Notes on monotonically metacompact generalized ordered spaces

Abstract: In this paper, we show that any generalized ordered space $X$ is monotonically (countably) metacompact if and only if the subspace $X - \{x\}$ is monotonically (countably) metacompact for every point $x$ of $X$ and monotone (countable) metacompact property is hereditary with respect to convex (open) subsets in generalized ordered spaces. In addition, we show the equivalence of two questions posed by H.R. Bennett, K.P. Hart and D.J. Lutzer.

Keywords: Monotonically metacompact, Monotonically countably metacompact, Generalized ordered spaces

MSC: 54F05, 54D20

1 Introduction

A topological space $X$ is monotonically (countably) metacompact if there is an operator $r$ which assigns to each (countable) open cover $U$ of $X$ an open point-finite cover $r(U)$ of $X$ that refines $U$ such that if $U$ refines $V$ then $r(U)$ refines $r(V)$. The operator $r$ is called a monotone (countable) metacompactness operator for $X$. The concepts were first introduced in [1, 5]. H.R. Bennett, K.P. Hart and D.J. Lutzer proved that any metacompact Moore space is monotonically metacompact, any monotonically (countably) metacompact generalized ordered space is hereditarily paracompact [1]. They also posed some questions on monotone (countable) metacompactness:

Question 1 ([1, Question 4.7]). Characterize monotone (countable) metacompactness in generalized ordered spaces.

Question 2 ([1, Question 4.9]). Is monotone (countable) metacompactness a hereditary property among generalized ordered spaces?

Question 3 ([1, Question 4.10]). Let $(X, \tau, <)$ be a monotonically (countably) metacompact generalized ordered space. If $S \subseteq X$ and $\tau^S = \tau \cup \{\{s\} : s \in S\}$, is the generalized ordered space $(X, \tau^S, <)$ also monotonically (countably) metacompact?

Throughout this paper, for a generalized ordered (GO) space $(X, \tau, <)$, we define $I_{\tau X} = \{x \in X : x$ is an isolated point of $X\}$, $L_{\tau X} = \{x \in X - I_{\tau X} : (\leftarrow, x) \in \tau\}$, $R_{\tau X} = \{x \in X - I_{\tau X} : (x, \rightarrow) \in \tau\}$, $E_{\tau X} = X - (I_{\tau X} \cup L_{\tau X} \cup R_{\tau X})$. For two collections $\mathcal{U}$ and $\mathcal{V}$, we write $\mathcal{U} \prec \mathcal{V}$ to mean that for each $U \in \mathcal{U}$ there is some $V \in \mathcal{V}$ with $U \subseteq V$.

In this paper, we show that any GO-space $(X, \tau, <)$ is monotonically (countably) metacompact if its subspace $X - \{x\}$ is monotonically (countably) metacompact for every point $x$ of $X$, which would be helpful for solving Question 1. We also show that any monotonically (countably) metacompact GO-space is hereditarily with respect to

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Research Article

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Throughout this paper, for a generalized ordered (GO) space $(X, \tau, <)$, we define $I_{\tau X} = \{x \in X : x$ is an isolated point of $X\}$, $L_{\tau X} = \{x \in X - I_{\tau X} : (\leftarrow, x) \in \tau\}$, $R_{\tau X} = \{x \in X - I_{\tau X} : (x, \rightarrow) \in \tau\}$, $E_{\tau X} = X - (I_{\tau X} \cup L_{\tau X} \cup R_{\tau X})$. For two collections $\mathcal{U}$ and $\mathcal{V}$, we write $\mathcal{U} \prec \mathcal{V}$ to mean that for each $U \in \mathcal{U}$ there is some $V \in \mathcal{V}$ with $U \subseteq V$.

In this paper, we show that any GO-space $(X, \tau, <)$ is monotonically (countably) metacompact if its subspace $X - \{x\}$ is monotonically (countably) metacompact for every point $x$ of $X$, which would be helpful for solving Question 1. We also show that any monotonically (countably) metacompact GO-space is hereditarily with respect to

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convex (open) subsets, which gives a partially answer to the question [1, Question 4.9]. Finally, we show that the question [1, Question 4.9] and the question [1, Question 4.10] are equivalent.

The set of all positive integers is denoted by $\mathbb{N}$ and $\omega$ denotes $\mathbb{N} \cup \{0\}$. For the undefined terms and notations we refer to [2, 4].

