Chaotic and hypercyclic properties of the quasi-linear Lasota equation

Abstract: In this paper we describe an explicit solution semigroup of the quasi-linear Lasota equation. By constructing the relationship of this solution semigroup with the translation semigroup we obtain some sufficient and necessary conditions for the solution semigroup of the quasi-linear Lasota equation to be hypercyclic or chaotic respectively.

Keywords: Lasota equation, $C_0$-semigroup

MSC: 35K30, 47D03

1 Introduction and preliminaries

In this paper we consider the first-order partial differential equations so called the quasi-linear Lasota equation. This equation has been developed as a model of the dynamics of a self-reproducing cell population, such as the population of developing red blood cells (erythrocyte precursors). It is expressed as

$$\frac{\partial}{\partial t} u + c(x) \frac{\partial}{\partial x} u = g(x, u), \quad t \geq 0, 0 \leq x \leq 1,$$

with an initial condition

$$u(0, x) = v(x), \quad 0 \leq x \leq 1,$$

where $v$ is a continuously differentiable function, and $c$ is a continuous function defined on $[0, 1]$ with

$$c(0) = 0, c(x) > 0 \quad \text{for} \quad x \in (0, 1], \quad \text{and} \quad \int_0^1 \frac{dx}{c(x)} = \infty.$$

In this model, a cell is characterized by a single, scalar variable $x$ to represent maturity, which is normalized to have values in $[0, 1]$. The state of the population at time $t$ is characterized by a density function $u(t, \cdot)$; i.e.,

$$\int_{x_1}^{x_2} u(t, x) \, dx$$

which measures the quantity of cells that have a maturity between $x_1$ and $x_2$ at time $t$. The coefficient $c(x)$ is the velocity of cell differentiation and the function $g(x, u)$ is defined by

$$g(x, u) = u \left( p(x, u) - \frac{d}{dx} v(x) \right).$$

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where \( p(x,u) \) is the proliferation rate (the relative increase of number of cells per unit time).

When \( v(x) > 0 \), the population is stable; but if \( v(x) = 0 \), the extreme instability can occur. Examples illustrating these behaviors can be found in [9].

Let \( c(x) = x \) and \( g(x,u) = \lambda u \) with \( \lambda \) be a constant, then (1)-(2) can be formed as

\[
\frac{\partial}{\partial t} u + x \frac{\partial}{\partial x} u = \lambda u, \quad t \geq 0, \quad 0 \leq x \leq 1
\]

with the initial condition

\[
u(0,x) = v(x), \quad 0 \leq x \leq 1.
\]

It is so-called Lasota equation, studied by many authors (see e.g., [3, 4, 6, 8, 14]).

The history of the problem studied in this paper is rather long and there were several approaches to it. In [7] Lasota proved that the equation (1) is chaotic (in the sense of Auslander-Yorke). The method of studying (1) by showing that the dynamical system generated by this equation is isomorphic to the translation semigroup was invented independently in the papers [2]. In [12], Rudnicki constructed invariant measures for this dynamical system having strong ergodic properties which leads to chaos. In [13], he also proved that the solution of equation (1) is chaotic in the sense of Devaney (by using ergodic theory approach) and has another chaotic property: the existence of turbulent trajectories in the sense of Bass.

Functional-analytic tools to study chaotic behaviour of linear semigroups defined on Banach spaces were developed in [5] and they were applied to a linear version of (1).

The models of the evolution of maturity of blood cells in the bone marrow, which lead to the equation (1), were described and studied in [15]. It was shown that chaos appears when the blood production system is not controlled from outside.

An interesting relation between hypercyclicity and existence of invariant mixing measure can be found in the paper [11]. If the reader is interested in the relations between different approaches to chaos she/he may look at the review paper [16].

