Abstract: In BL-algebras we introduce the concept of generalized co-annihilators as a generalization of co-annihilator and the set of the form $x^{-1}F$ where $F$ is a filter, and study basic properties of generalized co-annihilators. We also introduce the notion of involutory filters relative to a filter $F$ and prove that the set of all involutory filters relative to a filter with respect to the suit operations is a complete Boolean lattice and BL-algebra. We use the technology of generalized co-annihilators to give characterizations of prime filters and minimal prime filters, respectively. In particular, we give a representation of co-annihilators in the quotient algebra of a BL-algebra $L$ via a filter $F$ by means of generalized co-annihilators relative to $F$ in $L$.

Keywords: BL-algebras, Boolean algebras, Filter, Prime filter, Minimal prime filter, Generalized co-annihilator, Involutory filter

MSC: 08A72, 06B75

1 Introduction

BL-algebras (Basic Logic algebras) were introduced by Hajek [8] in order to provide an algebraic proof of the completeness theorem of a class of $[0, 1]$-valued logics familiar in fuzzy logic framework. Cignoli et al [1] proved that Hájek’s logic really is the logic of continuous $t$-norms as conjectured by Hájek. At the same time a systematic study of BL-algebras, and in particular, filter theory, was started. Filters (also called deductive systems) are basic tools in the study of BL-algebras. Filter theory is one of the important contents of BL-algebras, which has been studied by many researchers, such as ([9, 14, 16–20]). From logical point of view, filters correspond to sets of provable formulas. Hájek introduced the notions of filters and prime filters in BL-algebras and proved the completeness of Basic Logic using prime filters. Turunen ([19–21]) studied some properties of deductive systems and prime deductive systems. Haveshki et al ([9, 16]) introduce (positive) implicative, fantastic filters in BL-algebras and studied their properties. BL-algebras are further discussed by Di Nola et al [11], Leustean [13], Iorgulescu [10], and so on. Recent investigations are concerned with non-commutative generalizations for these structures (see [2, 3, 5, 6, 13]). Georgescu and Iorgulescu [6] introduced the concept of pseudo MV-algebras as non-commutative generalization of MV-algebras. Several researchers studied the properties of pseudo MV-algebras (see [2, 3]). Pseudo BL-algebras are a common extension of BL-algebras and pseudo MV-algebras (see [4, 7, 12]). These structures seem to be a very general algebraic concept in order to express the non-commutative reasoning. Turunen [19] and Leustean [15] introduced and studied a special class of filters, co-annihilators.

In the present paper we introduce the concept of generalized co-annihilators as a generalization of co-annihilator and the set of the form $x^{-1}F$ where $F$ is a filter, and study basic properties of generalized co-annihilators. We also introduce the notion of involutory filters relative to a filter $F$ and prove that the set of all involutory filters relative to a filter with respect to the suit operations is a complete Boolean lattice and BL-algebra. We use the technology
of generalized co-annihilators to give characterizations of prime filters and minimal prime filters, respectively. In particular, we give a representation of co-annihilators in the quotient algebra of a BL-algebra $L$ via a filter $F$ by means of generalized co-annihilators relative to $F$ in $L$.

## 2 Preliminaries

Let us recall some of concepts and results which will be used in the sequel.

**Definition 2.1** ([8]). An algebra $(L; \land, \lor, \ast, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called a BL-algebra if it satisfies the following conditions:

1. $(L; \land, \lor, 0, 1)$ is a bounded lattice,
2. $(L; \ast, 1)$ is a commutative monoid,
3. $x \ast z \leq y$ if and only if $z \leq x \to y$ (residuation),
4. $x \land y = x \ast (x \to y)$, thus $x \ast (x \to y) = y \ast (y \to x)$ (divisibility),
5. $(x \to y) \lor (y \to x) = 1$ (prelinearity).

In this paper $L$ will always be a BL-algebra without mentioned otherwise.

**Proposition 2.2** (see [14, 16, 19, 22]). For any $x, y, z \in L$, the following assertions are true:

1. $x \ast (x \to y) \leq y$,
2. $x \leq y \to (x \ast y)$,
3. $x \leq y$ if and only if $x \to y = 1$,
4. $x \to (y \to z) = (x \ast y) \to z = y \to (x \to z)$,
5. $x \leq y$ implies $z \to x \leq z \to y$, $y \to z \leq x \to z$ and $x \ast z \leq y \ast z$,
6. $y \leq (y \to x) \to x$,
7. $(x \to y) \ast (y \to z) \leq x \to z$,
8. $y \to x \leq (z \to y) \to (z \to x)$,
9. $x \to y \leq (y \to z) \to (x \to z)$,
10. $x \lor y = [(x \to y) \to y] \land [(y \to x) \to x]$.
11. $x \leq y$ implies $y^- \leq x^-$,
12. $1 \to x = x$, $x \to x = 1$, $x \to 1 = 1$,
13. $x \leq (y \to x)$, or equivalently, $x \to (y \to x) = 1$,
14. $((x \to y) \to y) \to y = x \to y$,
15. $1^- = 0$, $0^- = 1$,
16. $x \ast (y \lor z) = (x \ast y) \lor (x \ast z)$,
17. $x \land (y \lor z) = (x \land y) \lor (x \land z)$, i.e., $(L; \land, \lor)$ is a distributive lattice,
18. $(y \lor z) \to x = (y \to x) \land (z \to x)$, in particular, $(y \lor z)^- = y^- \land z^-,
19. x \to (y \lor z) = (x \to y) \land (x \to z),
20. x \to (y \lor z) = (x \to y) \lor (x \to z),
21. (y \land z) \to x = (y \to x) \lor (z \to x)$, in particular, $(y \land z)^- = y^- \lor z^-,
22. x \ast (y \land z) = (x \ast y) \land (x \ast z),
23. $x \lor y = 1$ implies $x \ast y = x \land y$.

where $x^- = x \to 0$.

**Definition 2.3** ([8]). Let $F$ be a nonempty subset of $L$. $F$ is said to be a filter of $L$ if it satisfies:

1. $x, y \in F$ implies $x \ast y \in F$,
2. $x \in F$ and $x \leq y$ imply $y \in F$.

A nonempty subset $D$ of $L$ is said to be a deductive system of $L$ if it satisfies:

1. $1 \in D$,
2. $x \in D$ and $x \to y \in D$ imply $y \in D$.
**Proposition 2.4** ([19]). Let \( F \) be a nonempty subset of \( L \). Then \( F \) is a deductive system of \( L \) if and only if \( F \) is a filter of \( L \).

The set of all filters of \( L \) is denoted by \( \mathcal{F}(L) \). It is obvious that \( \{1\}, L \in \mathcal{F}(L) \). The following proposition will be repeatedly used.

**Proposition 2.5** ([19]). If \( X \) is a nonempty subset of \( L \), then

\[
[X] = \{a \in L \mid x_1 \cdots x_n \leq a \text{ for some } x_1, \ldots, x_n \in X\}
\]

is a filter of \( L \) and \( X \subseteq [X] \). \( [X] \) is called the filter generated by \( X \). If \( X = \{x\} \), we simply denote \([x] = \{x\}\). If \( x, y \in L \) with \( x \leq y \), then \([y] \subseteq [x]\). For any \( x, y \in L \), \([x \lor y] = [x] \cap [y]\).

Turunen [19] and Leustean [15] introduced the notion of co-annihilators of \( L \) and investigated some of its important properties, essentially Leustean [15] also studied the sets of the form \( x^{-1}A \) for any element \( x \in L \) and any nonempty subset \( A \) of \( L \). We will introduce the concept of generalized co-annihilators of \( L \), which is a generalization of these two ones, and also is a unifying treatment of them.

Given a nonempty subset \( A \) of \( L \), the set

\[
\perp A = \{x \in L \mid x \lor a = 1 \text{ for all } a \in A\}
\]

is said to be a co-annihilator of \( A \). It is proved that \( \perp A \in \mathcal{F}(L) \), \( \perp \{0\} = \perp L = \{1\}, \perp \{1\} = L, A \subseteq \perp \perp A, \perp A = \perp \perp \perp A, \perp A = \perp A \), where \( \perp \perp \perp \) means \( \perp (\perp A) \) (the double co-annihilator). For short we write \( \perp(a) \) instead of \( \perp\{a\} \) (see [19]).

