Research Article

Khaledoun Al-Zoubi*, Imad Jaradat, and Mohammed Al-Dolat

On graded $P$-compactly packed modules

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Abstract: Let $G$ be a group with identity $e$. Let $R$ be a $G$-graded commutative ring and $M$ a graded $R$-module. In this paper, we introduce the concept of graded $P$-compactly packed modules and we give a number of results concerning such graded modules. In fact, our objective is to investigate graded $P$-compactly packed modules and examine in particular when graded $R$-modules are $P$-compactly packed. Finally, we introduce the concept of graded finitely $P$-compactly packed modules and give a number of its properties.

Keywords: Graded primary submodules, Graded $P$-compactly packed modules, Graded finitely $P$-compactly packed modules

MSC: 13A02, 16W50

1 Introduction and Preliminaries

Graded primary ideals in a commutative graded ring have been introduced and studied by M. Refai and K. Al-Zoubi in [8]. Graded primary submodules of graded modules over graded commutative rings have been studied in [3, 4]. Graded primary radical of a graded submodule over graded commutative rings have been introduced and studied by K. Al-Zoubi in [1]. Also, the concept of graded compactly packed modules was introduced by F. Farzalipour and P. Ghiassvand in [4]. Here, we generalize this concept to the concept of graded $P$-compactly packed modules and give a number of its properties. We also introduce the concept of graded finitely $P$-compactly packed modules and give some results about it.

Before we state some results, let us introduce some notations and terminologies. Let $G$ be a group with identity $e$ and $R$ be a commutative ring with identity $1_R$. Then $R$ is a $G$-graded ring if there exist additive subgroups $R_g$ of $R$ such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. We denote this by $(R, G)$ (see [6].) The elements of $R_g$ are called to be homogeneous of degree $g$ where the $R_g$’s are additive subgroups of $R$ indexed by the elements $g \in G$. If $x \in R$, then $x$ can be written uniquely as $\sum_{g \in G} x_g$, where $x_g$ is the component of $x$ in $R_g$. Moreover, $h(R) = \bigcup_{g \in G} R_g$. Let $I$ be an ideal of $R$. Then $I$ is called a graded ideal of $(R, G)$ if $I = \bigoplus_{g \in G} (I \cap R_g)$. Thus, if $x \in I$, then $x = \sum_{g \in G} x_g$ with $x_g \in I$. An ideal of a $G$-graded ring need not be $G$-graded (see [6].)

Let $R$ be a $G$-graded ring and $M$ an $R$-module. We say that $M$ is a $G$-graded $R$-module (or graded $R$-module) if there exists a family of subgroups $\{M_g\}_{g \in G}$ of $M$ such that $M = \bigoplus_{g \in G} M_g$ (as abelian groups) and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Here, $R_g M_h$ denotes the additive subgroup of $M$ consisting of all finite sums of elements $r_g s_h$ with $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = \bigcup_{g \in G} M_g$ and the elements of $h(M)$ are called to be homogeneous.

*Corresponding Author: Khaledoun Al-Zoubi: Department of Mathematics and Statistics, Jordan University of Science and Technology, P.O.Box 3030, Irbid 22110, Jordan, E-mail: kazoobi@just.edu.jo
Imad Jaradat: Department of Mathematics and Statistics, Jordan University of Science and Technology, P.O.Box 3030, Irbid 22110, Jordan
Mohammed Al-Dolat: Department of Mathematics and Statistics, Jordan University of Science and Technology, P.O.Box 3030, Irbid 22110, Jordan

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Let \( M = \bigoplus_{g \in G} M_g \) be a graded \( R \)-module and \( N \) a submodule of \( M \). Then \( N \) is called a graded submodule of \( M \) if \( N = \bigoplus_{g \in G} N_g \) where \( N_g = N \cap M_g \) for \( g \in G \). In this case, \( N_g \) is called the \( g \)-component of \( N \) (see [6].)

