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Only 3-generalized metric spaces have a compatible symmetric topology

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Abstract: We prove that every 3-generalized metric space is metrizable. We also show that for any \( v \) with \( v \geq 4 \), not every \( v \)-generalized metric space has a compatible symmetric topology.

Keywords: \( v \)-generalized metric space, Metrizability, Topology, Symmetrizable, Semimetrizable

MSC: 54E99

1 Introduction

Throughout this paper we denote by \( \mathbb{N} \) the set of all positive integers.

In 2000, Branciari in [2] introduced the following, very interesting concept. See also [1, 4–6] and others.

Definition 1.1 (Branciari [2]). Let \( X \) be a set, let \( d \) be a function from \( X \times X \) into \([0, \infty)\) and let \( v \in \mathbb{N} \). Then \( (X, d) \) is said to be a \( v \)-generalized metric space if the following hold:

(N1) \( d(x, y) = 0 \) iff \( x = y \) for any \( x, y \in X \).
(N2) \( d(x, y) = d(y, x) \) for any \( x, y \in X \).
(N3) \( d(x, y) \leq d(x, u_1) + d(u_1, u_2) + \cdots + d(u_v, y) \) for any \( x, u_1, u_2, \ldots, u_v, y \in X \) such that \( x, u_1, u_2, \ldots, u_v, y \) are all different.

To be precise, we give some definitions.

Definition 1.2. Let \( (X, d) \) be a \( v \)-generalized metric space. Then \( X \) is called metrizable iff there exists a metric \( \rho \) on \( X \) such that

\[
\lim_{\alpha} d(x, x_\alpha) = 0 \quad \text{and} \quad \lim_{\alpha} \rho(x, x_\alpha) = 0
\]

are equivalent for any net \( \{x_\alpha\} \) in \( X \) and \( x \in X \).

Definition 1.3. Let \( X \) be a topological space with topology \( \tau \). Let \( d \) be a function from \( X \times X \) into \([0, \infty)\) satisfy (N1)–(N3) with some \( v \in \mathbb{N} \). Then \( \tau \) is compatible with \( d \) iff the following are equivalent for any net \( \{x_\alpha\} \) in \( X \) and \( x \in X \):

\[\begin{align*}
&\lim_{\alpha} d(x, x_\alpha) = 0. \\
&\{x_\alpha\} \text{ converges to } x \text{ in } \tau.
\end{align*}\]

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Remark. It is obvious that there exists at most one topology which is compatible with $d$. We sometimes say that the topology $\tau$ is a compatible symmetric topology.

It is obvious that $(X, d)$ is a metric space if and only if $(X, d)$ is a 1-generalized metric space. That is, we can tell that every 1-generalized metric space is metrizable. Of course, this statement is trivial.

Very recently, in [6], we found that not every 2-generalized metric space has a compatible symmetric topology.

Motivated by these facts, in this paper, we prove that every 3-generalized metric space is metrizable. Thus, every 3-generalized metric space has a compatible symmetric topology. We also show that for any $v$ with $v \geq 4$, not every $v$-generalized metric space has a compatible symmetric topology. Therefore we can tell that only 1- and 3-generalized metric spaces always have a compatible symmetric topology.

## 2 Metrization

In this section, we prove that every 3-generalized metric space $(X, d)$ is metrizable.

**Lemma 2.1.** Let $(X, d)$ be a 3-generalized metric space. Let $\varepsilon > 0$ and let $x, u_1, u_2, v_1, v_2, y \in X$ such that $x, u_j, v_j, y$ ($j = 1, 2$) are all different and

\[ d(x, u_1) < \varepsilon, \quad d(x, u_2) < \varepsilon, \quad d(u_1, v_1) < \varepsilon, \quad d(u_2, v_2) < \varepsilon, \quad d(v_1, y) < \varepsilon. \]

Then $d(x, y) < 7\varepsilon$ holds.