For any \( x \in L \) and any \( F \in \mathcal{F}(L) \), define \( x^{-1}F := \{y \in L \mid y \lor x \in F\} \). If \( F = \{1\} \), then \( x^{-1}\{1\} = \{y \in L \mid y \lor x \in \{1\}\} = \{y \in L \mid y \lor x = 1\} = \perp(x) \). It has been proved that \( F \subseteq x^{-1}F \in \mathcal{F}(L) \) (see [15]).

In what follows we give an inequality, which is very useful in the sequel.

**Proposition 2.6.** For all \( x, y, z \in L \), the following inequality is true:

\[
x \lor (y \rightarrow z) \leq (x \lor y) \rightarrow (x \lor z).
\]

*Proof.* Since \( (x \lor y) \ast (y \rightarrow z) = [x \ast (y \rightarrow z)] \lor [y \ast (y \rightarrow z)] \leq x \lor z \), it follows that

\[
(y \rightarrow z) \leq (x \lor y) \rightarrow (x \lor z),
\]

(\(\ast\))

Also,

\[
x \leq x \lor z \leq (x \lor y) \rightarrow (x \lor z).
\]

(\(\ast\ast\))

By (\(\ast\)) and (\(\ast\ast\)) we obtain \( x \lor (y \rightarrow z) \leq (x \lor y) \rightarrow (x \lor z) \).

\[\square\]

The above definitions and results will be used and we will not cite them every time they are used.

## 3 Generalized co-annihilators

In this section we introduce the concept of generalized co-annihilators of \( BL \)-algebras, which is a generalization of co-annihilators, and investigate its basic properties. For any \( x \in L \) and any subsets \( A, B \subseteq L \), denote

\[
A \lor B := \{a \lor b \mid a \in A, b \in B\}, \quad x \lor B := \{x \lor b \mid b \in B\}.
\]

If \( A, B \) are filters of \( L \), then \( A \lor B = A \cap B \).

**Definition 3.1.** Let \( F \) be a filter of \( L \) and \( A \) be a nonempty subset of \( L \), then the set

\[
(F : A) := \{x \in L \mid x \lor A \subseteq F\} = \{x \in L \mid x \lor a \in F \text{ for all } a \in A\}
\]

is called the generalized co-annihilator of \( A \) relative to \( F \).
It is easily to see the following facts:

Obviously, \((F : A) \neq \emptyset\) since \(1 \in (F : A)\).

If \(F = \{1\}\), then \(\{1\} : A = \{x \in L | x \lor a = 1 \text{ for all } a \in A\} = A\).

If \(A = \{y\}\), then we simply denote

\[
y^{-1}F = \{x \in L | x \lor y \in F\} = (F : \{y\}).
\]

It is clear that \(0^{-1}F = F\) and \(1^{-1}F = L\) for any filter \(F \in \mathcal{F}(L)\).

It is obvious that \(x^{-1}\{1\} = \downarrow(x)\).

If \(x \in (F : A)\), then \(x \lor A \subseteq F\), and so \(A \subseteq x^{-1}F\). Thus

\[
(F : A) = \{x \in L | A \subseteq x^{-1}F\}.
\]

Example 3.2 ([16, Example 1]). Let \(L = [0, 1]\). Define \(*\) and \(\to\) as follows:

\[
x \ast y = \min\{x, y\},
\]

and

\[
x \to y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{if } x > y. \end{cases}
\]

Then \((L; \land, \lor, \ast, \to, 0, 1)\) is a BL-algebra and it is clear that \(F = [1/4, 1]\) is a filter. Let \(A = [1/2, 1]\), then

\[
(F : A) = L, \quad \downarrow F = \{1\}, \quad \downarrow \downarrow F = L \neq F.
\]

Theorem 3.3. Let \(F\) be a filter of \(L\) and \(A\) be a nonempty subset of \(L\). Then

(i) \((F : A)\) is a filter of \(L\).

(ii) \(F \subseteq (F : A)\).

Proof. Suppose \(x \to y, x \in (F : A)\), that is, \((x \to y) \lor A \subseteq F, x \lor A \subseteq F\), thus for all \(a \in A\), \((x \to y) \lor a \in F, x \lor a \in F\), by Proposition 2.6, \((x \to y) \lor a \leq (x \lor a) \to (y \lor a)\). It follows from \((x \to y) \lor a \in F\) that \((x \lor a) \to (y \lor a) \in F\). Observe \(x \lor a \in F\) and \(F\) is a filter, we have \(y \lor a \in F\). Hence \(y \lor A \subseteq F\), i.e., \(y \in (F : A)\). Therefore \((F : A)\) is a filter of \(L\). (i) holds.

Now we prove that \(F \subseteq (F : A)\). If \(x \in F\), then for all \(a \in A\), \(x \lor a \in F\) since \(F\) is a filter. Hence \(x \lor A \subseteq F\), i.e., \(x \in (F : A)\), and so \(F \subseteq (F : A)\). (ii) is true.

Corollary 3.4. Let \(F\) be a filter of \(L\) and \(A\) be a nonempty subset of \(L\). Then:

(i) \(\downarrow A\) is a filter of \(L\) (see [19]).

(ii) For any \(y \in L\), \(y^{-1}F\) is a filter of \(L\) and \(F \subseteq y^{-1}F\) (see [15]).

Proof. (i) follows by taking \(F = \{1\}\) in Theorem 3.3. (ii) is a special case of Theorem 3.3 when \(A = \{y\}\).

In what follows we give a relation of \((F : A)\) with \(y^{-1}F\).

Proposition 3.5. Let \(F\) be a filter of \(L\) and \(A\) be a nonempty subset of \(L\). Then

(i) \((F : A) = \bigcap_{y \in A} y^{-1}F\),

(ii) \((F : (F : A)) = \bigcap_{y \in (F : A)} y^{-1}F\).

Proof. Obviously \((F : A) \subseteq \bigcap_{y \in A} y^{-1}F\). To show the converse inclusion, let \(x \in \bigcap_{y \in A} y^{-1}F\), then for all \(y \in A\), \(x \in y^{-1}F\), i.e., \(y \in A, x \lor y \in F\). Hence \(x \lor A \subseteq F\), and so \(x \in (F : A)\). Thus \(\bigcap_{y \in A} y^{-1}F \subseteq (F : A)\). (i) holds.

(ii) follows from replacing \(A\) by \((F : A)\) in (i).

Before further results on generalized co-annihilators we need the following proposition.
Proposition 3.6. Let $F$ be a filter of $L$ and $A$ be a nonempty subset of $L$. Then $(x_1 \cdots \cdots x_n) \vee y \in F$ for any $y \in (F : A)$ and $x_i \in A(i = 1, \cdots , n)$.

Proof. Since $y \in (F : A)$ and $x_i \in A(i = 1, \cdots , n)$, it follows that $y \vee x_i \in F(i = 1, \cdots , n)$. We now demonstrate, by induction on $n$, that also $(x_1 \cdots \cdots x_n) \vee y \in F$. If $n = 1$, then the conclusion is clearly true. Now assume it is true for $n = k$. Let $n = k + 1$ and denote $x = x_1 \cdots \cdots x_{k+1}$. Let us calculate

$$(y \vee x) \vee (y \vee x_{k+1})$$

$$(y \vee x) \vee (y \vee x_{k+1})$$

$$(y \vee x) \vee (y \vee x_{k+1})$$

$$(y \vee x) \vee (y \vee x_{k+1})$$

$$(y \vee x) \vee (y \vee x_{k+1})$$

Since $y \vee x \in F$ by induction hypothesis and $y \vee x_{k+1} \in F$, it follows from (F1) that $(y \vee x) \vee (y \vee x_{k+1}) \in F$, and so $y \vee (x_1 \cdots \cdots x_{k+1}) \in F$. Thus the conclusion is true for all natural numbers $n$.