Let \( R \) be a \( G \)-graded ring and \( M \) a graded \( R \)-module. The graded radical of a graded ideal \( I \), denoted by \( Gr(I) \), is the set of all \( x = \sum_{g \in G} x_g \in R \) such that for each \( g \in G \) there exists \( n_g > 0 \) with \( x_g^{n_g} \in I \). Note that, if \( r \) is a homogeneous element, then \( r \in Gr(I) \) if and only if \( r^n \in I \) for some \( n \in \mathbb{N} \). A proper graded ideal \( P \) of \( R \) is said to be graded primary ideal if whenever \( r, s \in h(R) \) with \( rs \in P \), then either \( r \in P \) or \( s \in Gr(P) \) (see [8].) A proper graded submodule \( N \) of a graded \( R \)-module \( M \) is said to be graded prime submodule if whenever \( r \in h(R) \) and \( m \in h(M) \) with \( rm \in N \), then either \( r \in (N :_R M) = \{ r \in R : rM \subseteq N \} \) or \( m \in N \). A proper graded submodule \( N \) of a graded \( R \)-module \( M \) is said to be graded primary submodule if whenever \( r \in h(R) \) and \( m \in h(M) \) with \( rm \in N \), then either \( m \in N \) or \( r \in Gr((N :_R M)) \) (see [7].) The graded primary and primary submodules are different concepts (see [8, Example 1.6].) The graded radical of a graded submodule \( N \) of a graded \( R \)-module \( M \), denoted by \( Gr_M(N) \), is defined to be the intersection of all graded primary submodules of \( M \) containing \( N \). If \( N \) is not contained in any graded prime submodule of \( M \), then \( Gr_M(N) = N \) (see [7].) A graded \( R \)-module \( M \) is called graded finitely generated if there exist \( x_{g_1}, x_{g_2}, \ldots, x_{g_n} \in h(M) \) such that \( M = Rx_{g_1} + \cdots + Rx_{g_n} \). A graded \( R \)-module \( M \) is called graded cyclic if \( M = Rm_g \) where \( m_g \in h(M) \).

## 2 Graded \( P \)-compactly packed modules

In this section, we define the graded \( P \)-compactly packed modules and give a number of its properties. We also find the necessary and sufficient conditions for any graded \( R \)-module \( M \) to be graded \( P \)-compactly packed.

**Definition 2.1.** Let \( R \) be a \( G \)-graded ring, \( M \) a graded \( R \)-module and \( N \) a proper graded submodule of \( M \). \( N \) is called graded \( P \)-compactly packed if whenever \( N \) is contained in the union of a family of graded primary submodules of \( M \), \( N \) is contained in one of the graded primary submodules of the family. \( M \) is called graded \( P \)-compactly packed if every proper graded submodule of \( M \) is graded \( P \)-compactly packed.

**Lemma 2.2** ([4, Lemma 2.1]). Let \( R \) be a \( G \)-graded ring and \( M \) a graded \( R \)-module. Then the following hold:

(i) If \( I \) and \( J \) are graded ideals of \( R \), then \( I + J \) and \( I \cap J \) are graded ideals.

(ii) If \( N \) is a graded submodule of \( M \), \( r \in h(R) \), \( x \in h(M) \) and \( I \) is a graded ideal of \( R \), then \( Rx \), \( IN \) and \( rN \) are graded submodules of \( M \).

(iii) If \( N \) and \( K \) are graded submodules of \( M \), then \( N + K \) and \( N \cap K \) are also graded submodules of \( M \) and \( (N :_R M) = \{ r \in R : rM \subseteq N \} \) is a graded ideal of \( R \).

(iv) Let \( \{N_\lambda\} \) be a collection of graded submodules of \( M \). Then \( \sum_{\lambda} N_\lambda \) and \( \cap_{\lambda} N_\lambda \) are graded submodules of \( M \).