**Proof.** Since

\[ d(v_1, v_2) \leq d(v_1, u_1) + d(u_1, x) + d(x, u_2) + d(u_2, v_2) < 4\varepsilon, \]

we have

\[ d(x, y) \leq d(x, u_2) + d(u_2, v_2) + d(v_2, v_1) + d(v_1, y) < 7\varepsilon \]

by (N3). \(\square\)

**Theorem 2.2.** Let $(X, d)$ be a 3-generalized metric space. Define a function $\rho$ from $X \times X$ into $[0, \infty)$ by

\[ \rho(x, y) = \inf \left\{ \sum_{j=0}^{n} d(u_j, u_{j+1}) : n \in \mathbb{N} \cup \{0\}, u_0 = x, u_1, \ldots, u_n \in X, u_{n+1} = y \right\}. \]

Then $(X, \rho)$ is a metric space; and for every $x \in X$ and for every net $\{x_\alpha\}_{\alpha \in D}$ in $X$, $\lim_{\alpha} d(x, x_\alpha) = 0$ if and only if $\lim_{\alpha} \rho(x, x_\alpha) = 0$.

**Proof.** We first show that $(X, \rho)$ is a metric space, that is, we show the following:

(D1) $\rho(x, y) \geq 0$, $\rho(x, y) = 0$ iff $x = y$.

(D2) $\rho(x, y) = \rho(y, x)$.

(D3) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

(D2) and (D3) are obvious. We shall show (D1). It is also obvious that $\rho(x, y) \geq 0$; and $\rho(x, y) = 0$ if $x = y$.

Before proving that $\rho(x, y) = 0$ implies $x = y$, we show that we can rewrite $\rho$ as follows:

\[ \rho(x, y) = \min \left\{ d(x, y), \inf \left\{ d(x, u) + d(u, y) : u \notin \{x, y\} \right\}, \inf \left\{ d(x, u) + d(u, v) + d(v, y) : u, v \notin \{x, y\}, u \neq v \right\} \right\}. \]

In the case where $x = y$, the left hand side and right hand side are obviously 0. We consider the other case, where $x \neq y$. Let $n \in \mathbb{N} \cup \{0\}$ and $u_0(= x), u_1, \cdots, u_{n+1}(= y) \in X$. If $u_k = u_\ell$ for some $k, \ell$ with $k < \ell$, then

\[ \sum_{j=0}^{n} d(u_j, u_{j+1}) \geq \sum_{j=0}^{k-1} d(u_j, u_{j+1}) + \sum_{j=\ell}^{n} d(u_j, u_{j+1}) \]
obviously holds, where \(\sum_{j=0}^{n-1} d(u_{j},u_{j+1}) = \sum_{j=n+1}^{n} d(u_{j},u_{j+1}) = 0\). So we only consider the case where \(u_0, \ldots, u_{n+1}\) are all different. If \(n \geq 3\), then we have by (N3)

\[
\sum_{j=0}^{n} d(u_{j},u_{j+1}) \geq d(u_0, u_4) + \sum_{j=4}^{n} d(u_{j},u_{j+1}).
\]

So we only consider the case where \(n < 3\). Thus, we have shown (1). Let us prove that \(\rho(x, y) = 0\) implies \(x = y\). We assume \(\rho(x, y) = 0\). Arguing by contradiction, we assume \(x \neq y\). By (1), we only consider the following case:

- There exists a sequence \(\{u_n\}\) in \(X \setminus \{x, y\}\) such that \(\lim_n \{d(x, u_n) + d(u_n, y)\} = 0\).
- There exist sequences \(\{u_{n_1}\} \) and \(\{v_{n_1}\} \) in \(X\) such that and \(x, u_{n_1}, v_{n_1}\) and \(y\) are all different for any \(n \in \mathbb{N}\); and \(\lim_n \{d(x, u_n) + d(u_n, v_n) + d(v_n, y)\} = 0\).