Theorem 3.7. Let $F$ be a filter of $L$ and $B$, $C$ be nonempty subsets of $L$. Then the following hold:

(i) $B \subseteq C \Rightarrow (F : C) \subseteq (F : B)$,

(ii) $(F : B) = (F : [B])$,

(iii) $B \subseteq (F : (F : B))$,

(iv) $(F : B) = (F : (F : (F : B)))$,

(v) If $B$ is also a filter of $L$, then $(F : B) \cap B \subseteq F$,

(vi) If $B$ and $C$ are also filters of $L$, then $B \cap C \subseteq F$ if and only if $C \subseteq (F : B)$,

(vii) If $B$ is also a filters of $L$ with $F \subseteq B$, then $(F : B) \cap B = F$,

(viii) $(F : B) \cap (F : (F : B)) = F$,

(ix) $(F : L) = F$,

(x) $(F : B)$ is $B$ if and only if $F \subseteq B$. In particular, $(F : F) = L$,

(xi) $(F : (F : F)) = F$,

(xii) $(F : (F : L)) = L$.

Proof. Let $x \in (F : C)$, then $x \vee C \subseteq F$. Since $B \subseteq C$ implies $x \vee B \subseteq x \vee C$, we have $x \vee B \subseteq F$. Thus $(F : C) \subseteq (F : B)$, (i) holds.

Since $B \subseteq [B]$, it follows from (i) that $(F : [B]) \subseteq (F : B)$. To prove the converse inclusion, assume $y \in (F : B)$. Then, for any $x \in B$, $x \vee y \in F$. If now $z \in [B]$, then there are $x_i \in (i = 1, \cdots , n)$ such that $x_1 \cdots \cdots x_n \subseteq z$. Therefore, by proposition 3.6, $y \vee (x_1 \cdots \cdots x_n) \in F$. Since $y \vee (x_1 \cdots \cdots x_n) \subseteq y \wedge z$, it follows from $F$ being a filter of $L$, that $y \wedge z \in F$, and $y \in (F : [B])$. (ii) holds.

If $b \in B$, then by definition of $(F : B)$ we have $x \vee b \in F$ for all $x \in (F : B)$. This shows $b \in (F : (F : B))$, and $B \subseteq (F : (F : B))$, (iii) holds.

$(F : B) \subseteq (F : (F : (F : B)))$ follows directly from replacing $B$ by $(F : B)$ in (iii). Since $B \subseteq (F : (F : B))$, (iv) follows. (F : B) \subseteq (F : (F : B)) because $F \subseteq (F : B)$. Hence (iv) holds.

Let $x \in (F : B) \cap B$, then $x \in B$ and $x \in (F : B)$. Therefore $x = x \vee x \in F$, and $(F : B) \cap B \subseteq F$, (v) holds.

Suppose that $B$, $C$ are filters of $L$. Let $B \cap C \subseteq F$. By $B \cap C = B \cap C$ we have $B \cap C \subseteq F$. Hence $C \subseteq (F : B)$. Conversely, let $C \subseteq (F : B)$. By (v),

$$B \cap C \subseteq (F : B) \cap B \subseteq F.$$ (vi) is true.

Suppose that $B$ is a filters of $L$. By (v), $(F : B) \cap B \subseteq F$. On the other hand, if $F \subseteq B$, then $F = F \cap B \subseteq (F : B) \cap B$ because $F \subseteq (F : B)$. Therefore $(F : B) \cap B = F$. (vii) holds.

By Theorem 3.3, $F \subseteq (F : B)$, and $(F : B)$ is a filter of $L$. Replacing $B$ by $(F : B)$ in (vii) we obtain (viii).

In (vii) let $B = L$ we obtain $(F : L) = (F : B) \cap L = F$. (ix) holds.
If \(B \subseteq F\), then \(x \vee B \subseteq F\) for all \(x \in L\), it follows that \((F : B) = L\). Conversely, if \((F : B) = L\), then for any \(b \in B\), we have \(b = b \vee b \in F\), and \(B \subseteq F\). (x) is true.

(xi) and (xii) are immediate consequences of (ix) and (x).

By Theorem 3.7(v) and (vi) it follows that for filters \(F\) and \(A\) of \(L\), \((F : A)\) is the greatest filter \(B\) of \(L\) such that \(A \cap B \subseteq F\). By Theorem 3.7(ii) we know that for a filter \(F\) of \(L\) and any \(a \in L\), \(a^{-1} F = (F : [a])\).

We now give another description of the generalized co-annihilator of a set of \(L\).

**Proposition 3.8.** Let \(F\) be a filter of \(L\) and \(A\) be a nonempty subset of \(L\). Then

\[
(F : A) = \{x \in L | [x] \cap [A] \subseteq F\}.
\]

**Proof.** Suppose that \(x \in L\) satisfies \([x] \cap [A] \subseteq F\). Since \([x]\) and \([A]\) are filters of \(L\), from Theorem 3.7(ii) and (vi) it follows that

\[
x \in [x] \subseteq (F : [A]) = (F : A).
\]

Therefore

\[
\{x \in L | [x] \cap [A] \subseteq F\} \subseteq (F : A).
\]

Conversely, let \(x \in (F : A)\). Since \((F : A) = (F : [A])\), it follows that \(x \in (F : [A])\), and so \([x] \subseteq (F : [A])\). Thus \([x] \cap [A] \subseteq F\) by Theorem 3.7(vi).

**Corollary 3.9 ([15, 19]).** Let \(F\) be a filter of \(L\) and \(B, C\) be nonempty subsets of \(L\). Then the following hold:

(i) \(B \subseteq C \Rightarrow \downarrow C \subseteq \downarrow B\),
(ii) \(\downarrow B = \downarrow [B]\),
(iii) \(B \subseteq \downarrow \downarrow B\),
(iv) \(\downarrow B = \downarrow \downarrow \downarrow B\),
(v) If \(B\) is a filter of \(L\), then \(\downarrow B \cap B = \{1\}\),
(vi) If \(B, C\) are filters of \(L\), then \(B \cap C = \{1\}\) if and only if \(C \subseteq \downarrow B\),
(vii) \(\downarrow L = \{1\}\),
(viii) \(\downarrow B = \downarrow \downarrow B\) if and only if \(B = \{1\}\),
(ix) If \(a, b \in L\) with \(a \leq b\), then \(a^{-1} F \subseteq b^{-1} F\),
(x) \(a^{-1} F = L\) if and only if \(a \in F\),
(xi) \(\downarrow (a) \subseteq a^{-1} F\) for all \(a \in L\),
(xii) \((a \lor b)^{-1} F = a^{-1} (b^{-1} F) = b^{-1} (a^{-1} F)\) for all \(a, b \in L\); in particular, \(a^{-1} F = a^{-1} (a^{-1} F)\) for all \(a \in L\),
(xiii) If \(F\) is a filter of \(L\), then for any \(a, b \in L\),

\[
(a^{-1} F) \cap (b^{-1} F) = (a \land b)^{-1} F = (a \land b)^{-1} F.
\]

**Proof.** We only prove (xii) and (xiii). The other conclusions are easy and omitted.

Since

\[
x \in (a \lor b)^{-1} F \iff x \lor (a \lor b) \in F
\]

\[
\iff x \lor a \in b^{-1} F
\]

\[
\iff x \in a^{-1} (b^{-1} F),
\]

we have \((a \lor b)^{-1} F = a^{-1} (b^{-1} F)\). If \(a = b\), then \(a^{-1} F = a^{-1} (a^{-1} F)\). (xii) holds. Since \(a \land b \leq a \land b \leq a, b\), it follows from (ix) that \((a \land b)^{-1} F \subseteq (a \land b)^{-1} F \subseteq \downarrow a^{-1} F, b^{-1} F\), then \((a \land b)^{-1} F \subseteq (a \land b)^{-1} F \subseteq (a^{-1} F) \cap (b^{-1} F)\). Conversely, if \(x \in (a^{-1} F) \cap (b^{-1} F)\), then \(x \lor a \in F\) and \(x \lor b \in F\). Since \((x \lor a) \land (x \lor b) \leq x \lor (a \land b)\) and \(F\) is a filter, we have \(x \lor (a \land b) \in F\), hence \(x \in (a \land b)^{-1} F\). This shows that \((a^{-1} F) \cap (b^{-1} F) \subseteq (a \land b)^{-1} F\). Therefore \((a^{-1} F) \cap (b^{-1} F) = (a \land b)^{-1} F = (a \land b)^{-1} F\). (xiii) holds.