2 Main results

**Definition 2.1** ([6]). Let $L$ be a compact LOTS. For $x \in L$, put

$$0-\text{cf}(x) = \min\{|C| : C \text{ is a cofinal subset of } (\leftarrow, x)\}$$

and

$$1-\text{cf}(x) = \min\{|C| : C \text{ is a coinitial subset of } (x, \rightarrow)\},$$

$0-\text{cf}(x)$ denotes the left cofinality of $x$, and $1-\text{cf}(x)$ denotes the right one of $x$.

It is well known that a GO-space $X$ can be embedded as dense subspace into the compact LOTS $l(X)$ that is called a minimal linearly ordered compactification of $X$ (see [3]). For a GO-space $X$ and $x \in X$, the left cofinality $0-\text{cf}(x)$ means the cofinality defined in its minimal linearly ordered compactification $l(X)$. $0-\text{cf}(x) = 0$ if $x$ is the left endpoint of $X$ and $0-\text{cf}(x) = 1$ if $x$ is not the left endpoint of $X$ and the set $[x, \rightarrow)$ is open in $X$. If $[x, \rightarrow)$ is not open, then $0-\text{cf}(x)$ is the smallest cardinal $\kappa$ such that $(\leftarrow, x)$ contains a cofinal subset that is order isomorphic to a cofinal subset of $\kappa$. The right cofinality $1-\text{cf}(x)$ can be discussed similarly.

**Definition 2.2.** Let $x$ be a point of a space $X$. Then $X$ is said to be monotonically compact at $x$, if there exists an operator $r_x$ that assigns to every non-empty family $\mathcal{F}$ of neighborhoods of $x$ a non-empty finite family $r_x(\mathcal{F})$ of neighborhoods of $x$ so that $r_x(\mathcal{F})$ refines $\mathcal{F}$ and $r_x(\mathcal{G})$ refines $r_x(\mathcal{G})$ provided that $\mathcal{F}$ refines $\mathcal{G}$. In this case, $r_x$ is called a monotone compact operator at the point $x$ of $X$.

Note that any nonvoid (open) subset of a GO-space $X$ can be uniquely represented as a union of its maximal (open) convex subsets. With a slight modification of the proof of [7, Proposition 2.1], we have the following proposition.

**Proposition 2.3.** For a GO-space $X$, the following conditions are equivalent.

1. $X$ is monotonically metacompact.
2. For any cover $\mathcal{U}$ of $X$ consisting of open convex subsets, there exists a point-finite open cover $r(\mathcal{U})$ refining $\mathcal{U}$ such that if $\mathcal{V}$ is also such an open convex cover of $X$ that refines $\mathcal{U}$, then $r(\mathcal{V})$ refines $r(\mathcal{U})$.
3. For any cover $\mathcal{U}$ of $X$ consisting of open convex subsets, there exists a point-finite open cover $r(\mathcal{U})$ which also consists of convex subsets refining $\mathcal{U}$ such that if $\mathcal{V}$ is also such an open convex cover of $X$ that refines $\mathcal{U}$, then $r(\mathcal{V})$ refines $r(\mathcal{U})$.

**Proposition 2.4** ([8]). Suppose that $X$ is a GO-space and $x \in X$. Then $X$ is monotonically compact at $x$.

**Theorem 2.5.** Let $X$ be a GO-space. Then $X$ is monotonically (countably) metacompact if and only if the subspace $X - \{x\}$ is monotonically (countably) metacompact for every point $x \in X$.

**Proof.** Necessity. Suppose that $X$ is monotonically (countably) metacompact. For every point $x \in X$, both $(\leftarrow, x)$ and $(x, \rightarrow)$ are closed and open subsets of $X - \{x\}$. To prove that $X - \{x\}$ is monotonically (countably) metacompact, we only need show that $(\leftarrow, x)$ is monotonically (countably) metacompact, the other case can be proved similarly. There are three cases to consider:

(i) $0-\text{cf}(x) = 0$ or $0-\text{cf}(x) = 1$. It is clear.

(ii) $0-\text{cf}(x) = \omega$. Take a cofinal increasing sequence $\{x_i \in X | i \in \mathbb{N}\}$ of $(\leftarrow, x)$. For each $i \in \mathbb{N}$, $(\leftarrow, x_i]$ is monotonically (countably) metacompact since $(\leftarrow, x_i]$ is a closed subset of $X$. Let $\mathcal{U}$ be any open cover of $(\leftarrow, x)$
and $r_{x_i}$ a monotone (countable) metacompactness operator of the subspace $(\leftarrow, x_i)$ of $X$, $i \in \mathbb{N}$. Define

$$r_1(\mathcal{U}) = \{ V \cap (\leftarrow, x_2) \mid V \in r_{x_2}(\mathcal{U} \cap (\leftarrow, x_2)) \},$$

and

$$r_j(\mathcal{U}) = \{ V \cap (x_{j-1}, x_{j+1}) \mid V \in r_{x_{j+1}}(\mathcal{U} \cap (\leftarrow, x_{j+1})) \}, \quad i \in \mathbb{N} - \{1\},$$

and

$$r(\mathcal{U}) = \bigcup \{ r_i(\mathcal{U}) \mid i \in \mathbb{N} \},$$

where

$$\mathcal{U} \cap (\leftarrow, x_i) = \{ U \cap (\leftarrow, x_i) \mid U \in \mathcal{U} \}, \quad i \in \mathbb{N}.$$