Here we considered the special condition as following:

\[
\begin{align*}
\frac{\partial}{\partial t} v + c(x) \frac{\partial}{\partial x} v &= k(x) v, \quad x \in [0,1], \quad t > 0; \\
u(0,x) &= f(x), \quad x \in [0,1].
\end{align*}
\]

where \( c(x) \) satisfied (3) and \( k(x) \) is a bounded continuous function on an interval \( I = [0,1] \). Moreover, if \( c(x) = -1 \) and \( k(x) = 0 \) for all \( x \in I = [0,\infty) \), then the solution semigroup \( \{ W(t) \} \) of this case is the translation semigroup, that is

\[
W(t) f(x) = f(x+t).
\]

The condition for the translation semigroup \( \{ W(t) \} \) to be hypercyclic or chaotic for different spaces is well known which was proved in [18, Theorem A]. Before we state this theorem, we would like to introduce some terminologies which will be used latter.

A \( C_0 \)-semigroup \( \{ T(t) \}_{t \geq 0} \) is called hypercyclic if there exists \( x \in X \) such that the set \( \{ T(t)x : t \geq 0 \} \) is dense in \( X \). The \( C_0 \)-semigroup \( \{ T(t) \}_{t \geq 0} \) is called chaotic if \( \{ T(t) \}_{t \geq 0} \) is hypercyclic and the set of periodic points

\[
X_{per} = \{ x \in X : \text{there exists a } t > 0 \text{ such that } T(t)x = x \}
\]

is dense in \( X \).

The function spaces concerned in Theorem A are defined by admissible weight functions.

An admissible weight function on \([0,\infty)\) is a measurable function \( \rho : [0,\infty) \to \mathbb{R} \) satisfying the following conditions:

(i) \( \rho(x) > 0 \) for all \( x \in [0,\infty) \);

(ii) there exist constant \( M \geq 1 \) and \( \omega \in \mathbb{R} \) such that \( \rho(x) \leq Me^{\omega t} \rho(t+x) \) for all \( x \in [0,\infty) \) and \( t > 0 \).

With an admissible weight function \( \rho \), we considered following function spaces:

\[
C_{0,\rho}([0,\infty), C) = \left\{ f : [0,\infty) \to C : f \text{ continuous, } \lim_{x \to \infty} \rho(x)f(x) = 0 \right\}
\]
Lemma 2.1

The following result is known.

If

(iii)

Theorem A

Let \( X \) be a measure space and consider the translation semigroup \( \{ W(t) \} \) on \( C([0, \infty), X) \) or \( L^p([0, \infty), X) \) is defined by

\[
W(t)f(x) = f(x + t) \quad \text{for} \quad f \in C_{0,\rho}([0, \infty), X) \quad \text{or} \quad f \in L^p([0, \infty), X).
\]

Under these notations, the well-known Theorem A can be stated as follows.

Theorem A ([18, Theorem A]). Let \( X \) be \( C_{0,\rho}([0, \infty), X) \) or \( L^p([0, \infty), X) \) with an admissible weight function \( \rho \) and consider the translation semigroup \( \{ W(t) \} \) on \( X ). Then

(i) \( \{ W(t) \} \) is hypercyclic if and only if \( \liminf_{x \to \infty} \rho(x) = 0 \);

(ii) if \( X \) be \( C_{0,\rho}([0, \infty), X) \), then \( \{ W(t) \} \) is chaotic if and only if \( \limsup_{x \to \infty} \rho(x) = 0 \);

(iii) if \( X \) be \( L^p([0, \infty), X) \), then \( \{ W(t) \} \) is chaotic if and only if for all \( \varepsilon > 0 \) and for all \( l > 0 \), there exists \( P > 0 \) such that \( \sum_{n=1}^{\infty} (\rho(l + nP))^p < \varepsilon \).

In Section 2, we will give the solution semigroup of (4). In Section 3, by constructing the relationship between the translation semigroup \( \{ W(t) \} \) and the solution semigroup \( \{ S(t) \} \) of the quasi-linear Lasota equation, we get the sufficient and necessary conditions for the solution semigroup \( \{ S(t) \} \) to be hypercyclic or chaotic respectively.