In the first case, since \(\lim_n d(x, u_n) = 0\), without loss of generality, we may assume that \(\{d(x, u_n)\}\) is strictly decreasing, which implies that \(u_n\) are all different. We have by (N3)

\[
\lim_{n \to \infty} d(u_n, u_{n+1}) \leq \lim_{n \to \infty} \left( d(u_n, x) + d(x, u_{n+2}) + d(u_{n+2}, y) + d(y, u_{n+1}) \right) = 0 \quad \text{and} \quad d(x, y) \leq \lim_{n \to \infty} \left( d(x, u_n) + d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) + d(u_{n+2}, y) \right) = 0.
\]

This is a contradiction. In the second case, we put \(\epsilon = d(x, y)/7\). We can choose \(n_1\) and \(n_2\) such that

\[
d(x, u_{n_1}) + d(u_{n_1}, v_{n_1}) + d(v_{n_1}, y) < \epsilon
\]

\[
d(x, u_{n_2}) + d(u_{n_2}, v_{n_2}) + d(v_{n_2}, y) < \min \{d(x, u_{n_1}), d(x, v_{n_1}), d(u_{n_1}, y), d(u_{n_2}, y)\}.
\]

We note that \(x, u_{n_1}, v_{n_1}, y\) are all different. So by Lemma 2.1, we have

\[
d(x, y) < 7\epsilon = d(x, y),
\]

which implies a contradiction. We have shown (D1). Therefore \(\rho\) is a metric on \(X\). Let \(x \in X\) and \(\{x_\alpha\}_{\alpha \in D}\) be a net in \(X\). Since \(\rho \leq d\), \(\lim_\alpha d(x, x_\alpha) = 0\) implies \(\lim_\alpha \rho(x, x_\alpha) = 0\). Let us prove the converse implication. We assume \(\lim_\alpha \rho(x, x_\alpha) = 0\). Arguing by contradiction, we assume \(\epsilon := \lim_\alpha d(x, x_\alpha)/11 > 0\). Then by (1) at least one of the following holds:

- There exist \(u_{1\beta_1}, u_{2\beta_2}, u_{2\beta_3}, u_{3\beta_3} \in X\) such that \(x, u_{1\beta_1}, x_{1\beta_1} \) are all different for any \(j\);

\[
d(x, u_{1\beta_1}) + d(u_{1\beta_1}, x_{1\beta_1}) < \epsilon,
\]

\[
d(x, u_{2\beta_2}) + d(u_{2\beta_2}, x_{2\beta_2}) < \min \{\rho(x, u_{1\beta_1}), \rho(x, x_{1\beta_1})\},
\]

\[
d(x, u_{3\beta_3}) + d(u_{3\beta_3}, x_{3\beta_3}) < \min \{\rho(x, u_{2\beta_2}), \rho(x, x_{2\beta_2})\} \quad \text{and}
\]

\[
d(x, x_{3\beta_3}) > 10\epsilon \quad \text{for} \quad j = 1, 2, 3.
\]

- There exist \(u_{1\beta_1}, v_{1\beta_1}, u_{2\beta_2}, v_{2\beta_2}, u_{2\beta_3} \in X\) such that \(x, u_{1\beta_1}, v_{1\beta_1}, x_{1\beta_1} \) are all different for any \(j\);

\[
d(x, u_{1\beta_1}) + d(u_{1\beta_1}, v_{1\beta_1}) + d(v_{1\beta_1}, x_{1\beta_1}) < \epsilon,
\]

\[
d(x, u_{2\beta_2}) + d(u_{2\beta_2}, v_{2\beta_2}) + d(v_{2\beta_2}, x_{2\beta_2}) < \min \{\rho(x, u_{1\beta_1}), \rho(x, v_{1\beta_1}), \rho(x, x_{1\beta_1})\} \quad \text{and}
\]

\[
d(x, x_{3\beta_3}) > 10\epsilon \quad \text{for} \quad j = 1, 2.
\]