By Corollary 3.9(v) and (vi) it follows that for a filter \(B\) of \(L\), \(\downarrow B\) is the greatest filter \(C\) of \(L\) such that \(B \cap C = \{1\}\). Let \(a \land b = 0\) in (xiii), we obtain \(F = (a^{-1} F) \cap (b^{-1} F)\), which is a decomposition of filters in \(BL\)-algebras.
Proposition 3.10. Let $F$ be a filter of $L$ and $B_\lambda (\lambda \in \Lambda \neq \emptyset)$ be nonempty subsets of $L$. Then

$$(F : \bigcup_{\lambda \in \Lambda} B_\lambda) = \bigcap_{\lambda \in \Lambda} (F : B_\lambda).$$

Proof. Since $B_\lambda \subseteq \bigcup_{\lambda \in \Lambda} B_\lambda$ for any $\lambda \in \Lambda$, it follows from Theorem 3.7(i) that

$$(F : \bigcup_{\lambda \in \Lambda} B_\lambda) \subseteq (F : B_\lambda).$$

therefore

$$(F : \bigcup_{\lambda \in \Lambda} B_\lambda) \subseteq \bigcap_{\lambda \in \Lambda} (F : B_\lambda).$$

To prove the converse inclusion take $c \in \bigcap_{\lambda \in \Lambda} (F : B_\lambda)$. Then for all $\lambda \in \Lambda$, $c \in (F : B_\lambda)$. Hence for all $\lambda \in \Lambda$, we have $c \vee B_\lambda \subseteq F$. Thus for all $\lambda \in \Lambda$, $c \vee \bigcup_{\lambda \in \Lambda} B_\lambda \subseteq F$, and $c \in (F : \bigcup_{\lambda \in \Lambda} B_\lambda)$. Hence

$$(\bigcap_{\lambda \in \Lambda} (F : B_\lambda) \subseteq (F : \bigcup_{\lambda \in \Lambda} B_\lambda).$$

Therefore we obtain

$$(F : \bigcup_{\lambda \in \Lambda} B_\lambda) = \bigcap_{\lambda \in \Lambda} (F : B_\lambda).$$

This completes the proof.

Corollary 3.11. Let $F$ be a filter of $L$ and $B$ be a nonempty subset of $L$. Then:

(i) $(F : B) = \bigcap_{b \in B} b^{-1} F$,

(ii) $\perp B = \bigcap_{b \in B} \perp (b)$.

Proof. By Proposition 3.10,

$$(F : B) = (F : \bigcup_{b \in B} \{b\}) = \bigcap_{b \in B} (F : \{b\}) = \bigcap_{b \in B} b^{-1} F,$$

(i) holds.

Let $F = \{1\}$ in (i) we obtain (ii).

Proposition 3.12. Let $F$, $A$, $B$ be filters of $L$. Then

$$(F : (F : A \bigcap B)) = (F : (F : A)) \bigcap (F : (F : B)).$$

Proof. By $A \bigcap B \subseteq A, B$, it is easy to see that

$$(F : (F : (A \bigcap B))) \subseteq (F : (F : A)) \bigcap (F : (F : B)).$$

In order to prove the converse inclusion, let $z \in (F : (F : A)) \bigcap (F : (F : B))$. For any $x \in A$ and $y \in B$ we have $x \vee y \in A \bigcap B$. By definition, for all $u \in (F : (A \bigcap B))$, $u \vee x \vee y \in F$, so $z \vee u \vee x \vee y \in F$. This shows that $z \vee u \vee x \in (F : B)$ since $z \leq z \vee u \vee x$ and $z \in (F : (F : B))$, it follows that $z \vee u \vee x \in (F : (F : B))$. This means $z \vee u \vee x \in (F : B) \bigcap (F : (F : B)) = F$, and so $z \vee u \in (F : (A))$. On the other hand, since $z \in (F : (F : A))$ and $(F : (F : A))$ is a filter of $L$, we have $z \vee u \in (F : (F : A))$. Therefore $z \vee u \in (F : (A) \bigcap (F : (F : A))) = F$. Thus $z \in (F : (F : (A \bigcap B)))$, and

$$(F : (F : A)) \bigcap (F : (F : B)) \subseteq (F : (F : (A \bigcap B))).$$

This completes the proof.
Proposition 3.13. Let $A_{\lambda}(\lambda \in \Lambda \neq \emptyset)$ be filters of $L$ and $B$ be a nonempty subset of $L$. Then

$$(\bigcap_{\lambda \in \Lambda} A_{\lambda} : B) = \bigcap_{\lambda \in \Lambda} (A_{\lambda} : B).$$

Proof. Since $x \in (\bigcap_{\lambda \in \Lambda} A_{\lambda} : B)$ if and only if $x \vee B \subseteq \bigcap_{\lambda \in \Lambda} A_{\lambda}$ if and only if $x \vee B \subseteq A_{\lambda}$ for all $\lambda \in \Lambda$ if and only if $x \in (A_{\lambda} : B)$ for all $\lambda \in \Lambda$ if and only if $x \in \bigcap_{\lambda \in \Lambda} (A_{\lambda} : B)$. This completes the proof of $(\bigcap_{\lambda \in \Lambda} A_{\lambda} : B) = \bigcap_{\lambda \in \Lambda} (A_{\lambda} : B)$. $\square$

Proposition 3.14. Let $A$ and $B$ be filters of $L$ satisfying $A \subseteq B$. Then for any nonempty subset $C$ of $L$, we have $(A : C) \subseteq (B : C)$. In particular, $\perp C \subseteq (A : C)$.

Proof. Since $A \subseteq B$, then for any $x \in L$, $x \vee C \subseteq A$ implies $x \vee C \subseteq B$, it follows that $(A : C) \subseteq (B : C)$. $\square$

In the above Proposition let $C = \{x\}$ we obtain

Corollary 3.15 ([15]). Let $A$ and $B$ be filters of $L$ satisfying $A \subseteq B$. Then for any $x \in L$, we have $x^{-1}A \subseteq x^{-1}B$.

## 4 Involutory filters

In this section, we will investigate involutory filters.

Definition 4.1. Let $A, F$ be filters of $L$. $F$ is said to be involutory relative to $A$ if $F = (A : (A : F))$. If every filter of $L$ is involutory relative to $A$, then $L$ is called an involutory BL-algebra relative to $A$. If $A = \{1\}$, we will simply call $F$ to be an involutory filter, i.e., $F$ is an involutory filter if $F = \perp F$. If every filter of $L$ is involutory, then $L$ is called an involutory BL-algebra. The set of all involutory filters relative to $A$ of $L$ is denoted by $S_A(L)$. The set of all involutory filters of $L$ is denoted by $S(L)$.

Remark. Let $A$ is a filter of $L$. It is easy to see that for any filter $B$ of $L$, $B \in S_A(L)$ if and only if there is a filter $C$ such that $A \subseteq C$ and $F = (A : C) \in S_A(L)$ by Theorem 3.7(4). Thus $S_A(L) = \{(A : C) \mid C \in \mathcal{F}(L) \text{ and } A \subseteq C\}$.

Example 4.2. Let $(L; \wedge, \vee, *, \rightarrow, 0, 1)$ be defined as in Example 3.2. Then $F = [1/2, 1]$ is a filter. Obviously, $\perp F = \{1\}$, $\perp \perp F = \perp \{1\} = [0, 1]$. Therefore $\perp \perp F \neq F$, and $F$ is not an involutory filter of $L$.