2 The solution semigroup

We will use semigroup modus operandi to find the solution of (4). For this purpose, we will apply the results of Batty [1], who considered the differential equation

\[
\frac{\partial T}{\partial t} = \lambda \circ T
\]

where \( \lambda : R \to R \) is a continuous function. Batty showed that the operator \( A = \lambda(x)D \) generated a \( C_0 \)-semigroup on \( C_0(R) \), where \( D \) denotes the differential operator on \( C^1(R) \).

Let

\[
Z(\lambda) = \{ x \in R : \lambda(x) = 0 \},
\]

\[
U(\lambda) = R \setminus Z(\lambda) = \{ x \in R : \lambda(x) \neq 0 \}
\]

and let \( A^+_1(\lambda) \) (respectively, \( A^-_1(\lambda) \)) be the set of all points \( x \) in \( Z(\lambda) \) such that for some \( y < x \), \( \lambda(\tau) \geq 0 \) (respectively, \( \lambda(\tau) \leq 0 \)) for every \( \tau \) in the interval \( (y,x) \), and \( \frac{1}{\lambda(\tau)} \) is integrable over \( (y,x) \). Let \( A^+_1(\lambda) \) (respectively, \( A^-_1(\lambda) \)) be the set of all points \( x \) in \( Z(\lambda) \) such that for some \( z > x \), \( \lambda(\tau) \geq 0 \) (respectively, \( \lambda(\tau) \leq 0 \)) for every \( \tau \) in the interval \( (x,z) \), and \( \frac{1}{\lambda(\tau)} \) is integrable over \( (x,z) \).

Denote

\[
A_I(\lambda) = A^+_1(\lambda) \cup A^-_1(\lambda),
\]

\[
A_F(\lambda) = A^+_1(\lambda) \cup A^-_1(\lambda),
\]

\[
A(\lambda) = A_F(\lambda) \cup A_I(\lambda).
\]

The following result is known.

Lemma 2.1 ([1, Proposition 2.7, p. 218]). Let \( \lambda : R \to R \) be continuous. Then the following are equivalent:

(i) \( \lambda D | C^\infty_c(R) \) generates a \( C_0 \)-semigroup on \( C_0(R) \), where \( C_c^\infty(R) \) is the space of all \( C^\infty \) functions on \( R \) whose support is compact and contained in \( R \).
(ii) \( A^+ (\lambda) = A^- (\lambda) = \emptyset \),

where \( D \) denotes the differential operator on \( C^1 (R) \).

For the normalized case, \( I = [0, 1] \), the equivalent conditions of Lemma 2.1 can be modified as follows:

(i) \( \lambda D \) generates a \( C_0 \)-semigroup on \( C [0, 1] \),

(ii) \( A^+ (\lambda) = A^- (\lambda) = \emptyset, \lambda (0) \geq 0, \lambda (1) \leq 0 \).

We know that (4) can be treated as the following Cauchy problem

\[
\begin{align*}
\frac{\partial}{\partial t} u &= Au + Bu; \\
\frac{\partial}{\partial x} c.0/u &= u_0 \\
\end{align*}
\]

where \( A = -cD \), with \( D (A) = \{ f : f \in C^1 [0, 1] \} \), \((cD) f (x) = c (x) f' (x) \) and \( Bu = k (x) u \).