The graded primary radical of a graded submodule \( N \) of a graded \( R \)-module \( M \), denoted by \( P-Gr_M(N) \), is defined to be the intersection of all graded primary submodules of \( M \) containing \( N \). Should there be no graded primary submodule of \( M \) containing \( N \), then we put \( P-Gr_M(N) = M \). It is easy to see that \( P-Gr_M(N) \) is a graded submodule of \( M \) containing \( N \). We say \( N \) is a graded primary radical submodule if \( P-Gr_M(N) = N \) (see [1, Definition 2.2].)

**Theorem 2.3.** Let \( R \) be a \( G \)-graded ring and \( M \) a graded \( R \)-module. Then the following statements are equivalent:

(i) \( M \) is a graded \( P \)-compactly packed.

(ii) For each proper graded submodule \( N \) of \( M \), there exists \( n_g \in N \cap h(M) \) such that \( P-Gr_M(N) = P-Gr_M(Rn_g) \).

(iii) For each proper graded submodule \( N \) of \( M \), if \( \{P_\alpha\}_{\alpha \in \Delta} \) is a family of graded submodules of \( M \) and \( N \subseteq \bigcup_{\alpha \in \Delta} P_\alpha \), then \( N \subseteq P-Gr_M(P_\beta) \) for some \( \beta \in \Delta \).

(iv) For each proper graded submodule \( N \) of \( M \), if \( \{P_\alpha\}_{\alpha \in \Delta} \) is a family of graded primary radical submodules of \( M \) and \( N \subseteq \bigcup_{\alpha \in \Delta} P_\alpha \), then \( N \subseteq P_\beta \) for some \( \beta \in \Delta \).
Proof. (i)⇒(ii) Assume (i) holds and let \( N \) be a proper graded submodule of \( M \). By [1, Theorem 2.4], \( PGR_M (RN_g) \subseteq PGR_M (N) \) for each \( n_g \in N \cap h(M) \). Now, suppose that \( PGR_M (N) \not\subseteq PGR_M (RN_g) \) for each \( n_g \in N \cap h(M) \). Then for each \( n \in N \cap h(M) \) there exists a graded primary submodule \( P_{n_g} \) for which \( RN_g \subseteq P_{n_g} \) and \( N \not\subseteq P_{n_g} \). But \( N = \cup_{n_g \in N} Rn_g \subseteq \cup_{n_g \in N} P_{n_g} \), that is \( M \) is not \( P \)-compactly packed, a contradiction.

(ii)⇒(iii) Assume (ii) holds. Let \( N \) be a proper graded submodule of \( M \) and let \( \{ P_{\alpha} \}_{\alpha \in \Delta} \) be a family of graded submodules of \( M \) such that \( N \subseteq \cup_{\alpha \in \Delta} P_{\alpha} \) by (ii) there exists \( n_g \in N \cap h(M) \) such that \( PGR_M (N) = PGR_M (RN_g) \). Hence \( n_g \in \cup_{\alpha \in \Delta} P_{\alpha} \) and so \( n_g \in P_{\beta} \) for some \( \beta \in \Delta \). Hence \( RN_g \subseteq P_{\beta} \) and by [1, Theorem 2.4], we conclude that \( N \subseteq PGR_M (N) = PGR_M (RN_g) \subseteq PGR_M (P_{\beta}) \).

(iii)⇒(iv) Assume (iii) holds. Let \( N \) be a proper graded submodule of \( M \) and let \( \{ P_{\alpha} \}_{\alpha \in \Delta} \) be a family of graded primary radical submodules of \( M \) such that \( N \subseteq \cup_{\alpha \in \Delta} P_{\alpha} \) by (iii) there exists \( \beta \in \Delta \) such that \( N \subseteq PGR_M (P_{\beta}) \).

Since \( P_{\beta} \) is graded primary radical submodule, \( P_{\beta} = PGR_M (P_{\beta}) \). Hence \( N \subseteq \cup_{\alpha \in \Delta} P_{\alpha} = \cup_{\alpha \in \Delta} PGR_M (P_{\beta}) \). By (iv), there exists \( \beta \in \Delta \) such that \( N \subseteq PGR_M (P_{\beta}) = P_{\beta} \). Therefore, \( M \) is graded \( P \)-compactly packed.

\[ \square \]

Lemma 2.4. Let \( R \) be a \( G \)-graded ring and \( M \) a graded \( R \)-module. If every proper graded submodule of \( M \) is graded cyclic, then \( M \) is graded \( P \)-compactly packed.