Example 4.3 ([16, Example 2]). Let $L = \{0, a, b, 1\}$. Define $*$ and $\rightarrow$ as follows

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
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<th>1</th>
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<tr>
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<tr>
<td>a</td>
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</table>

The order relations are follows: $0 \leq a \leq 1, b \leq 1, a$ and $b$ are incomparable. Then $(L; \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra. It is easy to check that $F_1 = \{1\}$, $F_a = \{a\}$, $F_b = \{b\} = \{1, b\}$, $L$ are all filters of $L$.

(i) $(F_a : F_1) = L$.
(ii) $(F_a : F_a) = L$. 

}\]
(\(F_a : F_b\)) = F_a because \(b \lor F_b = F_b \not\subseteq F_a\).

(\(F_a : L\)) = F_a because \(b \lor L = F_b \not\subseteq F_a\).

\(\perp (i) \Rightarrow F_1 = L\).

\(\perp F_a = F_b\) because \(0 \lor a = a \lor a = a \neq 1\) and \(b \lor a = 1\).

\(\perp F_b = F_a\) because \(0 \lor b = b \lor b = b \neq 1\) and \(a \lor b = 1\).

\(\perp L = F_1\) because \(b \lor L = F_b \not\subseteq \{1\}\), \(a \lor L = F_a \not\subseteq \{1\}\).

(ii) \((F_a : (F_a : F_1)) = (F_a : L) = F_a \neq F_1\), so \(F_1\) is not involutory relative to \(F_a\).

(iii) \((F_a : (F_a : F_1)) = (F_a : L) = F_a\), so \(F_a\) is involutory relative to \(F_a\).

(iv) \((F_a : (F_a : F_1)) = (F_a : L) = F_a\), so \(F_a\) is involutory relative to \(F_a\).

Proposition 4.4. Let \(A\) be a filter of \(L\) and \(B\) be a nonempty subset of \(L\) such that \([B] \in S_A(L)\). Then \([B] = (A : (A : B))\).

Proof. By Theorem 3.7(ii), \((A : [B]) = (A : B)\). Since \((A : (A : [B]) = [B])\), it follows that

\([B] = (A : (A : [B])) = (A : (A : B))\).

The proof is complete.

Proposition 4.5. Let \(A\) be a filter of \(L\). If \(B, C \in S_A(L)\), then

(i) \(A \subseteq B\).

(ii) \((A : B) \cap B = A\).

(iii) \(B \cap C \subseteq A\) implies \(C \subseteq (A : B)\).

(iv) \(A, L \in S_A(L)\).

Proof. By Theorem 3.7(viii) it follows that

\(A = (A : B) \cap (A : (A : B)) = (A : B) \cap B \subseteq B\).

(i) holds. (ii) and (iii) are immediate consequences of (i) and Theorem 3.7(vi) and (vii). (iv) follows from Theorem 3.7(xi) and (xii).

Remark. This Proposition shows that \(A\) and \(L\) are the least element and the largest element in the \(S_A(L)\) with respect to the set-theoretic inclusion, respectively.

Proposition 4.6. Let \(A\) be a filter of \(L\) and \(\{X_\lambda \mid \lambda \in \Lambda\} \subseteq S_A(L)\) where \(\Lambda \neq \emptyset\). Then \(\bigcap_{\lambda \in \Lambda} X_\lambda \in S_A(L)\). Hence \(\bigcap_{\lambda \in \Lambda} X_\lambda\) is the infimum of the set \(\{X_\lambda \mid \lambda \in \Lambda\}\) in \(S_A(L)\) with respect to the set-theoretic inclusion.

Proof. By Proposition 3.10 we have

\(\bigcap_{\lambda \in \Lambda} X_\lambda = \bigcap_{\lambda \in \Lambda} (A : (A : X_\lambda)) = (A : \bigcup_{\lambda \in \Lambda} (A : X_\lambda)),\)

it follows that

\((A : (A : \bigcap_{\lambda \in \Lambda} X_\lambda)) = (A : (A : \bigcup_{\lambda \in \Lambda} (A : X_\lambda))) = (A : \bigcup_{\lambda \in \Lambda} (A : X_\lambda)) = \bigcap_{\lambda \in \Lambda} X_\lambda,\)

so \(\bigcap_{\lambda \in \Lambda} X_\lambda \in S_A(L)\).
Let \( A \) be a filter of \( L \). For any nonempty subsets \( X, Y, X_\lambda (\lambda \in \Lambda \neq \emptyset) \) of \( L \), define
\[
X \bigcup Y := (A : (A : X \bigcup Y)), \quad \bigcup_{\lambda \in \Lambda} X_\lambda := (A : (\bigcup_{\lambda \in \Lambda} X_\lambda)).
\]

By Proposition 3.10, we have
\[
X \bigcup Y = (A : (A : X) \bigcap (A : Y)), \quad \bigcup_{\lambda \in \Lambda} X_\lambda = (A : \bigcap_{\lambda \in \Lambda} (A : X_\lambda)).
\]

**Proposition 4.7.** Let \( A \) be a filter of \( L \) and \( \{X_\lambda | \lambda \in \Lambda \} \subseteq S_\Lambda(L) \) where \( \Lambda \neq \emptyset \). Then \( (A : (\bigcup_{\lambda \in \Lambda} X_\lambda)) \in S_\Lambda(L) \) and \( \bigcup_{\lambda \in \Lambda} X_\lambda := (A : (\bigcup_{\lambda \in \Lambda} X_\lambda)) \) is the supremum of the set \( \{X_\lambda | \lambda \in \Lambda \} \) in \( S_\Lambda(L) \) with respect to the set-theoretic inclusion.

**Proof.** It is clear that \( (A : (\bigcup_{\lambda \in \Lambda} X_\lambda)) \in S_\Lambda(L) \).

Since \( X_\lambda \subseteq \bigcup_{\lambda \in \Lambda} X_\lambda \subseteq (A : (\bigcup_{\lambda \in \Lambda} X_\lambda)) \) for all \( \lambda \in \Lambda \), it follows that \( (A : (\bigcup_{\lambda \in \Lambda} X_\lambda)) \) is an upper bound of the set \( \{X_\lambda | \lambda \in \Lambda \} \). We now take any \( X \in S_\Lambda(L) \) such that \( X_\lambda \subseteq X \) for all \( \lambda \in \Lambda \). Then \( (A : X) \subseteq (A : X_\lambda) \), and so \( (A : X) \subseteq \bigcap_{\lambda \in \Lambda} (A : X_\lambda) \). From Proposition 3.10 it follows that
\[
(A : (\bigcup_{\lambda \in \Lambda} X_\lambda)) = (A : \bigcap_{\lambda \in \Lambda} (A : X_\lambda)) \subseteq (A : (A : X)) = X.
\]

Therefore \( \bigcup_{\lambda \in \Lambda} X_\lambda = (A : (\bigcup_{\lambda \in \Lambda} X_\lambda)) \) is the supremum of the set \( \{X_\lambda | \lambda \in \Lambda \} \) in \( S_\Lambda(L) \) with respect to the set-theoretic inclusion. \( \square \)

**Theorem 4.8.** Let \( A \) be a filter of \( L \). Then \( (S_\Lambda(L); \bigcap, \bigcup, A, L) \) is a complete Boolean lattice, in which \( A \) is the least element, \( L \) is the largest element, respectively.

**Proof.** By Propositions 4.6 and 4.7, \( (S_\Lambda(L); \bigcap, \bigcup, A, L) \) is a complete lattice. Proposition 4.5 shows that \( A \) and \( L \) are the least element and the largest element in the \( S_\Lambda(L) \), respectively.

Since, for any \( X \in S_\Lambda(L) \), by Proposition 4.5(ii) we have
\[
X \bigcap (A : X) = A,
\]
\[
X \bigcup (A : X) = (A : (A : X) \bigcup (A : X))
= (A : (A : X) \bigcap X)
= (A : A) = L,
\]

hence the lattice \( (S_\Lambda(L); \bigcap, \bigcup, A, L) \) is a complemented lattice.