For solving (6), we first consider the Cauchy problem without the perturbation term, \( Bu = k (x) u \).

\[
\begin{align*}
\frac{\partial}{\partial t} u &= Au; \\
\frac{\partial}{\partial x} c.0/u &= u_0 \\
\end{align*}
\]

Let \( c (x) = -\lambda (x) \), from (3) we have \( Z (\lambda) = \{ 0 \} \) and \( A^+ (-\lambda) = \{ z \in (0, 1), -c (z) \geq 0 \} \). Since \( c (z) \geq 0 \) for \( 0 < z < 1 \) we have that \( A^+ (-\lambda) = \emptyset \). On the other hand, \( A^- (-\lambda) = \{ y : y < 0, -c (y) \geq 0 \} \). Since \( c \) is only defined on \([0, 1]\), which implies \( A^- (-\lambda) = \emptyset \). Now we have \( \lambda (0) = -c (0) = 0, \lambda (1) = -c (1) \leq 0 \) and \( A^+ (-\lambda) = A^- (-\lambda) = \emptyset \). This implies that \( c \) satisfies the condition (ii) of Lemma 2.1 and \( \lambda = - (cD) \) generates a \( C_0 \)-semigroup \( \{ T (t) \}_{t \geq 0} \) on \( C [0, 1] \).

To represent the \( C_0 \)-semigroup \( \{ T (t) \}_{t \geq 0} \) in explicit form, we define a function \( q (x) \) by

\[
q (x) = -\int_{0}^{x} \frac{ds}{c (s)} \quad \text{for all} \quad 0 < x \leq 1. 
\]

Since \( c (x) > 0 \), the function \( q \) is strictly increasing on the interval \((0, 1)\). It follows that \( q \) is an one-to-one mapping, and hence \( q^{-1} \) exists on \([0, \infty)\). From (7) it is obvious that \( q' (x) = \frac{1}{c (x)} \).

We also use a auxiliary function \( h_t : [0, 1] \to [0, 1] \) in the following way

\[
h_t 0 = 0 \quad \text{if} \quad c (x) = 0 \text{ at } x = 0, \\
h_t x = q^{-1} (q (x) - t) \quad \text{if} \quad c (x) \neq 0 \text{ for all } 0 < x \leq 1, t \geq 0.
\]

Then \( \{ T (t) \} \) on \( C [0, 1] \) can be written as

\[
T (t) f (x) = f (h_t x).
\]

Since \( \int_{0}^{1} \frac{ds}{c (s)} = \infty \), \( h_t \) is well-defined for all \( t \in [0, \infty) \), and hence \( T (t) \) exists on the interval \([0, \infty)\).

According to (8), we have that

\[
\frac{\partial}{\partial t} \left( \frac{\partial}{\partial x} (T (t) f) (x) \right) = f' (h_t x) \left( \frac{\partial}{\partial x} (h_t x) \right) = f' (h_t x) \frac{q'}{q (h_t x)} = -c (h_t x) f' (h_t x) = \lambda (h_t x) f' (h_t x) = \lambda (h_t x) f' (h_t x) = \lambda (h_t x) f' (h_t x) = \lambda (h_t x) f' (h_t x) = \lambda (h_t x) f' (h_t x) = \lambda (h_t x) f' (h_t x) = \lambda (h_t x) f' (h_t x) = \lambda (h_t x) f' (h_t x) = \lambda (h_t x) f' (h_t x) = \lambda (h_t x) f' (h_t x).
\]

for all \( t > 0 \) and \( f \) in \( C^1 [0, 1] \) and

\[
\frac{\partial}{\partial x} (T (t) f) (x) = f' (h_t x) \left( \frac{\partial}{\partial x} (h_t x) \right) = f' (h_t x) \frac{q'}{q (h_t x)}.
\]

From above, the operator \( \lambda D = - (cD) \) generates a \( C_0 \)-semigroup \( \{ T (t) \}_{t \geq 0} \) on \( C [0, 1] \).
By general perturbation theorem, the solution semigroup of (6) is given by

\[ (S (t) v) (x) = \sum_{n=0}^{\infty} T_n (t) v (x) \quad \text{for} \quad v (x) \in C [0,1], \] (9)

where \( T_n (\cdot) \) is defined recursively as

\[ T_0 (t) v (x) = T (t) v (x) = v (h_t x), \]

where \( T (t) \) and \( h_t x \) are defined by (8) and

\[ T_{n+1} (t) = \int_0^t T_0 (t-s) B T_n (s) \, ds. \]

where \( B \) is defined by

\[ B v (x) = k (x) v (x). \]

By induction and integration by part we have

\[ (S (t) v) (x) = \sum_{n=0}^{\infty} T_n (t) v (x) = \exp \left( \int_0^t k (h_{t-s} x) \, ds \right) v (h_t x) \]

and \{ \( S (t) \) \} is the solution semigroup of (6).