Proof. Let \( N \) be a proper graded submodule of \( M \) and let \( \{ P_{\alpha} \}_{\alpha \in \Delta} \) be a family of graded primary submodules of \( M \) such that \( N \subseteq \cup_{\alpha \in \Delta} P_{\alpha} \). Since \( N \) is a graded cyclic, \( N = Rn_g \) for some \( n_g \in N \cap h(M) \). Since \( n_g \in N \subseteq \cup_{\alpha \in \Delta} P_{\alpha} \), \( n_g \in P_{\beta} \) for some \( \beta \in \Delta \) it follows that \( N = Rn_g \subseteq P_{\beta} \). Therefore \( M \) is graded \( P \)-compactly packed.

A graded \( R \)-module \( M \) is said to be with graded primary decomposition if each of its proper graded submodules is an intersection, possibly infinite, of graded primary submodules of \( M \).

Lemma 2.5. Let \( R \) be a \( G \)-graded ring and \( M \) a graded \( R \)-module. \( M \) is a graded module with graded primary decomposition if and only if \( PGR_M (N) = N \) for all graded submodules \( N \) of \( M \).

Proof. Let \( N \) be a proper graded submodule of \( M \), then \( N \) has a graded primary decomposition \( N = \cap_{\alpha \in \Delta} P_{\alpha} \). Each of \( P_{\alpha} \) is containing \( N \). Since \( PGR_M (N) \) is the intersection of all graded primary submodules containing \( N \), \( PGR_M (N) \subseteq N \) and it is clear that \( N \subseteq PGR_M (N) \). Thus \( PGR_M (N) = N \). Conversely, assume that \( PGR_M (N) = N \) for all graded submodules \( N \) of \( M \). Then every proper graded submodule of \( M \) is an intersection of graded primary submodules of \( M \). Hence \( M \) is a graded module with graded primary decomposition.

\[ \square \]

Theorem 2.6. Let \( R \) be a \( G \)-graded ring and \( M \) a graded \( R \)-module with graded primary decomposition. Then the following statements are equivalent:

(i) \( M \) is a graded \( P \)-compactly packed.

(ii) Every proper graded submodule of \( M \) is graded cyclic.

Proof. (i)⇒(ii) Assume (i) holds and let \( N \) be a proper graded submodule of \( M \). By Theorem 2.3, there exists \( n_g \in N \cap h(M) \) such that \( PGr(N) = PGr(Rn_g) \) but \( M \) is graded module with graded primary decomposition, then by previous Lemma \( N = Rn_g \). Thus \( N \) is graded cyclic.

(ii)⇒(i) Lemma 2.4.

Recall that a proper graded submodule \( N \) of a graded \( R \)-module \( M \) is said to be graded maximal submodule if there is no graded submodule \( K \) of \( M \) such that \( N \not\subseteq K \not\subseteq M \) (see [2]).