In what follows, we prove the distributive law. For any \( X, Y, Z \in S_\Lambda(L) \), denote \( D = (X \bigcap Y) \bigcup (X \bigcap Z) \), then \( X \bigcap Y \subseteq D \) and \( X \bigcap Z \subseteq D \). Thus \( Y \bigcap (X \bigcap (A : D)) = A \) and \( Z \bigcap (X \bigcap (A : D)) = A \), and so
\[
Y \subseteq (A : X \bigcap (A : D)),
\]
\[
Z \subseteq (A : X \bigcap (A : D)).
\]

Hence
\[
Y \bigcup Z \subseteq (A : X \bigcap (A : D)).
\]

It follows that \( (Y \bigcup Z) \bigcap X \bigcap (A : D) = A \), and
\[
X \bigcap (Y \bigcup Z) \subseteq (A : (A : D)) = D = (X \bigcap Y) \bigcup (X \bigcap Z).
\]

The converse inclusion is clear. The lattice \( (S_\Lambda(L); \bigcap, \bigcup, A, L) \) is distributive, and a Boolean lattice. \( \square \)
Corollary 4.9 ([15]). Let \( L \) be a \( BL \)-algebra. Then \((S(L); \cap, \cup, \{1\}, L)\) is a complete Boolean lattice in which \( \{1\} \) is the least element, \( L \) is the largest element, respectively.

**Proof.** It follows immediately from letting \( A = \{1\} \) in Theorem 4.8. \( \square \)

**Theorem 4.10.** Let \( A \) be a filter of \( L \). For any \( X, Y \in S_A(L) \), define

\[
X \leq Y \text{ if and only if } X \subseteq Y,
\]

\[
X \rightarrow Y = Y \bigcup (A : X) \in S_A(L),
\]

\[
X \ast Y = X \cap Y \in S_A(L).
\]

Then \((S_A(L); \cap, \cup, \ast, \rightarrow, A, L)\) is a \( BL \)-algebra where \( A \) is the least element, \( L \) is the largest element.

**Proof.** In Theorem 4.8 we have proved that \((S_A(L); \cap, \cup, A, L)\) is a lattice with \( A \) as the least element, \( L \) as the largest element, where the order relation is the set-theoretic inclusion. Hence \((S_A(L); \cap, \cup, \ast, \rightarrow, A, L)\) satisfies (BL1).

It is obvious that \((S_A(L); \cap, \cup, \ast, \rightarrow, A, L)\) satisfies (BL2).

Let \( X, Y, Z \in S_A(L) \). If \( X \subseteq Y \rightarrow Z \), then \( X \subseteq Z \bigcup (A : Y) \), and

\[
X \ast Y = X \cap Y \subseteq (Z \bigcup (A : Y)) \cap Y = (Z \cap Y) \bigcup (A : Y \cap Y) = (Z \cap Y) \bigcup A = Z \cap Y \subseteq Z.
\]

Therefore \( X \leq Y \rightarrow Z \) implies \( X \ast Y \leq Z \).

Conversely, if \( X \ast Y \leq Z \) then

\[
X = X \cap (Y \bigcup (A : Y)) = (X \cap Y) \bigcup (X \cap (A : Y)) \subseteq Z \bigcup (A : Y) = Y \rightarrow Z.
\]

Therefore \( X \ast Y \leq Z \) implies \( X \leq Y \rightarrow Z \). Hence \((S_A(L); \cap, \cup, \ast, \rightarrow, A, L)\) satisfies (BL3).

Let \( X, Y \in S_A(L) \). Then

\[
X \ast (X \rightarrow Y) = X \cap (Y \bigcup (A : X)) = (X \cap Y) \bigcup (X \cap (A : X)) = (X \cap Y) \bigcup A = X \cap Y.
\]

Thus \((S_A(L); \cap, \cup, \ast, \rightarrow, A, L)\) satisfies (BL4).

For any \( X, Y \in S_A(L) \), we have

\[
(X \rightarrow Y) \bigcup (Y \rightarrow X) = (Y \bigcup (A : X)) \bigcup (X \bigcup (A : Y)) = (X \bigcup (A : X)) \bigcup (Y \bigcup (A : Y)) = L \bigcup L = L.
\]

Thus \((S_A(L); \cap, \cup, \ast, \rightarrow, A, L)\) satisfies (BL5).

So \((S_A(L); \cap, \cup, \ast, \rightarrow, A, L)\) is a \( BL \)-algebra. \( \square \)

**Corollary 4.11.** Let \( L \) be a \( BL \)-algebra. For any \( X, Y \in S(L) \), define

\[
X \leq Y \text{ if and only if } X \subseteq Y,
\]

\[
X \rightarrow Y = Y \bigcup \downarrow X \in S(L),
\]

\[
X \ast Y = X \cap Y \in S(L).
\]

Then \((S(L); \cap, \cup, \ast, \rightarrow, \{1\}, L)\) is a \( BL \)-algebra with \( \{1\} \) the least element, \( L \) the largest element.

**Proof.** It follows immediately from letting \( A = \{1\} \) in Theorem 4.10. \( \square \)

The following result gives a relation between the notion of involutory filters relative to a filter and that of involutory filters.
Proposition 4.12. Let $A, B$ be filters of $L$ with $A \subseteq B$ and $B = \perp B$. Then $B = (A : (A : B))$.

Proof. It follows from Theorem 3.7(iii) that $B \subseteq (A : (A : B))$. To prove the converse inclusion it suffices to show that $x \notin B$ implies $x \notin (A : (A : B))$. Now let $x \notin B = \perp B$. Then $x \vee y \neq 1$ for some $y \in \perp B$. But $y \in \perp B$ implies $x \vee y \in \perp B$, by $\perp B \subseteq (A : (A : B))$ we have

$$x \vee y \in (A : B).$$

(1)

On the other hand, $x \vee y \neq 1, x \vee y \in \perp B$ and $B \cap \perp B = \{1\}$ imply $x \vee y \notin B$. By $A \subseteq B$ we obtain

$$x \vee y \notin A.$$  

(2)

By (1), (2) and $(A : B) \cap (A : (A : B))) = A$ we have

$$x \vee y \notin (A : (A : B)).$$

(3)

Therefore $x \notin (A : (A : B))$. This prove that $x \notin B$ implies $x \notin (A : (A : B))$.  

Corollary 4.13. Let $L$ be an involutory BL-algebra. If $A, B$ are filters of $L$ with $A \subseteq B$, then $B$ is an involutory filter relative to $A$.

5 Prime filters

In this section, we will utilize generalized co-annihilators to give characterizations of prime filters and minimal prime filters, respectively. In particular, we give a representation of co-annihilators in the quotient algebra of a BL-algebra $L$ via a filter $F$ by means of generalized co-annihilators relative to $F$ in $L$.

Definition 5.1 ([8]). A proper filter $P$ of $L$ is said to be prime if, for any $a, b \in L$, $a \lor b \in P$ implies $a \in P$ or $b \in P$, or equivalently, $a \notin P$ and $b \notin P$ imply $a \lor b \notin P$.

Proposition 5.2. Let $P$ be a proper filter of $L$. Then $P$ is a prime filter if and only if for any $x, y \in L, x \lor y = 1$ implies $x \in P$ or $y \in P$.

Proof. The necessity is obvious, we just prove the sufficiency. Let $x \lor y \in P$, by $x \lor y \leq (x \rightarrow y) \lor (y \rightarrow x) \rightarrow x$ and $P$ is a filter then we have $(x \rightarrow y) \rightarrow y \in P$ and $(y \rightarrow x) \rightarrow x \in P$. Also for any $x, y \in L$, $(x \rightarrow y) \lor (y \rightarrow x) = 1$, by the assumption we have $x \rightarrow y \in P$ or $y \rightarrow x \in P$, hence $y \in P$ or $x \in P$.

Proposition 5.3. Let $\{P_\lambda : \lambda \in \Lambda\}$ be a family of prime filters of $L$, totally ordered by inclusion. Then $\cap\{P_\lambda : \lambda \in \Lambda\}$ is a prime filter of $L$.