3 Main results

Since the initial data of (4) is defined on \( I = [0,1] \), we need to slightly modify the terminologies in Section 1. Although original \( \rho \) is defined on \([0,\infty)\), we can use \( \rho \) to define an admissible weight function on \([0,1]\). We consider a measurable function \( \kappa : (0,1] \to [0,\infty) \), which is defined by

\[ \kappa (x) = \rho (-q (x)) \]

for all \( x \in (0,1] \) and \( q \) is defined by (7). Then \( \kappa (x) \) satisfies the following conditions:

(i) \( \kappa (x) > 0 \) for all \( x \in (0,1] \);
(ii) there exist constants \( M \geq 1 \) and \( \omega \in \mathbb{R} \) such that \( \kappa (x) \leq M e^{\omega t} \kappa (h_t x) \) for \( x \in (0,1] \) and \( t > 0 \); here \( h_t \) is defined by (8).

This implies that \( \kappa \) is an admissible weight function on \((0,1]\). In a similar way, as for \( C_{0,\rho} ([0,\infty), C) \) or \( L^p_\rho ([0,\infty), C) \), we can use \( \kappa \) to define the following function spaces on \((0,1]\). We define

\[ C_{0,\kappa} ((0,1], C) = \left\{ f \in C ((0,1], C) : \lim_{x \to 0} \kappa (x) f (x) = 0 \right\} \]

with \( \| f \|_\kappa = \sup_{x \in [0,1]} |\kappa (x) f (x)| \), and

\[ L^p_{\kappa} ([0,1], C) = \left\{ f : [0,1] \to C : f \text{ measurable and } \int_0^1 |\kappa (x) f (x) d x|^p < \infty \right\} \]

with \( \| f \|_{\rho,\kappa} = \left( \int_0^1 |\kappa (x) f (x) d x|^p \right)^{1/p} \) \((p \geq 1)\).
To found the relation between $W(t)$ in (5) and $T(t)$ in (8), we defined the linear map $Q : C((0,1],C) \to C([0,\infty),C)$ as

$$Q(f(x)) = f(y) = f(-q(x))$$

for $x \in (0,1]$, $y \in [0,\infty)$ and $y = -q(x)$, where $q$ is defined as in (7).

Since $q$ is an one-to-one function, the inverse of $Q$ exists, and it can be expressed as

$$Q^{-1}(g(y)) = g\left(q^{-1}(-y)\right).$$

Then we have

$$Q^{-1}W(t)Q(f(x)) = Q^{-1}W(t)f(y) = Q^{-1}f(t+y) = f\left(q^{-1}(-y-t)\right) = f(h_x) = T(t)f(x)$$

for every $f \in C_{0,x}((0,1],C)$. This shows that $Q^{-1} \circ W(t) \circ Q = T(t)$, which implies the equivalence of $W(t)$ on $C_{0,x}([0,\infty),C)$ and $T(t)$ on $C_{0,x}((0,1],C)$.

From above and Theorem A, we get following proposition.

**Proposition 3.1.** Let $X$ be $C_{0,x}((0,1],C)$ or $L^p([0,1],C)$ with an admissible weight function $\phi$ and consider the translation semigroup $\{T(t)\}$ on $X$. Then we have the following results:

(i) $T(t)$ is hypercyclic if and only if $\lim\inf_{t \to \infty} \phi\left(x(t)\right) = 0$.

(ii) If $X$ be $C_{0,x}((0,1],C)$, then $T(t)$ is chaotic if and only if $\lim_{t \to \infty} \phi\left(x(t)\right) = 0$.