Theorem 2.7. Let \( R \) be a \( G \)-graded ring and \( M \) a graded \( R \)-module. If \( M \) is graded \( P \)-compactly packed which has at least one graded maximal submodule, then \( M \) satisfies the ascending chain condition on graded primary radical submodules.
Proof. Let \( P_1 \subseteq P_2 \subseteq P_3 \subseteq \cdots \) be an ascending chain of graded primary radical submodules of \( M \). If \( P_k = M \) for some \( k \), then the result follows immediately, so assume that none of \( P_k \)'s is \( M \) and let \( P = \bigcup_{i=1}^{\infty} P_i \). We claim that \( P \) is a proper graded submodule of \( M \). Assume on contrary that \( P = M \) and let \( L \) be a graded maximal submodule of \( M \). Then \( L \subseteq \bigcup_{i=1}^{\infty} P_i \). Since \( M \) is graded \( P \)-compactly packed, by Theorem 2.3 \( L \subseteq P_k \) for some \( k \). Hence \( L = P_k \) and so \( P_k \) is graded maximal. Hence \( P_k = P_i \) for all \( i \geq k \) it follows that \( P_k = \bigcup_{i=1}^{\infty} P_i = M \), which is impossible. Thus \( P \) is a proper graded submodule of \( M \). Since \( M \) is graded \( P \)-compactly packed, by Theorem 2.3 \( P \subseteq P_s \) for some \( s \) and hence \( P_s = P_i \) for all \( i \geq s \). Therefore the ascending chain condition is satisfied on graded primary radical submodules.

By [2, Lemma 2.7], every graded finitely generated module over graded ring has a graded proper maximal submodule. Then we have the following Corollary.

**Corollary 2.8.** Let \( R \) be a \( G \)-graded ring and \( M \) a graded finitely generated \( R \)-module. If \( M \) is graded \( P \)-compactly packed, then \( M \) satisfies the ascending chain condition on graded primary radical submodules.

**Lemma 2.9.** Let \( R \) be a \( G \)-graded ring and \( M \) a graded \( R \)-module. If \( M \) satisfies the ascending chain condition on graded primary radical submodules, then every graded primary radical submodule is the graded primary radical of a graded finitely generated submodule.

**Proof.** Assume that there exists a graded primary radical \( P \) which is not graded primary radical of a graded finitely generated submodule. Let \( n_1 \in P \cap h(M) \) and let \( P_1 = P.GR(M)(n_1) \). Then \( P_1 \subsetneq P \). Hence there exists \( n_2 \in (P \cap h(M)) - P_1 \). Let \( P_2 = P.GR(M)(n_1 - n_2) \). Then \( P_1 \subsetneq P_2 \subsetneq P \) and hence there exist \( n_3 \in (P \cap h(M)) - P_2 \) etc. This gives an ascending chain of graded primary radical submodules \( P_1 \subsetneq P_2 \subsetneq P_3 \subsetneq \cdots \) which is a contradiction.

**Theorem 2.10.** Let \( R \) be a \( G \)-graded ring and \( M \) a graded \( R \)-module such that every graded finitely generated submodule of \( M \) is graded cyclic. If \( M \) satisfies the ascending chain condition on graded primary radical submodules, then \( M \) is a graded \( P \)-compactly packed.

**Proof.** Let \( N \) be a proper graded submodule of \( M \). By Lemma 2.9, there exists a graded finitely generated submodule \( P \) of \( M \) such that \( P.GR(M)(N) = P.GR(M)(P) \). By our assumption we conclude that \( P \) is a graded cyclic, it follows that there exists \( n_g \in N \cap h(M) \) such that \( P = Rn_g \). By Theorem 2.3, \( M \) is a graded \( P \)-compactly packed.

Let \( M \) and \( M' \) be two graded \( R \)-modules. A homomorphism of graded \( R \)-modules \( \varphi : M \to M' \) is a homomorphism of \( R \)-modules verifying \( \varphi(M_g) \subseteq M'_g \) for every \( g \in G \).

**Lemma 2.11.** Let \( R \) be a \( G \)-graded ring and \( M, M' \) be two graded \( R \)-modules and \( \varphi : M \to M' \) be an epimorphism of graded modules. If \( M \) is a graded \( P \)-compactly packed, then so is \( M' \).

