New interval oscillation criteria for second-order functional differential equations with nonlinear damping

Abstract: This paper concerns the oscillation problem of second-order nonlinear damped ODE with functional terms. We give some new interval oscillation criteria which is not only based on constructing a lower solution of a Riccati type equation but also based on constructing an upper solution for corresponding Riccati type equation. We use a recently developed pointwise comparison principle between those lower and upper solutions to obtain our results. Some illustrative examples are also provided to demonstrate our results.

Keywords: Differential equations, Functional term, Oscillation

MSC: 34C10, 34C15, 34K11

1 Introduction

This paper is concerned with the problem of oscillation of the second order forced nonlinear functional differential equations with nonlinear damping terms of the form

\[ r(t)k_1(x(t),x'(t))' + p(t)k_2(x(t),x'(t))x'(t) + F(t,x(t),x(\tau(t)),x'(t),x'(\tau(t))) = e(t), \]

on the half line \([t_0, \infty), t_0 \geq 0\). In what follows we assume with respect to (1) that \(r \in C^1([t_0, \infty), (0, \infty)), p, e \in C([t_0, \infty), \mathbb{R}), k_1, k_2, k_3 \in C(\mathbb{R}^2, \mathbb{R}), F \in C([t_0, \infty) \times \mathbb{R}^4, \mathbb{R}), \tau \in C([t_0, \infty), (0, \infty))\) with \(\lim_{t \to \infty} \tau(t) = \infty\).

We restrict our attention to solutions of Eq. (1) which exists on \([t_0, \infty)\). As usual, such a solution, \(x(t)\), is said to be oscillatory if it has arbitrarily zeros for all \(t_0 \geq 0\), otherwise, it is called nonoscillatory. Eq. (1) is called oscillatory if all solutions are oscillatory.

An interval oscillation criterion means that we do not require information about the coefficients or parameters of the equation on whole half-line \([t_0, \infty)\) but only on a pair of intervals. For some of well known interval oscillation criteria, we refer to papers [1, 5, 6, 13, 14, 16, 17, 19, 20, 22] for linear and nonlinear equations, [10, 15, 18, 23] for functional equations. In our oscillation criteria of interval type, we always suppose that the following condition holds:

\[(C_1) \text{ For any } T \geq t_0, \text{ there exists } T \leq s_1 < t_1 \leq s_2 < t_2 \text{ such that}\]

\[ e(t) \leq 0 \text{ for } t \in [s_1, t_1], \quad e(t) \geq 0 \text{ for } t \in [s_2, t_2], \]

\[ r(t) > 0 \text{ and } p(t) \geq 0 \text{ for } t \in [s_1, t_1] \cup [s_2, t_2]. \]
For the function $F$, we will suppose that

$$(C_2)\quad \text{There exist a function } q_1(t) > 0 \text{ and a constant } \gamma \geq 1 \text{ such that}$$

$$\frac{F(t,x,u,v,w)}{x} \geq q(t)|x|^\gamma - 1$$

holds for $t \in [s,1] \cup [s,2]$, and $x \neq 0, u,v,w \in \mathbb{R}$.

For the functions $k_1$ and $k_2$, we impose the following assumptions:

$$(C_3)\quad uk_1(u,v) \geq \alpha_1 |k_1(u,v)|^{\beta} |u|^{2-\beta} \quad \text{for some } \alpha_1 > 0, \beta > 1 \text{ and for all } u \neq 0, v \in \mathbb{R},$$

$$(C_4)\quad uk_2(u,v) \text{sgn}(u) |u|^\delta - 1 \geq \alpha_2 |k_2(u,v)|^\delta \quad \text{for some } \alpha_2 > 0, \delta > 1 \text{ and for all } u \neq 0, v \in \mathbb{R}.$$

In particular for $\beta = \delta = 2$, inequalities in the conditions $(C_3)$ and $(C_4)$ are equivalent with the following well-known basic ones:

$$(i)\quad uk_1(u,v) \geq \alpha_1 k_1^2(u,v),$$

$$(ii)\quad uvk_2(u,v) \geq \alpha_2 k_2^2(u,v).$$

The assumptions $(i) - (ii)$ were considered for the first time in the paper [10], also the equation (1) were considered in same paper for the first time (with absence of functional term).