In the first case, we have

\[
\max \{\rho(x, u_{j+1}), \rho(x, x_{j+1})\} < \min \{\rho(x, u_{j}), \rho(x, x_{j})\}
\]

for \(j = 1, 2\), which implies that \(x, u_{j}, x_{j} \) are all different. We have

\[
d(x_{1\beta_1}, x_{2\beta_2}) \leq d(x_{1\beta_1}, u_{1\beta_1}) + d(u_{1\beta_1}, x) + d(x, u_{2\beta_2}) + d(u_{2\beta_2}, x_{2\beta_2}) < 4\epsilon,
\]

\[
d(x_{2\beta_2}, x_{3\beta_3}) \leq d(x_{2\beta_2}, u_{2\beta_2}) + d(u_{2\beta_2}, x) + d(x, u_{3\beta_3}) + d(u_{3\beta_3}, x_{3\beta_3}) < 4\epsilon \quad \text{and}
\]

\[
d(x, x_{1\beta_1}) \leq d(x, u_{1\beta_1}) + d(u_{1\beta_1}, x_{1\beta_1}) + d(x_{1\beta_1}, x_{2\beta_2}) + d(x_{2\beta_2}, x_{3\beta_3}) + d(x_{3\beta_3}, x_{1\beta_1}) < 10\epsilon.
\]
This is a contradiction. In the second case, we have
\[
\max \{ \rho(x, u\beta), \rho(x, v\beta), \rho(x, x\beta) \} < \min \{ \rho(x, u\beta_1), \rho(x, v\beta_1), \rho(x, x\beta_1) \},
\]
which implies that \( x, u\beta, v\beta, x\beta \) \((j = 1, 2)\) are all different. So by Lemma 2.1, we have
\[
d(x, x\beta_1) < 7 \varepsilon < 10 \varepsilon.
\]
This is a contradiction. Therefore \( \lim_{\alpha} d(x, x_\alpha) = 0 \).

3 Topology

In the section, we discuss the representation of the compatible topology.

**Lemma 3.1.** Let \((X, d)\) be a 3-generalized metric space. Define subsets \(A\) and \(B\) of \(X\) as follows: \(x \in A\) iff there exists a sequence \(\{x_n\}\) in \(X \setminus \{x\}\) converging to \(x\). \(x \in B\) iff there exists a sequence \(\{x_n\}\) in \(A \setminus \{x\}\) converging to \(x\).

Then
\[
d(u_1, u_n) \leq \sum_{j=1}^{n-1} d(u_j, u_{j+1}) \tag{2}
\]
for \(u_1, u_2, \ldots, u_n \in X\) with \(\{u_1, u_2, \ldots, u_n\} \cap B \neq \emptyset\).

**Proof.** Let \(\rho\) be as in Theorem 2.2. In the case where \(n = 2\), the conclusion obviously holds. So we assume \(n \geq 3\). We will prove (2) in the following cases:

(i) \(n = 3\) and \(u_2 \in B\)
(ii) \(n = 3\) and either \(u_1 \in B\) or \(u_3 \in B\)
(iii) either \(u_1 \in B\) or \(u_n \in B\)
(iv) \(u_\kappa \in B\) for some \(\kappa\) with \(2 \leq \kappa \leq n - 1\)

Fix \(\varepsilon > 0\). We first consider the first case. In the case where either \(u_1 = u_2, u_2 = u_3\) or \(u_3 = u_1\) holds, (2) is obvious. So we assume that \(u_1, u_2, u_3\) are all different. Since \(u_2 \in B\), there exist \(v_1, w_1, v_2, w_2 \in X\) such that \(u_2 \neq v_1 \neq w_1, u_2 \neq v_2 \neq w_2\),

\[
2d(u_2, v_1) < \min \{ \varepsilon, \rho(u_1, u_2), \rho(u_2, u_3) \},
\]
\[
2d(v_1, w_1) < \rho(u_2, v_1) < \varepsilon,
\]
\[
4d(u_2, v_2) < \rho(u_2, v_1) < \varepsilon \quad \text{and}
\]
\[
2d(v_2, w_2) < \rho(u_2, v_2) < \varepsilon.
\]

Then we have
\[
2\rho(u_2, v_1) < \min \{ \varepsilon, \rho(u_1, u_2), \rho(u_2, u_3) \},
\]
\[
2\rho(v_1, w_1) < \rho(u_2, v_1),
\]
\[
4\rho(u_2, v_2) < \rho(u_2, v_1) \quad \text{and}
\]
\[
2\rho(v_2, w_2) < \rho(u_2, v_2).
\]