**Proof.** Assume that \( M \) is a graded \( P \)-compactly packed. Let \( N' \) be a proper graded submodule of \( M' \) and let \( \{ P'_\alpha \}_{\alpha \in \Delta} \) be a family of graded primary submodules of \( M' \) such that \( N' \subseteq \bigcup_{\alpha \in \Delta} P'_\alpha \). Since \( \varphi \) is an epimorphism of graded modules, \( \varphi^{-1}(N') \subseteq \bigcup_{\alpha \in \Delta} \varphi^{-1}(P'_\alpha) \). Hence \( \varphi^{-1}(N') \subseteq \bigcup_{\alpha \in \Delta} \varphi^{-1}(P'_\alpha) \). By [1, Lemma 2.14], \( \varphi^{-1}(P'_\alpha) \) is a graded primary submodule of \( M \) for each \( \alpha \in \Delta \). Since \( M \) is a graded \( P \)-compactly packed, there exists \( \beta \in \Delta \) such that \( \varphi^{-1}(N') \subseteq \varphi^{-1}(P'_\beta) \). Thus \( N' \subseteq P'_\beta \) for some \( \beta \in \Delta \). Therefore \( M' \) is a graded \( P \)-compactly packed.

**Theorem 2.12.** Let \( R \) be a \( G \)-graded ring and \( M, M' \) be two graded \( R \)-modules and \( \varphi : M \to M' \) be an epimorphism of graded modules such that \( \text{Ker}(\varphi) \subseteq P.GR(M)\{0\} \). Then \( M \) is a graded \( P \)-compactly packed if and only if \( M' \) is a graded \( P \)-compactly packed.

**Proof.** \((\Rightarrow) \) Lemma 2.11.

\((\Leftarrow)\) Assume that \( M' \) is a graded \( P \)-compactly packed. Let \( N \) be a proper graded submodule of \( M \) and let \( \{ P_\alpha \}_{\alpha \in \Delta} \) be a family of graded primary submodules of \( M \) such that \( N \subseteq \bigcup_{\alpha \in \Delta} P_\alpha \). Then \( \varphi(N) \subseteq \varphi(\bigcup_{\alpha \in \Delta} P_\alpha) \).
and hence $\varphi(N) \subseteq \bigcup_{\alpha \in \Delta} \varphi(P_{\alpha})$. Since $\ker(\varphi) \subseteq P_{\alpha}$ for each $\alpha \in \Delta$, by [1, Lemma 2.15], $\varphi(P_{\alpha})$ is a graded primary submodule of $M'$. Since $M'$ is a graded P-compactly packed, $\varphi(N) \subseteq \varphi(P_{\beta})$ for some $\beta \in \Delta$. Now, we show that $N \subseteq P_{\beta}$. Let $n = \sum_{g \in G} n_g \in N$. For $g \in G$, $n_g \in N$ and so $\varphi(n_g) \in \varphi(N) \subseteq \varphi(P_{\beta})$. Hence there exists $t \in P_{\beta} \cap \ker(\varphi)$ such that $\varphi(n_g) = \varphi(t)$. Hence $n_g - t \in \ker(\varphi) \subseteq P_{\beta}$, it follows that $n_g \in P_{\beta}$. So $N \subseteq P_{\beta}$. Therefore $M$ is a graded $P$-compactly packed.

Let $R$ be a $G$-graded ring and $M$ a graded $R$-module and $S \subseteq h(R)$ a multiplicatively closed subset of $R$. A non empty subset $S^*$ of $h(M)$ is said to be graded $S$-closed if $se \in S^*$ for every $s \in S$ and $e \in S^*$ (see [4, Definition 2.11].)

**Lemma 2.13.** Let $S \subseteq h(R)$ be a multiplicatively closed subset of graded ring $R$ and $S^* \subseteq h(M)$ be a graded $S$-closed of a graded $R$-module $M$. If $N$ is a graded submodule of $M$ contained in $M - S^*$, then $Gr((N :_R M)) \cap S = \Phi$.

**Proof.** Assume that $Gr((N :_R M)) \cap S \neq \Phi$ and let $r_k \in Gr((N :_R M)) \cap S$. Then $r_k^k M \subseteq N$ for some $k \in N$ and for any $e \in S^*$, $r_k^k e \in S^* \cap N$, which is contradiction with $N \subseteq M - S^*$.