In particular for $\beta = \delta = (\alpha + 1)/\alpha$ for some $\alpha > 0$, conditions $(C_3)$ and $(C_4)$ generalize in some sense the following assumptions which have been widely used recently:

$$(iii)\quad uk_1(u,v) \geq \alpha_1 k_1(u,v)(u)^{\alpha+1} |u|^{\alpha-1} \quad \text{for some } \alpha_1 > 0 \text{ and for all } u \neq 0, v \in \mathbb{R},$$

$$(iv)\quad uk_2(u,v) \text{sgn}(u) |u|^\delta - 1 \geq \alpha_2 |k_2(u,v)|^\delta \quad \text{for some } \alpha_2 > 0 \text{ and for all } u \neq 0, v \in \mathbb{R}.$$

Under assumptions including $(iii) - (iv)$, authors in [8] generalize the well-known interval criteria obtained in [1] and [20] (which are the earliest oscillation criteria of interval type in the literature) from linear forced second-order differential equations to general type of equations such as (1). Furthermore, authors in [3] have obtained Kamanov type interval oscillation criteria for the same equation under some assumptions including $(iii) - (iv)$. With aid of the assumptions $(iii) - (iv)$ many different type of interval oscillation criteria have been obtained (see for instance: [4, 6, 13, 21, 22]). When $e(t) = 0$, assumptions $(i) - (ii)$ have been widely considered (see for instance: [5, 7, 9, 11, 14, 19, 24]).

According to a discussion from [12], one can take $k_1(u,v) = v$ so that the following very particular case of assumption of $(C_3)$ for $\beta = 2$ holds: $uk_1(u,v) = k_1^2(u,v)$. Using this equality in $(C_6)$, we conclude that the function $k_2$ can be discontinuous at $u = 0$, which is a contradiction with the assumption that $k_2$ is a continuous function in all its variables. In such a case, instead of condition $(C_4)$ we propose the more general one:

$$(C_4^* \quad uvk_2(u,v) \geq 0 \quad \text{for all } (u,v) \in \mathbb{R}^2.$$
In some of our results, instead of the pair of assumptions \((C_3) - (C_4)\), we propose to consider \((C^*_3) - (C_4)\).

After this introductory section, we will state our main results, corollaries and some remarks in the next section. Some illustrative examples will be also provided in the next section. The proofs of our results will be given in the last section of this paper.

## 2 Main results, remarks and applications

**Theorem 2.1.** Let assumptions \((C_1 - C_4)\) hold. For a function \(C(t)\), denote \(c_i := \int_{s_i}^t C(t) \, dt > 0\) for \(i = 1, 2\). Then equation (1) is oscillatory provided there exist a real parameter \(\lambda > 0\) and a function \(C \in L^1((s_1, t_1) \cup (s_2, t_2), \mathbb{R}_+)\) such that

\[
\frac{1}{c_i} C(t) \leq \frac{\beta}{2\pi} \sin \frac{\pi}{\beta} \min \left\{ \frac{\alpha_1}{(\lambda r(t))^{\beta-1}}, \frac{\alpha_2 p(t)}{\lambda^{\delta-1} r(t)} \right\} \lambda Q(t)
\]

for \(t \in [s_i, t_i], i = 1, 2\), where

\[
Q(t) = \begin{cases} 
q(t), & \text{if } \gamma = 1 \\
\gamma (\gamma - 1) \frac{1 - \gamma}{\gamma} \frac{q''(t)}{q'(t)} \frac{1}{\gamma^2}, & \text{if } \gamma > 1.
\end{cases}
\]

**Theorem 2.2.** Let assumptions \((C_1 - C_3)\) and \((C^*_4)\) hold. Then equation (1) is oscillatory provided there exist a real parameter \(\lambda > 0\) and a function \(C \in L^1((s_1, t_1) \cup (s_2, t_2), \mathbb{R}_+)\) such that

\[
\frac{1}{c_i} C(t) \leq \frac{\beta}{2\pi} \sin \frac{\pi}{\beta} \min \left\{ \frac{\alpha_1}{(\lambda r(t))^{\beta-1}}, \lambda Q(t) \right\}
\]

for \(t \in [s_i, t_i], i = 1, 2\), where the numbers \(c_i\) and the function \(Q(t)\) are defined in the statement of Theorem 2.1.

