can be written as the following form

$$
x = a_1^{-1} c_1 u_1 (u_2 c_1 u_1)_R u_2 c_1 b_1 - L a_1 s^{-1} a_2 L_g a_2 u_5 (u_6 c_2 u_5)_R u_6 c_2 b_2 - a_1^{-1} c_1 u_1 (u_2 c_1 u_1)_R u_2 c_1 b_1 h b_2 - g^{-1} g_{u_7 b_2 c_2 a_2 u_7}_R u_8$$

$$+ L a_1 (u_9 - s^{-1} u_9 b_2 b_2) - L a_1 s^{-1} a_2 L_g u_9 b_2^2 + (u_9 - g^{-1} g_{u_9 t} b_2^2) R b_2 - L a_1 u_9 R b_1 + L a_1 s^{-1} u_9 b_2^2,$$

where $u_1, u_2, u_5, u_6, u_9 \in R$ are arbitrary.

Proof. $a_1x_1 = c_1, a_2x_2 = c_2$ is equivalent to $d = R a_1 c_1 = 0, e = c_1 L b_1, f = R a_2 c_2$ and $h = c_2 L b_2 = 0$. Thus, (16) reduces to (18).

Corollary 3.3. Suppose $a_1, b_1, c_1, a_2, b_2$ and $c_2$ satisfy $a_1 R \subseteq c_1 R, Ra_1 R_1 \subseteq c_1 R, a_2 R \subseteq c_2 R$ and $R b_2 \subseteq R c_2$, then the general solution of the system (2) is

$$
x = u_3 (u_4 b_1 c_1 u_3)_R u_4 + L a_1 s^{-1} a_2 L_g [u_7 (u_8 b_2 c_2 a_2 u_7)_R u_8$$

$$- u_3 (u_4 b_1 c_1 a_1 u_3)_R u_4] b_2 b_2^2 + g^{-1} g_{u_7 (u_8 b_2 c_2 a_2 u_7)_R u_8} u_8$$

$$- u_3 (u_4 b_1 c_1 a_1 u_3)_R u_4] b_2^2 R b_1 + L a_1 (u_9 - s^{-1} u_9 b_2 b_2)$$

$$- L a_1 s^{-1} a_2 L_g u_9 b_2^2 + (u_9 - g^{-1} g_{u_9 t} b_2^2) R b_2$$

$$- L a_1 u_9 R b_1 + L a_1 s^{-1} u_9 b_2^2,$$

where $u_3, u_4, u_7, u_8, u_9 \in R$ are arbitrary.

Proof. Obviously, $a_1 R \subseteq c_1 R, Ra_1 R_1 \subseteq c_1 R, a_2 R \subseteq c_2 R$ and $R b_2 \subseteq R c_2$ are equivalent to $k = R c_1 a_1 = 0, i = b_1 L c_1 = 0, m = R c_2 a_2 = 0$ and $n = b_2 L c_2 = 0$. Hence, (17) simplifies to (19)

4 Conclusions

In Sections 2 and 3, we have derived the general solutions of systems (1) and (2) by using the generalized inverses of elements over von Neumann regular ring $R$. As applications, some corollaries are also presented. Moreover, we will present the general solutions of some other equations over von Neumann regular ring in another paper.

References