Hence by (D3), we obtain that \(u_1, u_2, u_3, v_1, w_1, v_2, w_2\) are all different. We have
\[
d(w_1, w_2) \leq d(w_1, v_1) + d(v_1, u_2) + d(u_2, v_2) + d(v_2, w_2) < 4 \varepsilon,
\]
\[
d(u_1, w_2) \leq d(u_1, u_2) + d(u_2, v_1) + d(v_1, w_1) + d(w_1, w_2) < d(u_1, u_2) + 6 \varepsilon \quad \text{and}
\]
\[
d(u_1, u_3) \leq d(u_1, u_2) + d(u_2, v_2) + d(v_2, u_2) + d(u_2, u_3) \leq d(u_1, u_2) + d(u_2, u_3) + 8 \varepsilon.
\]
Since $\epsilon > 0$ is arbitrary, we obtain (2). We next consider the second case. Without loss of generality, we may assume $u_1 \in B$. In the case where either $u_1 = u_2$, $u_2 = u_3$ or $u_3 = u_1$ holds, (2) is obvious. So we assume that $u_1$, $u_2$, $u_3$ are all different. Since $u_1 \in B$, as in the first case, there exist $v_3, w_3, v_4, w_4 \in X$ such that

$$d(u_1, v_3) < \epsilon, \quad d(v_3, w_3) < \epsilon, \quad d(u_1, v_4) < \epsilon, \quad d(v_4, w_4) < \epsilon$$

and $u_1, u_2, u_3, v_3, w_3, v_4, w_4$ are all different. We have

$$d(w_3, w_4) \leq d(w_3, v_3) + d(v_3, u_1) + d(u_1, v_4) + d(v_4, w_4) < 4\epsilon,$$

$$d(w_4, u_3) \leq d(w_4, u_3) + d(u_4, v_1) + d(u_1, u_2) + d(u_2, u_3) < d(u_1, u_2) + d(u_2, u_3) + 2\epsilon,$$

$$d(u_1, u_3) \leq d(u_1, u_3) + d(v_3, w_3) + d(w_3, w_4) + d(w_4, u_3) \leq d(u_1, u_2) + d(u_2, u_3) + 8\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we obtain (2). In the third case, without loss of generality, we may assume $u_1 \in B$. Considering the second case, we have

$$d(u_1, u_n) \leq d(u_1, u_{n-1}) + d(u_{n-1}, u_n)$$

$$\leq d(u_1, u_{n-2}) + d(u_{n-2}, u_{n-1}) + d(u_{n-1}, u_n)$$

$$\leq d(u_1, u_{n-3}) + d(u_{n-3}, u_{n-2}) + d(u_{n-2}, u_{n-1}) + d(u_{n-1}, u_n)$$

$$\vdots$$

$$\leq \sum_{j=1}^{n-1} d(u_j, u_{j+1}).$$

In the fourth case, considering the first and third cases, we have

$$d(u_1, u_n) \leq d(u_1, u_{\kappa}) + d(u_{\kappa}, u_n)$$

$$\leq \sum_{j=1}^{\kappa-1} d(u_j, u_{j+1}) + \sum_{j=\kappa}^{n-1} d(u_j, u_{j+1}).$$

We complete the proof. \[\square\]

**Theorem 3.2.** Let $(X, d)$ be a 3-generalized metric space. Let $A$ and $B$ be as in Lemma 3.1. Define $\delta_x > 0$ by

$$\delta_x = \begin{cases} \inf \{d(x, y) : y \in X \setminus \{x\}\} & \text{if } x \in X \setminus A \\ \inf \{d(x, y) : y \in A \setminus \{x\}\} & \text{if } x \in A \setminus B \\ \infty & \text{if } x \in B \end{cases}$$

for $x \in X$. Define a subset $N_x$ of $X$ by

$$N_x = \{S(x, r) : 0 < r < \delta_x\},$$

where

$$S(x, r) = \{y : d(x, y) < r\}.$$ 

Then the topology $\tau$ induced by a subbase $\cup\{N_x : x \in X\}$ is compatible with $d$. 