**Theorem 2.3.** Let assumptions \((C_1 - C_2)\), \((C^*_3)\) and \((C_4)\) hold. Then equation (1) is oscillatory provided there exist a real parameter \(\lambda > 0\) and a function \(C \in L^1((s_1, t_1) \cup (s_2, t_2), \mathbb{R}_+)\) such that

\[
\frac{1}{c_i} C(t) \leq \frac{\beta}{2\pi} \sin \frac{\pi}{\beta} \min \left\{ \frac{\alpha_2 p(t)}{\lambda^{\delta-1} r(t)} \right\} \lambda Q(t)
\]

for \(t \in [s_i, t_i], i = 1, 2\), where the numbers \(c_i\) and the function \(Q(t)\) are defined in the statement of Theorem 2.1.

**Corollary 2.4.** Let assumptions \((C_1 - C_4)\) hold and there exists a positive constant \(\eta_0\) such that \(t_i - s_i \geq \eta_0\) for \(i = 1, 2\). Let \(r_0, p_0\) and \(Q_0\) be positive constants such that \(r(t) \leq r_0, p(t) \geq p_0\) and \(Q(t) \geq Q_0\) for \(t \in [s_1, t_1] \cup [s_2, t_2]\). Then equation (1) is oscillatory provided one of the following hypotheses is satisfied:

- \((H_1)\) conditions \((C_3 - C_4)\) hold and there exists a real parameter \(\lambda > 0\) such that

\[
\frac{1}{\eta_0} \leq \frac{\beta}{2\pi} \sin \frac{\pi}{\beta} \min \left\{ \frac{\alpha_1}{(\lambda r_0)^{\beta-1}}, \frac{\alpha_2 p_0}{\lambda^{\delta-1} r_0^\delta} \lambda Q_0 \right\}:
\]

- \((H_2)\) conditions \((C_3)\), \((C^*_4)\) hold and there exists a real parameter \(\lambda > 0\) such that

\[
\frac{1}{\eta_0} \leq \frac{\beta}{2\pi} \sin \frac{\pi}{\beta} \min \left\{ \frac{\alpha_1}{(\lambda r_0)^{\beta-1}}, \lambda Q_0 \right\}:
\]

- \((H_3)\) conditions \((C^*_3)\), \((C_4)\) hold and there exists a real parameter \(\lambda > 0\) such that

\[
\frac{1}{\eta_0} \leq \frac{\beta}{2\pi} \sin \frac{\pi}{\beta} \min \left\{ \frac{\alpha_2 p_0}{\lambda^{\delta-1} r_0^\delta}, \lambda Q_0 \right\}.
\]
Corollary 2.5. Let assumptions \((C_1 - C_2)\) and either \((C_3 - C_4)\) or \((C_3^* - C_4^*)\) or \((C_3^* - C_4)\) hold. Let the positive constant \(\eta_0\) be defined as before and there exist \(r_0, p_0, Q_0, \rho, \mu\) and \(v\) such that \(r (t) \leq r_0 t^{-\rho}\), \(p (t) \geq p_0 t^\mu\) and \(Q (t) \geq Q_0 t^v\) for \(t \in [s_1, t_1] \cup [s_2, t_2]\). Then equation (1) is oscillatory.

Remark 2.6. Although the functions \(r (t), p (t)\) and \(q (t)\) are supposed to be nonnegative on \([s_1, t_1] \cup [s_2, t_2]\), they can change sign on \([0, \infty)\). For instance, if \(p (t) = q (t) = \sin t\) and \(e (t) = \cos t\), it is easy to check that the assumption \((C_1)\) holds for \(s_1 = \left(\frac{1}{2} + 2n\right) \pi, t_1 = (1 + 2n) \pi, s_2 = (2 + 2n) \pi\) and \(t_2 = \left(\frac{5}{2} + 2n\right) \pi, n \in \mathbb{N}\).