Recall that a graded $R$-module $M$ is called graded multiplication if for each graded submodule $N$ of $M$, $N = IM$ for some graded ideal $I$ of $R$. One can easily show that if $N$ is a graded submodule of a graded multiplication module $M$, then $N = (N :_R M)$, (see [7, Definition 2.1]). Also, a proper graded ideal $P$ of a $G$-graded ring $R$ graded $P$-compactly packed if whenever $P$ is contained in the union of a family of graded primary ideals of $R$, $P$ is contained in one of the graded primary ideals of the family. A graded ring $R$ is said to be graded $P$-compactly packed if every proper graded ideals of $R$ is graded $P$-compactly packed.

**Theorem 2.14.** Let $R$ be a $G$-graded ring, $M$ a graded multiplication $R$-module such that $GrM(N) = N$ for all graded submodules $N$ of $M$. If $R$ is a graded $P$-compactly packed and $M \neq \bigcup_{\alpha \in \Delta} P$ for each family $\{P_{\alpha}\}_{\alpha \in \Delta}$ of graded primary submodules of $M$, then $M$ is graded $P$-compactly packed.

**Proof.** Let $N$ be a proper graded submodule of $M$ and let $\{P_{\alpha}\}_{\alpha \in \Delta}$ be a family of graded primary radical submodules of $M$ such that $N \subseteq \bigcup_{\alpha \in \Delta} P_{\alpha}$. Put $S^* = h(M) - \bigcup_{\alpha \in \Delta} P_{\alpha}$. Then $S^*$ is graded $S$-closed of $M$ where $S = h(R) - \bigcup_{\alpha \in \Delta} Gr((P_{\alpha} :_R M))$. Since $N \subseteq S^* = \Phi$, by Lemma 2.13, $Gr((N :_R M)) \cap S = \Phi$. Hence $Gr((N :_R M)) \subseteq \bigcup_{\alpha \in \Delta} Gr((P_{\alpha} :_R M))$. By [1, Lemma 2.7], $Gr((P_{\alpha} :_R M))$ is graded primary ideals of $R$ for all $\alpha$. Since $R$ is a graded $P$-compactly packed, $Gr((N :_R M)) \subseteq Gr((P_{\beta} :_R M))$ for some $\beta$. By [7, Theorem 9], $N = GrM(N) = Gr((N :_R M))M \subseteq Gr((P_{\beta} :_R M))M = GrM(P_{\beta}) = P_{\beta}$. Therefore, $M$ is graded $P$-compactly packed.

3 Graded finitely $P$-compactly packed modules

In this section, we define the graded finitely $P$-compactly packed modules and give a number of its properties. Also, we find the conditions that make graded finitely $P$-compactly packed modules graded $P$-compactly packed.

**Definition 3.1.** Let $R$ be a $G$-graded ring and $M$ a graded $R$-module. A proper graded submodule $N$ of $M$ is called graded finitely $P$-compactly packed if for each family $\{P_{\alpha}\}_{\alpha \in \Delta}$ of graded primary submodules of $M$ with $N \subseteq \bigcup_{\alpha \in \Delta} P_{\alpha}$, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Delta$ such that $N \subseteq \bigcup_{i=1}^{n} P_{\alpha_i}$. A graded module $M$ is called graded finitely $P$-compactly packed if every proper graded submodule of $M$ is graded finitely $P$-compactly packed.

It is clear that if $M$ is graded $P$-compactly packed, then $M$ is graded finitely $P$-compactly packed.

**Theorem 3.2.** Let $R$ be a $G$-graded ring and $M$ a graded $R$-module in which every finite family of graded primary submodules of $M$ is totally ordered by inclusion. If $M$ is graded finitely $P$-compactly packed, then $M$ is graded $P$-compactly packed.
Proof. Let $N$ be a proper graded submodule of $M$ and let $\{P_\alpha\}_{\alpha \in \Delta}$ be a family of graded primary submodules of $M$ such that $N \subseteq \cup_{\alpha \in \Delta} P_\alpha$. Since $M$ is graded finitely $P$-compactly packed, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Delta$ such that $N \subseteq \cup_{i=1}^n P_{\alpha_i}$. Since $\{P_{\alpha_i}\}_{i=1}^n$ is totally ordered by inclusion, there exists $\beta \in \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ such that $\cup_{i=1}^n P_{\alpha_i} = P_\beta$. Thus $M$ is graded $P$-compactly packed. 