(iii) If $X$ be $L^p([0,1],C)$, then $T(t)$ is chaotic if and only if for all $\epsilon > 0$ and for all $l > 0$, there exists $P > 0$ and some $x_0 \in (0,1]$ such that $\sum_{n=1}^{\infty} \left(\phi\left(h_{n,P}x_0\right)\right)^P < \epsilon$.

**Proof.** From the fact that $y = -q(x) \to \infty$ as $x \to 0$ and because of the equivalence of $W(t)$ and $T(t)$, it is easy to obtain (i) and (ii).

Since $q$ is an one-to-one function, there exists some $x_0 \in (0,1]$ such that $l = -q(x_0)$, where $l$ is appeared in condition (iii) of Theorem A. Since $\rho(y) = \rho(-q(x)) = \phi\left(x(t)\right) = \phi\left(q^{-1}(y)\right)$, we have

$$\rho\left(l + nP\right) = \phi\left(q^{-1}(-l - nP)\right) = \phi\left(q^{-1}(q(x_0) - nP)\right) = \phi\left(h_{n,P}x_0\right)$$

and

$$\sum_{n=1}^{\infty} \left(\phi\left(h_{n,P}x_0\right)\right)^P = \sum_{n=1}^{\infty} \left(\rho\left(l + nP\right)\right)^P$$

This implies that condition (iii) is also truth.

For appling Theorem A to $S(t)$, we rewrite $S(t)$ as follows:

$$(S(t)v)(x) = \exp\left(\int_0^t k(h_{t-s}x)ds\right) v(h_tx) = \exp\left(\int_{h_tx}^x \frac{k(r)}{c(r)}dr\right) v(h_tx) = \frac{\eta(x)}{\eta(h_tx)} v(h_tx),$$

where $\eta(x) = \exp\left(-\int_0^x \frac{k(r)}{c(r)}dr\right)$.

It is obvious that $\kappa'(x) = \eta(x)$ is an admissible weight function on $(0,1]$.

Followed the same idea, we can obtain a necessary and sufficient condition or sufficient condition for $S(t)$ to be chaotic or hypercyclic by showing the equivalence of $S(t)$ and $T(t)$.

Let $\Phi : C_{0,x}((0,1],C) \to C_{0}([0,1],C) = \{f \in C([0,1],C) : f(0) = 0\} \text{ [resp. } \Phi : L^p([0,1],C) \to L^p([0,1],C)\}$ be defined by $\Phi(f)(x) = \kappa'(x)f(x)$ for $f \in C_{0,x}((0,1],C)$ [resp. $f \in L^p([0,1],C)$]. Then $\Phi$ is an isomorphism and

$$\Phi(T(t)f)(x) = \kappa'(x)f(h_tx) = S(t)\Phi(f)(x).$$

This implies that $T(t)$ is chaotic [hypercyclic] on $C_{0,x}((0,1],C)$ or $L^p([0,1],C)$ if only if $S(t)$ is chaotic [hypercyclic] on $C_{0,x}((0,1],C)$ or $L^p([0,1],C)$. From above and proposition 3.1 we have the following theorem.
Theorem 3.2. Let $X$ be the space $C_0([0, 1], C)$ or $L^p([0, 1], C)$. Consider the initial value problem:

$$
\begin{cases}
\frac{d^2 u}{dt^2} + c(x) \frac{du}{dx} = k(x) u, & x \in [0, 1], t > 0; \\
u(0, x) = f(x), & x \in [0, 1].
\end{cases}
$$

where $c(x)$ satisfies (3), $k \in C([0, 1], C)$ is bounded and $f \in X$.

Then the solution semigroup $\{S(t)\}_{t \geq 0}$ can be expressed as

$$(S(t)f)(x) = \exp \left( \int_0^t T(t-s)k(x) \, ds \right) T(t)f(x),$$

where $\{T(t)\}_{t \geq 0}$ is the semigroup defined by (8). Furthermore,

(i) $\{S(t)\}$ is hypercyclic if and only if $\liminf_{x \to 0} k'(x) = 0$.