Remark 2.7. We consider the following special case of Eq. (1):

\[
(r (t) \Phi_1 (x) x')' + p (t) \Phi_2 (x)x' + q (t) f (x) = e (t)
\]

for \(t \geq t_0 > 0, i.e. k_1 (u, v) = \Phi_1 (u) v, k_2 (u, v) = \Phi_2 (u) v\) and \(F (t, x, u, v, w) = q (t) f (x)\) in (1). Suppose that the assumption \((C_1)\) holds. It is easy to check that following assertions:

- Assumption \((C_2)\) holds provided \(q (t) > 0\) for \(t \in [s_1, t_1] \cup [s_2, t_2]\) and \(f (x) / x \geq |x|^{\gamma - 1}\) for \(x \in \mathbb{R} - \{0\}\).
- Assumption \((C_3)\) and \((C_4)\) hold with \(\beta = \delta = 2\) provided \(0 \leq \Phi_1 (u) \leq \frac{1}{\alpha} \Phi_1 (u)\) and \(u \Phi_2 (u) \geq \alpha \Phi_1 (u)\).
- Assumption \((C_4^*)\) holds with \(\beta = 2\) provided \(0 \leq \Phi_1 (u) \leq \frac{1}{\alpha} \Phi_1 (u)\) and \(u \Phi_2 (u) \geq 0\).
- Assumption \((C_3^*)\) holds with \(\delta = 2\) provided \(\Phi_1 (u) \geq 0\) and \(u \Phi_2 (u) \geq \alpha \Phi_1 (u)\).

Example 2.8. Consider the following second-order nonlinear functional differential equation

\[
\dot{x}'' (t) + x (t) \left(\dot{x}'' (t)\right)^2 + x (t) \left[1 + \sum_{k=1}^{m} b_k \left(x \left(\dot{x} (t)\right)^{2k} + \dot{x} \left(\dot{x} (t)\right)^{2k}\right)\right] = \sin t
\]

where \(t \geq t_0 > 0, b_k \geq 0\) and the integer \(m \geq 1\). Note that the functions \(r (t) = p (t) = 1\) and \(e (t) = \sin t\) satisfy the condition \((C_1)\) with \(s_1 = k \pi, t_1 = s_2 = (k + 1) \pi\) and \(t_2 = (k + 2) \pi\) for some sufficiently large integer \(k\). Condition \((C_2)\) is also satisfied for the function \(F (t, x, u, v, w) = \sin \left[\sum_{k=1}^{m} b_k \left(u^{2k} + w^{2k}\right)\right] - q (t) f (x)\) in (1) and \(\gamma = 1\). Furthermore, the functions \(k_1 (u, v) = v\) and \(k_2 (u, v) = v\) satisfy \((C_1 - C_4^*)\) with \(\alpha_1 = 1\) and \(\beta = 2\).

According to Theorem 2.2, (9) is oscillatory provided there exist a constant \(\lambda > 0\) and a function \(C \in L^1 (I, \mathbb{R}_+^+)\) such that the inequalities

\[
\frac{1}{\int_{k \pi}^{(k+1) \pi} C (s) \, ds} C (t) \leq \frac{1}{\pi} \min \left\{ \frac{1}{\lambda}, \frac{1}{\lambda} \right\}
\]

and

\[
\frac{1}{\int_{(k+1) \pi}^{(k+2) \pi} C (s) \, ds} C (t) \leq \frac{1}{\pi} \min \left\{ \frac{1}{\lambda}, \frac{1}{\lambda} \right\}
\]

where \(I := (k \pi, (k + 1) \pi) \cup ((k + 1) \pi, (k + 2) \pi)\). Choosing \(C (t) = 1\) and \(\lambda = 1\), we see that (9) is oscillatory from Theorem 2.2.