The set of all prime filters of $L$ is denoted by $\mathcal{PF}(L)$. A prime filter containing a filter $F$ is called a prime filter associated with $F$. The set of all prime filters associated with the $F$ of $L$ is denoted by $\mathcal{PF}_F(L)$. A minimal element of $\mathcal{PF}_F(L)$ with respect to set-theoretic inclusion is called a minimal prime filter associated with the filter $F$.

Proposition 5.4. Let $A$ be a filter of $L$ and $P$ be a prime filter with $A \subseteq P$. Then

(i) For any nonempty subset $B$ of $L \setminus P$, we have $(A : B) \subseteq P$.
(ii) If $B$ is a subset of $L$ with $B \cap (L \setminus P) \neq \emptyset$, then $(A : B) \subseteq P$.
(iii) $x^{-1} A \subseteq P$ for all $x \in L \setminus P$.

Proof. Let $B$ be any nonempty subset of $L \setminus P$. For any $y \in (A : B)$ we have $y \lor B \subseteq A \subseteq P$. Since $P$ is prime, it follows that $y \in P$, and so $(A : B) \subseteq P$. (i) holds.
Denote $C = B \cap (L \setminus P)$, then $C \subseteq B$ and $C \subseteq L \setminus P$. By (ii) and Theorem 3.7(1) we have

$$(A : B) \subseteq (A : C) \subseteq P.$$

(ii) is true. (iii) is a special case of (i).

**Corollary 5.5.** Let $P$ be a prime filter of $L$. Then the following holds

(i) For any nonempty subset $B$ of $L \setminus P$, we have $(P : B) = P$.

(ii) If $B$ is a subset of $L$ with $B \cap (L \setminus P) \neq \emptyset$, then $(P : B) = P$.

(iii) $x^{-1}P = P$ for all $x \notin P$.

**Proposition 5.6.** Suppose $A$ is a filter of $L$. Then $A$ is prime if and only if for any nonempty subset $B$ of $L$ with $(A : B) \neq L$, $(A : B) = A$.

**Proof.** Suppose $A$ is a prime filter of $L$ and $B$ is a nonempty subset of $L$ such that $(A : B) \neq L$, then $B \not\subseteq A$. By Corollary 5.5(i) we have $(A : B) = A$.

Conversely, suppose $(A : B) = A$ for any nonempty subset $B$ with $(A : B) \neq L$. To prove that $A$ is prime let $a \lor b \in A$ and $b \not\in A$, then $(A : \{b\}) \neq L$ since $b \lor b = b \not\in A$, and so $a \in (A : \{b\}) = A$. This shows that $A$ is prime.

**Proposition 5.7.** Let $A$ be a proper filter of $L$. Then $A$ is prime if and only if $x^{-1}A = A$ for all $x \in L \setminus A$.

**Proof.** Let $A$ be a prime filter of $L$, by Proposition 5.6 we have $x^{-1}A = (A : \{x\}) = A$ for all $x \in L \setminus A$.

Conversely, let $x^{-1}A = A$ for all $x \in L \setminus A$. If $a \lor b \in A$ and $b \not\in A$, then $a \in b^{-1}A = A$. This shows that $A$ is prime.

For a filter $F$ and a prime filter $P$ of $L$ denote $F_P := \{x \in L \mid x^{-1}F \not\subseteq P\}$. If $F = \{1\}$, then $\{1\}_P = \{x \in L \mid x \not\in P\}$. We will simply write $1_P$ instead of $\{1\}_P$.

**Proposition 5.8.** Let $F$ be a filter of $L$. If $P, Q \in \mathcal{P}_F(L)$ with $P \subseteq Q$, then $F_Q \subseteq F_P$.

**Proof.** Let $x \in F_Q$, then $x^{-1}F \not\subseteq Q$. It follows from $P \subseteq Q$ that $x^{-1}F \not\subseteq P$, so $x \in F_P$. Therefore $F_Q \subseteq F_P$.

**Proposition 5.9.** Suppose $F$ is a filter of $L$ and $P \in \mathcal{P}_F(L)$, then:

(i) $F \subseteq F_P$.

(ii) $x \in F_P$ if and only if there is $a \in L \setminus P$ such that $x \in a^{-1}F$ (equivalently, $x \lor a \in F$).

(iii) $F_P \subseteq P$.

(iv) $F_P$ is a filter of $L$.

**Proof.** (i). If $x \in F$ then by Corollary 3.9(x) we have $x^{-1}F = L \not\subseteq P$, hence $x \in F_P$, $F \subseteq F_P$.

(ii). Suppose $x \in F_P$, then $x^{-1}F \not\subseteq P$, thus there is $a \in x^{-1}F \setminus P$. While $a \in x^{-1}F$ implies $x \lor a \in F$. Conversely, if there is $a \in L \setminus P$ such that $x \lor a \in F$, then $a \in x^{-1}F$ and $a \not\in P$, hence $x^{-1}F \not\subseteq P$, i.e., $x \notin F_P$.

(iii). Let $x \in F_P$, by (ii) there is $a \in L \setminus P$ such that $x \lor a \subseteq F \subseteq P$. Since $P$ is prime, so we have $x \in P$, hence $F_P \subseteq P$.

(iv). It follows from $1^{-1}F = L$ that $1 \in F_P$. Also if $x \to y \in F_P$ and $x \in F_P$, then by (ii) there are $a, b \in L \setminus P$ such that $(x \to y) \lor a \in F$ and $x \lor b \in F$. Denote $c = a \lor b$, then $(x \to y) \lor c \in F$ and $x \lor c \in F$, so $x \to y \in c^{-1}F$ and $x \in c^{-1}F$. Because $c^{-1}F$ is a filter of $L$, we obtain $y \in c^{-1}F$. By $P \in \mathcal{P}_F(L)$ and $a, b \in L \setminus P$, it follows that $c \in L \setminus P$. This proves that $y \in F_P$ by (ii), therefore $F_P$ is a filter of $L$. (iv) holds.

We need the following result.
Proposition 5.10 ([7]). Let $F$ be a filter of $L$ and $S$ be a nonempty subset of $L$. If $F \cap S = \emptyset$ and $S$ is $\vee$-closed, then there is $P \in \mathcal{P}\mathcal{F}_F(L)$ such that $P \cap S = \emptyset$. In particular, for any $x \notin F$, there is a prime filter $P$ such that $x \notin P$ and $F \subseteq P$.

Theorem 5.11. Let $F$ be a filter of $L$ and $P \in \mathcal{P}\mathcal{F}_F(L)$. Then $P$ is a minimal prime filter associated with $F$ if and only if $F_P = P$.

Proof. Suppose $P$ is a minimal prime filter associated with $F$. In order to show $F_P = P$, it suffices to verify $P \setminus F_P = \emptyset$ since $F_P \subseteq P$ by Proposition 5.9(iii). Suppose there is $x_0 \in P \setminus F_P$, denote $S_0 := \{x_0 \vee x \mid x \in L \setminus P\}$ and $S := S_0 \cup (L \setminus P)$. We have the following assertions:

I: $x_0 \in S$. Indeed, there exists $x \in L \setminus P$ since $P$ is prime. From $x_0 \wedge x \leq x$ we have $x_0 \wedge x \notin P$. Hence $x_0 = x_0 \vee (x_0 \wedge x) \in S_0$, and $x_0 \in S$.

II: $S$ is $\vee$-closed. For any $a, b \in S$, consider the following cases.

(i) $a, b \in S_0$. Then there are $x, y \in L \setminus P$ such that $a = x_0 \vee x$ and $b = x_0 \vee y$. $P$ being prime implies $x \vee y \notin P$, and so

$$a \vee b = (x_0 \vee x) \vee (x_0 \vee y) = x_0 \vee (x \vee y).$$

Hence $a \vee b \in S_0 \subseteq S$.

(ii) $a, b \in L \setminus P$. $P$ being prime implies $a \vee b \in L \setminus P$. Hence $a \vee b \in S$.

(iii) $a \in S_0$ and $b \in L \setminus P$. Then there is $x \in L \setminus P$ such that $a = x_0 \vee x$. Notice that $x \vee b \in L \setminus P$ and

$$a \vee b = (x_0 \vee x) \vee b = x_0 \vee (x \vee b) \in S_0,$$

and so $a \vee b \in S$. Therefore $S$ is $\vee$-closed.