(ii) If $X$ be $C_{0,k'}((0, 1], C)$, then $\{S(t)\}$ is chaotic if and only if $\liminf_{x \to 0} k'(x) = 0$.

(iii) If $X$ be $L^p_k((0, 1], C)$, then $\{S(t)\}$ is chaotic if and only if for all $\varepsilon > 0$ and for all $l > 0$, there exists $P > 0$ and some $x_0 \in (0, 1]$ such that

$$\sum_{n=1}^{\infty} (\kappa(h_nP,x_0))^{P} < \varepsilon.$$

Denoting $\Re z$ be the real part of $z$, we have following results.

Corollary 3.3. $\{S(t)\}_{t \geq 0}$ is chaotic if and only if $\lim_{x \to 0} \int_0^1 \frac{\Re k(s)}{c(s)} \, ds = \infty$. Consequently, if $\Re k(0) > 0$, then $\{S(t)\}_{t \geq 0}$ is chaotic.

Proof. Since $\eta(x) = \exp(-\int_0^1 \frac{k(r)}{c(r)} \, dr)$ and $k'(x)$ is an admissible weight function on $(0, 1]$, we have $\lim_{x \to 0} \int_0^1 \frac{\Re k(s)}{c(s)} \, ds = \infty$ if and only if $\lim_{x \to 0} k'(x) = 0$. By Theorem 3.2, $\{S(t)\}$ is chaotic.

Theorem 3.4. If $\inf \{\Re (k(s)) : s \in [0, 1]\} = \alpha > 0$, then $\{S(t)\}$ is chaotic on $C_{0,k'}((0, 1], C)$ or $L^p((0, 1], C)$ with $p \geq 1$.

Proof. In the case $X = C_0([0, 1], C)$, consider the generator $A$ of $\{S(t)\}$ on $X$ with $D(A) = \{f \in X : c(x) f'(x) \in X\}$. Let $U = \{\lambda \in C : \Re \lambda < \alpha\}$. Then for each $\lambda \in U$, $f_\lambda(x) = \exp\left(\int_x^1 \frac{\lambda - k(s)}{c(s)} \, ds\right)$ belongs to $X$ and $f_\lambda$ is a nonzero eigenvector, i.e. $Af_\lambda = \lambda f_\lambda$. So $U$ is an open subset of the point spectrum of $A$, which intersects imaginary axis. For each $\phi \in X^*$ we defined a function $F_\phi : U \to C$ by $F_\phi(\lambda) = (\phi, x_\lambda)$. Then with the same way as in the proof of [10, Theorem 2], $F_\phi$ is analytic, and $F_\phi(\lambda) = 0$ for any $\lambda \in U$ implies that $\phi = 0$. So by applying [18, Theorem C], $\{S(t)\}$ is chaotic.

When $X = L^p((0, 1], C)$, we just need to check $f_\lambda$ belongs to $L^p((0, 1], C)$. Since $p (\Re (\lambda) - \alpha) < 0$, and $0 \leq \int_x^1 \frac{1}{c(s)} \, ds \leq \infty$ for $x \in [0, 1]$, we have

$$\int_0^1 (f_\lambda(x))^p \, dx = \int_0^1 \left( \exp\left(\int_x^1 \frac{\lambda - k(s)}{c(s)} \, ds\right) \right)^p \, dx \leq \int_0^1 \left( \exp p (\Re (\lambda) - \alpha) \left(\int_x^1 \frac{1}{c(s)} \, ds\right) \right) \, dx \leq \int_0^1 dx < \infty.$$

This implies $f_\lambda(x)$ belongs to $L^p((0, 1], C)$. Following the same method as for the case $C_0([0, 1], C)$, we get the conclusion.

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