Furthermore, according to Corollary 2.4 (H2) for constants \(\eta_0 = \pi, r_0 = p_0 = Q_0 = 1\), (9) is oscillatory provided there exists a constant \(\lambda > 0\) such that the inequality

\[
\frac{1}{\pi} \leq \frac{1}{\pi} \min \left\{ \frac{1}{\lambda}, \frac{1}{\lambda} \right\}
\]

holds. Note that the inequalities (10) and (11) coincide with the equality (12) with choosing \(C (t) = 1\). Thus, with choosing \(\lambda = 1\), oscillation of (9) also comes from Corollary 2.4.

Remark 2.9. Since the functional term contained in (9) has a very general form, oscillation of the Eq. (9) cannot be demonstrated by other oscillation criteria given in the papers cited in this study, except [15, 23]. In Example 2.8, we have considered a very simple case \(k_1 (u, v) = v\), but if we consider Eq. (9) replacing \(k_1 (u, v) = v\) with \(k_1 (u, v) = \frac{v}{1+e^v}\), i.e.,

\[
\left[\frac{x'' (t)}{1 + (x'' (t))^2}\right] + x (t) \left(x'' (t)\right)^2 + x (t) \left[1 + \sum_{k=1}^{m} b_k \left(x \left(\dot{x} (t)\right)^{2k} + \dot{x} \left(\dot{x} (t)\right)^{2k}\right)\right] = \sin t
\]
we see that the condition $(C_3)$ holds with $\alpha_1 = 1$ and $\beta = 2$. Thus, oscillation of this equation comes from Theorem 2.2 or Corollary 2.4 again. But the results given in the papers [15, 23] are not applicable to this equation, since they have considered very special cases of the term $k_1(u, v)$.

Example 2.10. Consider the differential equation
\[
[k_1(x(t), x'(t))'] + \sin(2nt)k_2(x(t), x'(t))x'(t) + q_0 \sin(2nt)f(x(t), x(\tau(t)), x'(t), x'(\tau(t))) = e_0 \sin t
\]  
(13)
where $t \geq t_0 > 0$, $q_0, e_0 > 0$, $b_k \geq 0$, $n \in \mathbb{N}$, the integer $m \geq 1$, the functions $k_1(u, v)$ and $k_2(u, v)$ satisfy the conditions $(C_3 - C_4)$ with $\alpha_1 = \alpha_2 = 1$, $\beta = \delta = 2$, and the function $f(x, u, v, w)$ satisfy $f(x, u, v, w)/x \geq K$ for some positive constant $K$ and all $x \neq 0, u, v, w \in \mathbb{R}$. The functions $r(t) = 1$, $p(t) = \sin(2nt)$ and $e(t) = e_0 \sin t$ satisfy the condition $(C_1)$ with $s_1 = \frac{6n+1}{6n} \pi + 2k\pi$, $t_1 = \frac{3n+1}{3n} \pi + 2k\pi$, $s_2 = \frac{12n+1}{6n} \pi + 2k\pi$ and $t_2 = \frac{6n+1}{3n} \pi + 2k\pi$.

More general, if the functions $k_1(u, v)$ and $k_2(u, v)$ satisfy the conditions $(C_3 - C_4)$ with different $\alpha_1$, $\alpha_2$, $\beta$ and $\delta$, then (13) is oscillatory provided
\[
q_0 \geq \frac{48n^2}{K}.
\]

According to Corollary 2.4 $(H_1)$ for constants $\eta_0 = \frac{\pi}{6n}$, $r_0 = 1$, $p_0 = \frac{\sqrt{3}}{\lambda_0}$ and $Q_0 = q_0K\frac{\sqrt{3}}{2}$, it is easy to check that (13) is oscillatory proved