Let $N_1, N_2, \ldots, N_n$ be graded submodules of a graded $R$-module $M$. We call a covering $N \subseteq N_1 \cup N_2 \cup \cdots \cup N_n$ efficient if $N$ is not contained in the union of any $n-1$ of the graded submodules $N_1, N_2, \ldots, N_n$. Any covering of a union of graded submodules can be reduced to an efficient one, called an efficient reduction, by deleting any unnecessary terms, (see [3]).

Theorem 3.3. Let $R$ be a $G$-graded ring and $M$ a graded multiplication $R$-module such that $Gr_M(N) = N$ for all graded submodules $N$ of $M$. If $M$ is graded finitely $P$-compactly packed, then $M$ is graded $P$-compactly packed.

Proof. Let $N$ be an efficient graded primary submodule of $M$ and let $\{P_\alpha\}_{\alpha \in \Delta}$ be a family of graded primary submodules of $M$ such that $N \subseteq \cup_{\alpha \in \Delta} P_\alpha$. Since $M$ is graded finitely $P$-compactly packed, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Delta$ such that $N \subseteq \cup_{i=1}^n P_{\alpha_i}$; we may assume that the covering is efficient. We show that $Gr((P_j :_R M)) \subseteq Gr((P_k :_R M))$ whenever $j \neq k$. Assume on contrary that $Gr((P_j :_R M)) \not\subseteq Gr((P_k :_R M))$ for some $j \neq k$. By [7, Theorem 9], $P_j = Gr_M(P_j) = Gr((P_j :_R M))M \subseteq Gr((P_k :_R M))M = Gr_M(P_k) = P_k$, a contradiction. Thus $Gr((P_j :_R M)) \not\subseteq Gr((P_k :_R M))$ whenever $j \neq k$. By [3, Theorem 2.6], $N \subseteq P_\beta$ for some $\beta$. Therefore $M$ is graded $P$-compactly packed.

Theorem 3.4. Let $R$ be a $G$-graded ring and $M$ a graded $R$-module. If $M$ is graded finitely $P$-compactly packed which has at least one graded maximal submodule, then $M$ satisfies the ascending chain condition on graded primary submodules.

Proof. Let $P_1 \subseteq P_2 \subseteq P_3 \subseteq \cdots$ be an ascending chain of graded primary submodules of $M$ and let $P = \cup_{i=1}^\infty P_i$. We claim that $P$ is a proper graded submodule of $M$. Assume on contrary that $P = M$ and let $L$ be a graded maximal submodule of $M$. Then $L \subseteq \cup_{i=1}^\infty P_i$. Since $M$ is graded finitely $P$-compactly packed, there exist $m_1, m_2, \ldots, m_k$ such that $L \subseteq \cup_{j=1}^k P_{m_j} = P_m$ where $m = \max\{m_1, m_2, \ldots, m_k\}$. Since $L$ is graded maximal, $L = P_m$. Hence $P_m$ is graded maximal, it follows that $P_i = P_m$ for all $i \geq m$. Thus $P_m = \cup_{i=1}^\infty P_i = M$ which is impossible. Thus $P$ is a graded proper submodule of $M$. Since $M$ is graded finitely $P$-compactly packed, there exist $t_1, t_2, \ldots, t_n$ such that $P \subseteq \cup_{i=1}^n P_{t_i} = P_t$ where $t = \max\{t_1, t_2, \ldots, t_n\}$. Hence $P_i \subseteq P_t$ for all $i$, thus $P_i = P_t$ for all $i \geq t$. Then the ascending chain condition is satisfied on graded primary submodules.

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References


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