Obviously, $r(\mathcal{U})$ is an open cover of $(\leftarrow, x)$. Since every point of $(\leftarrow, x)$ is covered by at most two collections of $r_j(\mathcal{U})$, $i \in \mathbb{N}$, $r(\mathcal{U})$ is point-finite. Then $r$ is a monotone (countable) metacompactness operator for $(\leftarrow, x)$.

(iii) \(0\)-cf\(x) \geq \omega_1. \) Take a cofinal increasing sequence $A = \{ x_\alpha(\alpha) \in l(X) \mid \alpha < 0\text{-cf}(x) \}$ with $x_0(\gamma) = \sup \{ x_\alpha(\alpha) : \alpha < \gamma \}$ for each limit ordinal $\gamma < 0\text{-cf}(x)$. Let $S_0(x) = \{ x_\alpha(\alpha) \in X \mid x_0(\alpha) < x \text{ and } x_0(\alpha) \in A \}$. Then $S_0(x)$ is homeomorphic to a subspace $H$ of $[0, 0\text{-cf}(x)]$ and a closed subset of the subspace $(\leftarrow, x)$ of $X$. If $H$ is a stationary subset, it is a contradiction since $X$ is hereditarily paracompact by [1, Proposition 3.4]. Therefore $H$ is not stationary in $[0, 0\text{-cf}(x)]$. Hence there exists a closed cofinal subset $D$ of $[0, 0\text{-cf}(x)]$ such that $D \cap H = \emptyset$. It follows that $D$ is homeomorphic to a subset $B$ of $A$ and $B \subseteq (l(X) - X$. Thus $(\leftarrow, x)$ can be presented as a union of $0\text{-cf}(x)$ many pairwise disjoint closed and open subsets of $X$. Consequently we easily obtain that $(\leftarrow, x)$ is monotonically (countably) metacompact since $X$ is monotonically (countably) metacompact.

Sufficiency. Suppose that $X - \{ x \}$ is monotonically (countably) metacompact for some point $x$ of $X$. By [8, Proposition 2.3], the GO-space $X$ is monotonically compact at $x$. Let $r_x$ be a monotone (countable) compactness operator at $x$ and $r_1$ be a monotone (countable) metacompactness operator for $X - \{ x \}$. Suppose that $\mathcal{U}$ is an open cover of $X$. Define $\mathcal{U}_x = \{ U \in \mathcal{U} \mid x \in U \}$ and $\mathcal{U}_1 = \{ U \cap (X - \{ x \}) \mid U \in \mathcal{U} \}$. Obviously $\mathcal{U}_1$ is an open collection of $X$. Define $r(\mathcal{U}) = r_x(\mathcal{U}_x) \cup r_1(\mathcal{U}_1)$. Then $r$ is a monotone (countable) metacompactness operator for $X$.

**Corollary 2.6.** If a GO-space $X$ is monotonically (countably) metacompact, then every convex subspace of $X$ is also monotonically (countably) metacompact.

**Proof.** Suppose that $X$ is monotonically (countably) metacompact and $C$ is a convex subspace of $X$. Then $\text{cl}_X(C)$ is monotonically (countably) metacompact. In addition, $\text{cl}_X(C) - C$ has at most two points since $C$ is convex in $X$. By Theorem 2.5, $C$ is monotonically (countably) metacompact.

By Corollary 2.6, we have the following corollary since any open subset of a GO-space is a union of open convex subsets.

**Corollary 2.7.** If a GO-space $X$ is monotonically (countably) metacompact, then every open subspace of $X$ is also monotonically (countably) metacompact.

Suppose $(X, \tau, <)$ is a GO-space and $S \subseteq X$. Then the family $\tau \cup \{ \{ s \} : s \in S \}$ is a base for the GO-space $(X, \tau^S, <)$. Obviously, $\tau \subseteq \tau^S$, $I_{\tau^S} = I_{\tau} \cup S$, $L_{\tau^S} = L_{\tau} - S$, $R_{\tau^S} = R_{\tau} - S$ and $E_{\tau^S} = E_{\tau} - S$. In addition we have the following proposition.