Example 2.11. By similar calculations with previous example, we observe the following conclusion. Consider the equation
\[
[k_1(x(t), x'(t))'] + \cos(2nt)k_2(x(t), x'(t))x'(t) + q_0 \cos(2nt)f(x(t), x(\tau(t)), x'(t), x'(\tau(t))) = e_0 \cos t
\]  
(14)
where $t \geq t_0 > 0$, $q_0, e_0 > 0$, $b_k \geq 0$, $n \in \mathbb{N}$, the integer $m \geq 1$, the functions $k_1(u, v)$ and $k_2(u, v)$ satisfy the conditions $(C_3 - C_4)$ with $\alpha_1 = \alpha_2 = 1$, $\beta = \delta = 2$, and the function $f(x, u, v, w)$ satisfy $f(x, u, v, w)/x \geq 1$ for all $x \neq 0, u, v, w \in \mathbb{R}$. According to Corollary 4, (14) is oscillatory proved
\[
q_0 \geq 9n^2.
\]

3 Proof of the main results

In this section, we give the proof of our main results stated in the previous section. First we state two lemmas which will be used as important tools in our proof.

Lemma 3.1 (Young’s inequality, [2]). If $A$ and $B$ are non-negative constants and $m, n \in \mathbb{R}$ such that $\frac{1}{m} + \frac{1}{n} = 1$, then
\[
\frac{1}{m}A + \frac{1}{n}B \geq A^{1/m}B^{1/n}.
\]

Lemma 3.2 ([9]). Let $T_1$ and $T_2$ be two arbitrary real numbers such that $T_1 < T_2$. Let $\psi(t)$ and $\phi(t)$ be functions such that $\psi, \phi \in C^1([T_1, T_2], \mathbb{R}) \cap C([T_1, T_2], \mathbb{R})$ and satisfy
\[
\psi' \leq h(t, \psi) \quad \text{and} \quad \phi' \geq h(t, \phi), t \in (T_1, T_2),
\]
where $h(t, u)$ is a locally Lipschitz function in the second variable. Then we have:
\[
\psi(T_1) \leq \phi(T_1) \quad \text{implies} \quad \psi(t) \leq \phi(t) \quad \text{for all} \quad t \in [T_1, T_2).
\]
Let us remark that, as usually, a function \( h : [0, \infty) \times \mathbb{R} \to \mathbb{R} \) is said to be locally Lipschitz in second variable if for any interval \([a, b] \subset [0, \infty)\) and \(M > 0\), there exists a function \( L := L(t) \in C^1((a, b), \mathbb{R}_+)\) depending on \([a, b], M\) and \(h\) such that

\[
|h(t, u_1) - h(t, u_2)| \leq L(t)|u_1 - u_2|
\]

for all \(t \in [a, b], u_1, u_2 \in \mathbb{R}, |u_1| \leq M\) and \(|u_2| \leq M\). Now we can state the proof of Theorem 2.1.

**Proof of Theorem 2.1.** On the contrary, suppose that (1) has a nonoscillatory solution \(x(t)\). Then \(x(t)\) eventually must have one sign, i.e. \(x(t) \neq 0\) on \([T_0, \infty)\) for some large \(T_0 \geq t_0\). Define

\[
w(t) = -\frac{\lambda r(t) k_1(x(t), x'(t))}{x(t)}
\]

for \(t \geq T_0\) and \(\lambda > 0\). Differentiating (15) and using (1) we obtain

\[
w'(t) = \frac{\lambda r(t)}{x^2(t)} k_1(x(t), x'(t)) x'(t) - \frac{\lambda}{x(t)} \left[r(t) k_1(x(t), x'(t))\right]
\]

\[
= \frac{\lambda r(t)}{x^2(t)} k_1(x(t), x'(t)) x'(t) + \frac{\lambda p(t)}{x(t)} k_2(x(t), x'(t)) x'(t)
\]

\[
+ \frac{\lambda F(t, x(t), x'(t), x''(t))}{x(t)} - \frac{\lambda e(t)}{x(t)}
\]