**Proof.** In the case where $x \in X \setminus A$, we note $N_x = \{\{x\}\}$. In the case where $x \in A \setminus B$, we note $S(x, \delta_x) \cap A = \{x\}$. We shall show that the topology $\tau$ induced by a subbase $\cup\{N_x : x \in X\}$ is compatible with $d$. It is obvious that if a net $\{x_\alpha\}_{\alpha \in D}$ converges to $x$ in $\tau$, then $\lim_\alpha d(x, x_\alpha) = 0$ holds. In order to prove the converse implication, we show the following:

- For $y \in X$, $G_y \subseteq N_y$ and $x \in G_y$, there exists $G_x \subseteq N_x$ such that $G_x \subset G_y$.
If \( x = y \), then we put \( G_x = G_y \). So we assume \( x \neq y \). In the case where \( y \in X \setminus A, x = y \) always holds. In the case where \( y \in A \setminus B, x \in X \setminus A \) holds. So, putting \( G_x = \{x\} \), we have \( G_x \subseteq G_y \). In the other case, where \( y \in B \), there exists \( \gamma > 0 \) such that \( G_y = \{y\} \). Let \( s \) be a real number with \( 0 < s < \min \{r - d(y, x), \delta_x\} \) and put \( G_x = S(y, s) \). Then we have \( G_x \in N_x \) and \( S(x, s) \subseteq G_y \) by Lemma 3.1. Let us prove the converse implication. We assume \( \lim_{i \to \infty} d(x, y_i) = 0 \) and let \( G \) be an open neighborhood of \( x \) in \( r \). Then there exist \( y_1, \ldots, y_n \in X \) and \( G_i \subseteq N_{y_i} \) such that \( G = \bigcap_{i=1}^n G_i \). For every \( i \), there exists \( \delta_i \) such that \( 0 < \delta_i < \delta_x \) and \( S(x, \delta_i) \subseteq G_i \). Let \( \delta = \min \{\delta_i : i = 1, 2, \ldots, n\} \). Then for sufficiently large \( \alpha \in D \), we have

\[
\begin{align*}
x_\alpha \in S(x, \delta) = \bigcap_{i=1}^n S(x, \delta_i) \subseteq \bigcap_{i=1}^n G_i = G.
\end{align*}
\]

Thus, \( \{x_\alpha\} \) converges to \( x \) in \( r \).

\[\square\]

### 4 Example

In this section, we give an example of \( v \)-generalized metric space for \( v \geq 4 \), which does not have a compatible symmetric topology.

**Lemma 4.1.** Let \( X \) be a set. Let \( a \in X \) and let \( B \) and \( C \) be two nonempty subsets of \( X \) with

\[
X = \{a\} \cup B \cup C.
\]

\( a \notin B \), \( a \notin C \) and \( B \cap C = \emptyset \). Let \( S \) be a mapping from \( C \) into \( B \). Let \( M \) be a positive real number and let \( f \) be a function from \( B \cup C \) into \( (0, M] \). Define a function \( d \) from \( X \times X \) into \([0, \infty)\) by

\[
\begin{align*}
d(x, x) &= 0 \\
d(a, x) &= d(x, a) = f(x) \quad \text{if } x \in B \\
d(Sx, x) &= d(x, Sx) = f(x) \quad \text{if } x \in C \\
d(x, y) &= M \quad \text{otherwise}.
\end{align*}
\]

Then \( (X, d) \) is a \( v \)-generalized metric space for \( v \geq 4 \).