**Proposition 2.8.** Suppose $(X, \tau, <)$ is a GO-space and $S \subseteq X$. Let $Y$ be any subset of $X - S$, $\tau \cap Y = \{ U \cap Y : U \in \tau \}$ and $\tau^S \cap Y = \{ U \cap Y : U \in \tau^S \}$. Then $\tau \cap Y = \tau^S \cap Y$, i.e. the two topologies defined on $Y$, the topology $\tau \cap Y$ of the subspace $Y$ of $(X, \tau)$ and the topology $\tau^S \cap Y$ of the subspace $Y$ of $(X, \tau^S)$, coincide.

**Proof.** Let $Y \subseteq X - S$. Then

$$\tau^S \cap Y = Y \cap (\tau \cup \{ \{ x \} : x \in S \}) = \{ Y \cap U : U \in \tau \cup \{ \{ x \} : x \in S \} \} = \{ Y \cap U : U \in \tau \} = \tau \cap Y.$$

The proof is completed.
The proof of Proposition 2.8 is simplified by a referee.

**Corollary 2.9.** Suppose \((X, \tau, <)\) is a GO-space and \(S \subseteq X\). Let \(Y = X - (I_{\tau X} \cup S)\), \(\tau \cap Y = \{U \cap Y : U \in \tau\}\) and \(\tau^S \cap Y = \{U \cap Y : U \in \tau^S\}\). Then \(\tau \cap Y = \tau^S \cap Y\), i.e. the two topologies defined on \(Y\), the topology \(\tau \cap Y\) of the subspace \(Y\) of \((X, \tau)\) and the topology \(\tau^S \cap Y\) of the subspace \(Y\) of \((X, \tau^S)\), coincide.

**Lemma 2.10** ([8]). Let \((X, \tau, <)\) be a GO-space. If its subspace \(Y = X - I_{\tau X}\) is monotonically (countably) metacompact, then \(X\) is monotonically (countably) metacompact.

**Theorem 2.11.** If \((X, \tau, <)\) is a monotonically (countably) metacompact GO-space, then the following conditions are equivalent:

1. For every \(Y \subseteq X\), the subspace \(Y\) of \((X, \tau, <)\) is monotonically (countably) metacompact.
2. For every \(S \subseteq X\), the GO-space \((X, \tau^S, <)\) is monotonically (countably) metacompact.

**Proof.** (1) \(\Rightarrow\) (2) Let \(S\) be any subset of \(X\) and \(Y = X - (S \cup I_{\tau X})\). Then the subspace \(Y\) of \((X, \tau, <)\) is monotonically (countably) metacompact. By Corollary 2.9, the subspace \(Y\) of \((X, \tau^S, <)\) is monotonically (countably) metacompact. Hence the GO-space \((X, \tau^S, <)\) is monotonically (countably) metacompact by Lemma 2.10.

(2) \(\Rightarrow\) (1) Let \(Y\) be any subset of \(X\) and \(S = X - Y\). Then the subspace \(Y\) of \((X, \tau^S, <)\) is monotonically (countably) metacompact since the monotone (countably) metacompact property is hereditary with respect to closed subsets. By Corollary 2.9, the subspace \(Y\) of \((X, \tau, <)\) is monotonically (countably) metacompact.

**Proposition 2.12.** Let \((X, \tau, <)\) be a monotonically (countably) metacompact GO-space and \(S \subseteq X\). If \(S\) is open (or closed, or convex) in \((X, \tau, <)\), then \((X, \tau^S, <)\) is monotonically (countably) metacompact.

**Proof.** By the proof of Theorem 2.11, we only need to prove that the subspace \(Y = X - (S \cup I_{\tau X})\) of \((X, \tau, <)\) is monotonically (countably) metacompact. We consider three cases:

(a) \(S\) is open in \((X, \tau, <)\). Then \(Y\) is monotonically (countably) metacompact since \(Y\) is a closed subset of \((X, \tau, <)\).

(b) \(S\) is closed in \((X, \tau, <)\). Then \(X - S\) is open in \((X, \tau, <)\). Hence \(X - S\) is monotonically (countably) metacompact by Corollary 2.7. It follows that \(Y\) is monotonically (countably) metacompact since \(Y\) is closed in \(X - S\).

(c) \(S\) is convex in \((X, \tau, <)\). Then \(X - S\) is a union of at most two disjoint convex subsets of \((X, \tau, <)\). It is easy to prove that \(X - S\) is monotonically (countably) metacompact by Corollary 2.6. Hence \(Y\) is monotonically (countably) metacompact since \(Y\) is closed in \(X - S\).

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