(16)

for \(t \geq T_0\). By assuming (C1), if \(x(t) > 0\), then we can choose \(s_1, t_1 \geq T_0\) such that \(e(t) \leq 0\) for \(t \in [s_1, t_1]\). Similarly if \(x(t) < 0\), then we can choose \(s_2, t_2 \geq T_0\) such that \(e(t) \geq 0\) for \(t \in [s_2, t_2]\). So \(e(t)/x(t) \leq 0\) (i.e. \(-e(t)/x(t) = e(t)/x(t)\)) for \(t \in [s_i, t_i], i = 1, 2\) and from (16) one can deduce

\[
w'(t) = \frac{\lambda r(t)}{x^2(t)} k_1(x(t), x'(t)) x'(t) + \frac{\lambda p(t)}{x(t)} k_2(x(t), x'(t)) x'(t)
\]

\[
+ \frac{\lambda F(t, x(t), x'(t), x''(t))}{x(t)}
\]

(17)

for \(t \in [s_1, t_1] \text{ or } t \in [s_2, t_2]\). Then, by assuming (C2 - C4) we easily obtain:

\[
w'(t) \geq \frac{\alpha_1}{(\lambda r(t))^{\beta-1}} |w(t)|^{\beta} + \frac{\alpha_2 p(t)}{\lambda^{\delta-1} r^{\delta}(t)} |w(t)|^{\delta} + \frac{\lambda}{x(t)} \left(q(t) |x(t)|^{\delta-1} + \frac{e(t)}{x(t)} \right)
\]

(18)

for \(t \in [s_1, t_1] \text{ or } t \in [s_2, t_2]\). For \(\gamma > 1\), by setting \(m = \gamma, n = \frac{\gamma}{\gamma-1}, A = \gamma q(t) |x(t)|^{\gamma-1}, B = \frac{\gamma}{\gamma-1} \left|\frac{e(t)}{x(t)}\right|\) and using Lemma 3.1, we obtain

\[
q(t) |x(t)|^{\gamma-1} + \frac{\gamma}{\gamma-1} \left|\frac{e(t)}{x(t)}\right| \geq Q(t).
\]

(19)

Note that inequality (19) trivially holds for \(\gamma = 1\). So, by combining (18) and (19), we obtain

\[
w'(t) \geq \frac{\alpha_1}{(\lambda r(t))^{\beta-1}} |w(t)|^{\beta} + \frac{\alpha_2 p(t)}{\lambda^{\delta-1} r^{\delta}(t)} |w(t)|^{\delta} + \lambda Q(t)
\]

(20)

for \(t \in [s_1, t_1] \text{ or } t \in [s_2, t_2]\).

On the other hand, for \(\beta > 1\) and two arbitrary real numbers \(R_1\) and \(R_2\) let the numbers \(a_1\) and \(a_2\) be defined with

\[
a_i \in (-\pi\beta, \pi\beta), \pi\beta = \frac{\pi}{\sin \frac{\pi}{\beta}} \text{ and } y(a_i) = R_i, i = 1, 2
\]

(21)

where \(y : (-\pi\beta, \pi\beta) \to \mathbb{R}\) is an injective (increasing) function which satisfies

\[
\begin{align*}
y & \in C^1((-\pi\beta, \pi\beta), \mathbb{R}) \\
y'(t) & = 1 + |y(t)|^{\beta}, t \in (-\pi\beta, \pi\beta), \\
y(0) & = 0 \text{ and } y(\pi\beta) = \infty.
\end{align*}
\]

(22)
For $\beta = 2$, $\pi_\beta$ becomes $\frac{\pi}{2}$ and we can take $y (t) = \tan t$. For general, such a function exists and can be determined explicitly by $y (t) = z^{-1} (t)$, where $z^{-1}$ is the inverse function of $z = z (t)$ and $z : \mathbb{R} \to (-\pi_\beta, \pi_\beta)$ is a bijective function defined by $z (t) = \int_{t_0}^t \frac{1}{1+|s|^\beta} \, ds$.