**Proof.** (N1) and (N2) are obvious. Let us prove (N3). Let \( x, u_1, \ldots, u_v, y \in X \) be all different. We will show

\[
t = d(x, u_1) + d(u_1, u_2) + \cdots + d(u_{v-1}, u_v) + d(u_v, y) \geq M.
\]

Arguing by contradiction, we assume \( t < M \). For example, we consider the case where \( v = 4 \) and \( x \in C \). Then we have \( u_1 = Sx \in B \). If \( u_2 \in C \), then we have \( Su_2 = u_1 \) and hence \( u_3 = Su_2 = u_1 \), which implies a contradiction. So \( u_2 = a \) holds. Then \( u_3 \in B, u_4 \in C \) and \( Su_4 = u_3 \). Hence \( y = u_3 \) holds, which implies a contradiction. We can similarly prove \( t \geq M \) in the other cases, where \( v \geq 5 \) or \( x \notin C \). Therefore \( d(x, y) \leq M \leq t \). Thus (N3) holds.

\[\square\]

**Example 4.2.** Let

\[
X = \{(0, 0)\} \cup ((0, 2) \times [0, 2])
\]

Define a function \( d \) from \( X \times X \) into \([0, \infty)\) by

\[
\begin{align*}
d(x, x) &= 0 \\
d((0, 0), (s, 0)) &= d((s, 0), (0, 0)) = s \quad \text{if } s \in (0, 2) \\
d((s, 0), (s, t)) &= d((s, t), (s, 0)) = t \quad \text{if } s, t \in (0, 2) \\
d(x, y) &= 6 \quad \text{otherwise}.
\end{align*}
\]

Then the following hold:
(i) $(X, d)$ is not a $v$-generalized metric space for $v = 1, 2, 3$.
(ii) $(X, d)$ is a $v$-generalized metric space for $v \geq 4$.
(iii) $X$ does not have a topology which is compatible with $d$.

**Proof.** Since
\[
d((1, 1), (1, 0)) + d((1, 0), (0, 0)) + d((0, 0), (2, 0)) + d((2, 0), (2, 1))
= 1 + 1 + 2 + 1 = 5 < 6 = d((1, 1), (2, 1)),
\]
\[
d((1, 1), (1, 0)) + d((1, 0), (0, 0)) + d((0, 0), (2, 0))
= 1 + 1 + 2 = 4 < 6 = d((1, 1), (2, 0))
\]
and
\[
d((1, 1), (1, 0)) + d((1, 0), (0, 0))
= 1 + 1 = 2 < 6 = d((1, 1), (0, 0)).
\]

(N3) does not hold for $v = 1, 2, 3$. We have shown (i). Put $M = 6, \ a = (0, 0), \ B = (0, 2] \times \{0\}$ and $C = (0, 2] \times (0, 2]$. Define $f$ and $S$ by
\[
f((s, t)) = t \quad \text{if } s, t \in (0, 2]
\]
\[
f((s, 0)) = s \quad \text{if } s \in (0, 2]
\]
\[
S(s, t) = (s, 0) \quad \text{if } s, t \in (0, 2]
\]
By Lemma 4.1, $(X, d)$ is a $v$-generalized metric space for $v \geq 4$. In order to show (iii), arguing by contradiction, we assume that a compatible symmetric topology exists. Then the following must hold:
- If a net $x_n$ assumes that a compatible symmetric topology exists. Then the following must hold:

- If a net $x_n$ converges to $x$ and for every $\alpha \in D$ a net $x_{n, \alpha}$ converges to $x_\alpha$, then $\{x_{n, \alpha}\}_{(\alpha, \gamma) \in D \times \prod E_\alpha : \alpha \in D}$ has a subnet converging to $x$; see page 77 of [7].

We have that $\{(1/\ell, 0)\}_\ell$ converges to $(0, 0)$ and $\{(1/\ell, 1/m)\}_m$ converges to $(1/\ell, 0)$ for every $\ell \in \mathbb{N}$. However, since $d((0, 0), (1/\ell, 1/m)) = 6$ for $(\ell, m) \in \mathbb{N}^2$, a net $\{(1/\ell, 1/\gamma(\ell))\}_{(\ell, \gamma)}$ does not converge to $(0, 0)$. This is a contradiction. Therefore there does not exist a topology which is compatible with $d$. 