From (2) and (21), it follows that the functions $W_1$ and $W_2$ defined by $W_i (t) := a_i + \frac{2\pi_\beta}{c_i} \int_{s_i}^t C (\tau) \, d\tau$ for $t \in [s_i, t_i]$ and $i = 1, 2$ satisfy

$W_i (s_i) \leq \pi_\beta$ and $W_i (t_i) \geq \pi_\beta$, $i = 1, 2$. (23)

Since $C \in L^1 ([s_1, t_1) \cup (s_2, t_2), \mathbb{R}_+)$, we conclude that $W_i \in AC ([s_i, t_i], \mathbb{R})$ which together $C (t)/c_i \geq 0$ and (23) ensures the existence of two points $T_i^* \in (s_i, t_i)$ such that

$W_i (T_i^*) = \pi_\beta$ and $W_i : [s_i, T_i^*] \to [a_i, \pi_\beta] \subset (-\pi_\beta, \pi_\beta)$, $i = 1, 2$. (24)

Hence, according to (21), (22) and (24) we conclude that two functions $w_1$ and $w_2$ defined with

$w_i (t) := y (W_i (t))$ for $t \in [s_i, T_i^*)$ and $i = 1, 2$ (25)

are well defined and satisfy

$w_i (s_i) = y (W_i (s_i)) = y (a_i) = R_i$ (26)

and

\[ \lim_{t \to T_i^*} w_i (t) = y (W_i (T_i^*)) = y (\pi_\beta) = \infty \] (27)

for $i = 1, 2$.

Next, from the definition of $W_i$, we have $W_i' (t) = \frac{2\pi_\beta}{c_i} C (t)$ and hence from (2) and (22) we have for $[s_i, T_i^*)$:

$W_i' (t) = y' (W_i (t)) W_i' (t) = \frac{2\pi_\beta}{c_i} C (t) \left( 1 + |w_i (t)|^\beta \right)$

\[ \leq \frac{2\pi_\beta}{c_i} C (t) \left( 1 + |w_i (t)|^\beta + |w_i (t)|^\delta \right) \]

\[ \leq \frac{\alpha_1}{(\lambda r (t))^{\beta-1}} |w_i (t)|^\beta + \frac{\alpha_2 p (t)}{\lambda_1^{\delta-1} r (t)} |w_i (t)|^\delta + \lambda Q (t). \] (28)

Now, define the function

$\hat{h} (t, u) := \frac{\alpha_1}{(\lambda r (t))^{\beta-1}} |u|^\beta + \frac{\alpha_2 p (t)}{\lambda_1^{\delta-1} r (t)} |u|^\delta + \lambda Q (t). \] (29)

Since $\frac{\alpha_1}{(\lambda r (t))^{\beta-1}}$, $\frac{\alpha_2 p (t)}{\lambda_1^{\delta-1} r (t)}$ and $\lambda Q (t)$ are continuous functions, it is easy to check that $\hat{h} (t, u)$ is locally Lipschitz function in second variable. Thus, From (20) and (28) we have

$w' (t) \geq \hat{h} (t, u)$ for $t \in [s_1, t_1]$ or $t \in [s_2, t_2]$, (30)

$w' (t) \leq \hat{h} (t, u)$ for $t \in [s_1, T_1^*)$, (31)

$w'_2 (t) \leq \hat{h} (t, u)$ for $t \in [s_2, T_2^*)$, (32)

where the functions $w, w_1$ and $w_2$ are defined with (15) and (25).

Now, by using Lemma 3.2 with (30), (31), (32) and taking (27) into account, we obtain

$\infty = \lim_{t \to T_1^*} w_1 (t) \leq \lim_{t \to T_1^*} w (t)$ for $t \in [s_1, t_1]$

$\infty = \lim_{t \to T_2^*} w_2 (t) \leq \lim_{t \to T_2^*} w (t)$ for $t \in [s_2, t_2]$, which contradicts to the fact that $w \in C ([s_i, t_i], \mathbb{R})$ for $i = 1, 2$. In conclusion, there is no nonoscillatory solution of Eq. (1). The proof is complete.

\[ \square \]

Remark 3.3. The proofs of Theorems 2.2 and 2.3 is similar with the proof of Theorem 2.1, hence omitted. The proofs of Corollaries 2.4 and 2.5 are immediately from Theorems 2.1, 2.2 and 2.3 in particular for $C (t) = 1$. 

Unauthenticated
References