**Remark.**
(i) For $(\alpha, \gamma) \in D \times \prod E_\alpha : \alpha \in D$, $x_{(\alpha, \gamma)} = x_{(\alpha, \gamma(\alpha))}$. For $(\alpha_1, \gamma_1), (\alpha_2, \gamma_2) \in D \times \prod E_\alpha : \alpha \in D$, $(\alpha_1, \gamma_1) \leq (\alpha_2, \gamma_2)$ iff $\alpha_1 \leq \alpha_2$ and $\gamma_1(\alpha) \leq \gamma_2(\alpha)$ for any $\alpha \in D$.
(ii) Indeed, let $\delta_x \in (0, \infty)$ for any $x \in X$ and let $\tau_1$ be the topology induced by a subbase
\[
\{S(x, r) : x \in X, \ 0 < r < \delta_x\}.
\]
Let $s, t \in (0, 1)$ satisfy $0 < 2s < \delta(0, 0)$ and $0 < t < \delta(s, 0)$. Then we have
\[
S((0, 0), 2s) \cap S((s, 0), t) = \left(\left[0, 2s\right] \times \{0\}\right) \cap \left(\{s\} \times [0, t]\right) = \{(s, 0)\}.
\]
Hence $\{(s, 0)\}$ is an open neighborhood of $(s, 0)$. So a sequence $\{(s, 1/n)\}$ does not converge to $(s, 0)$ in $\tau_1$. Since $\lim_n d((s, 0), (s, 1/n)) = 0$, $\tau_2$ is not compatible with $d$.
(iii) Define a topology $\tau_2$ on $X$ as follows: A subset $U$ of $X$ is open iff for any $x \in U$, there exists $\delta > 0$ such that $S(x, \delta) \subset U$; see Section 5. Then a typical basic open set containing $(0, 0)$ has the following form:
\[
\{(0, 0)\} \cup \bigcup_{0 < s < 6} \{s\} \times [0, t_s),
\]
where $0 < \varepsilon \leq 2$ and $0 < t_s \leq 2$ for any $s$. This shows that $(0, 0)$ is not a point of first countability. Also, $(0, 0)$ belongs to the closure of $A := \left(\left[0, 2\right] \times \{0, 2\}\right)$, however $d(x, A) = 6 > 0$ holds.
5 Symmetric space and semimetric spaces

In this section, we mention symmetric spaces and semimetric spaces. See Section 9 in [3]. We give some concepts and theorems.

– Let $X$ be a set. Then a function $d$ from $X \times X$ into $[0, \infty)$ is called symmetric if the following holds:
  (i) $d(x, y) = 0$ implies $x = y$.
  (ii) $d(x, y) = d(y, x)$.
– Let $X$ be a topological space. Then $X$ is called symmetrizable if there exists a symmetric $d$ on $X$ and satisfying the following: A subset $U \subset X$ is open iff for any $x \in U$, there exists $\delta > 0$ such that $S(x, \delta) \subset U$.
– Let $X$ be a topological space. Then $X$ is called semimetrizable if there exists a symmetric $d$ on $X$ such that for each $x \in X$, $\{S(x, r) : r > 0\}$ forms a neighborhood base at $x$.
– Let $X$ be a topological space. Then the following are equivalent:
  – $X$ is semimetrizable.
  – $X$ is symmetrizable and first countable.
– Let $X$ be a topological space and let $d$ be a symmetric $d$ on $X$. Then the following are equivalent:
  – $X$ is semimetrizable.
  – For any $x \in X$ and $A \subset X$, $d(x, A) := \inf\{d(x, y) : y \in A\} = 0$ iff $x$ belongs to the closure of $A$.

Remark.
(i) $\nu$-generalized metric spaces $(X, d)$ are symmetrizable, $d$ is a symmetric on $X$.
(ii) Let $(X, d)$ be a $\nu$-generalized metric space. Then $X$ has a topology which is compatible with $d$ in the sense of Definition 1.3 iff $X$ is semimetrizable.

Finally, the referee raises the following question.

Problem 5.1. Let $(X, d)$ be a $\nu$-generalized metric space. Assume that $X$ has a topology which is compatible with $d$ in the sense of Definition 1.3. Then, is $X$ metrizable?

Conflict of interests
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