III: $L \setminus S \subseteq P$ but $L \setminus S \neq P$. By I and the construction of $S$ this is clear.

Now we prove that $F \cap S = \emptyset$. If there is $a \in F \cap S$, then $a \in F \subseteq P$ and $a \in S = S_0 \cup (L \setminus P)$, hence $a \in S_0$, then there is $x \in L \setminus P$ such that $a = x_0 \vee x \in F$, i.e., $x \in x_0^{-1} F$. Since $x \notin P$, thus $x_0^{-1} F \subseteq P$, then $x_0 \in F_P$, a contradiction, therefore $F \cap S = \emptyset$.

Since $S$ is $\vee$-closed (II), it follows from Proposition 5.10 that there is a prime filter $P_0$ such that $F \subseteq P_0$ and $P_0 \cap S = \emptyset$. So $F \subseteq P_0 \subseteq P$, which contradicts the minimality of $P$ in $\mathcal{P}\mathcal{F}_F(L)$. Therefore $F_P = P$.

Conversely, let $F_P = P$. If $P$ is not a minimal element in $\mathcal{P}\mathcal{F}_F(L)$, then there is a $P_0 \in \mathcal{P}\mathcal{F}_F(L)$ such that $F \subseteq P_0 \subseteq P$. Hence by Proposition 5.8 we have $F_P \subseteq F_{P_0} \subseteq P_0 \subseteq P$. By $F_P = P$ we obtain $P_0 = P$, a contradiction. Therefore $P$ is a minimal element in $\mathcal{P}\mathcal{F}_F(L)$.

\[\square\]

Let $F = \{1\}$ in Theorem 5.11 we obtain the following.

Corollary 5.12. Let $P \in \mathcal{P}\mathcal{F}(L)$. Then $P$ is a minimal prime filter if and only if $1_P = P$.

Proposition 5.13. Let $A$ be a nonempty set of $L$ and $F$ be a filter of $L$. Then $(F : A) = \cap\{P \in \mathcal{P}\mathcal{F}_F(L) : P \not\supseteq A\}$.

Proof. Denote $C := \cap\{P \in \mathcal{P}\mathcal{F}_F(L) : P \not\supseteq A\}$. Let $x \in (F : A)$, $P$ is a prime filter and $P \not\supseteq A$, then $x \vee A \subseteq F \subseteq P$, hence $x \in (P : A) = P$ by Corollary 5.5(ii), i.e., $x \in P$. Therefore $x \in C$, $(F : A) \subseteq C$. On the other hand, suppose $x \in C$, then for any $P \in \mathcal{P}\mathcal{F}_F(L)$ and $P \not\supseteq A$, $x \in P = (P : A)$. Since $F \subseteq P$, then $P = (P : A) \subseteq (F : A)$, thus $x \in (F : A)$, hence $C \subseteq (F : A)$. By the above proving we have $(F : A) = C$. \[\square\]

Proposition 5.14. Let $P$ be a prime filter of $L$. Then $P$ contains a minimal prime filter.

Proof. It is easily obtained by Zorn’s Lemma. \[\square\]

By the above two Propositions we have the following result, which shows that every relative involutary filter of $L$ can be represented by the intersection of minimal prime filters of $L$.

Theorem 5.15. Let $A$ be a nonempty set of $L$ and $F$ be a filter of $L$. Then $(F : A)$ is the intersection of the minimal prime filter of $L$ not containing $A$. \[\square\]
The next result characterizes minimal prime filter of $L$.

**Theorem 5.16.** Let $F$ be a proper filter of $L$. Then the following are equivalent:

1. $F$ is a minimal prime filter,
2. $F = \bigcup \{ \frac{x}{L} : x \notin F \}$,
3. $F$ is prime and for all $y \in F$, $\frac{y}{L} \not\in F$.

**Proof.** (1) $\Rightarrow$ (2). Suppose that $F$ is a minimal prime filter, then $1_{F} = F$ by Corollary 5.12. For any $y \in F = 1_{F}$, then $\frac{y}{L} \not\in F$, hence there is $x \in \frac{y}{L}$ but $x \notin F$, i.e., $y \notin \frac{1}{x}$ and $x \notin F$, hence $y \in \bigcup \{ \frac{x}{L} : x \notin F \}$, $F \subseteq \bigcup \{ \frac{x}{L} : x \notin F \}$. On the other hand, assume that for some $x \notin F$, $y \in \frac{1}{x}$, i.e., $x \in \frac{y}{L}$ and $x \notin F$, then we have $\frac{y}{L} \not\in F$. So $y \in 1_{F} = F$, the converse inclusion holds. Thus we have $F = \bigcup \{ \frac{x}{L} : x \notin F \}$.

(2) $\Rightarrow$ (3). Suppose that $F = \bigcup \{ \frac{x}{L} : x \notin F \}$. For any $a \vee b = 1$ and $a \notin F$, then $b \in \frac{1}{a}$ and so $b \in F$, hence $F$ is prime filter by Proposition 5.2. Also if $y \in F$, then for some $x \notin F$, $y \in \frac{x}{L}$, i.e., $x \in \frac{y}{L}$, hence $\frac{y}{L} \not\in F$.

(3) $\Rightarrow$ (1). Assume that $F$ is a prime filter and for all $y \in F$, $\frac{y}{L} \not\in F$, then $y \in 1_{F}$, so $F \subseteq 1_{F}$. Since $1_{F} \subseteq F$, hence $1_{F} = F$, (1) holds by Theorem 5.11.

Now let us recall a quotient algebra of a BL-algebra via a filter [8]. Let $F$ be a filter of a BL-algebra $L$. Define: $x \sim_{F} y$ if and only if $x \rightarrow y \in F$ and $y \rightarrow x \in F$. Then $\sim_{F}$ is a congruence relation on $L$. The set of all congruence classes is denoted by $L/F$, i.e., $L/F := \{ [x] \mid x \in L \}$, where $[x] := \{ y \in L \mid x \sim_{F} y \}$. Define $*, \rightarrow, \wedge, \vee$ on $L/F$ as follows:

$[x] \ast [y] = [x \ast y]$, $[x] \rightarrow [y] = [x \rightarrow y]$,

$[x] \wedge [y] = [x \wedge y]$, $[x] \vee [y] = [x \vee y]$.

Then $(L/F; \wedge, \vee, *, \rightarrow, [0], [1])$ is a BL-algebra and $[1] = F$. For any nonempty subset $B$ of $L$, denote $B/F = \{ [x] \mid x \in B \}$.

Observe that $x \in F$ then $x^{-1} = L$ by Corollary 3.9(x).

$L$ is called cancellative if $x \vee y = 1$ implies $x = 1$ or $y = 1$ for any $x, y \in L$. For $L/F$ we have the following result.

**Proposition 5.17.** A filter $F$ is prime if and only if $L/F$ is cancellative.

**Proof.** Let $F$ be prime and $[x] \vee [y] = F$ for any $[x], [y] \in L/F$. Then $[x \vee y] = F$, and so $x \vee y \in F$. The filter $F$ being prime implies $x \in F$ or $y \in F$, hence $[x] = F$ or $[y] = F$. Therefore $L/F$ is cancellative.

Conversely, let $L/F$ be cancellative and $x \vee y \in F$. Then $[x] \vee [y] = F$. While $L/F$ being cancellative implies $[x] = F$ or $[y] = F$, that is, $x \in F$ or $y \in F$. This shows that $F$ is a prime filter of $L$.  

**Theorem 5.18.** Suppose $F$ is a filter of $L$ and $B$ is a nonempty subset of $L$. Then:

(i) $\perp (B/F) = (F : B)/F$,
(ii) $[x] = (x^{-1}/F)/F$,
(iii) If $F$ is a prime filter of $L$ and $[x] \neq F$ then $\perp [x] = F$,
(iv) $\perp (B/F) = (F : (F : B))/F$.

**Proof.** Let us calculate

\[
\perp (B/F) = \{ [y] \in L/F \mid [y] \vee [x] = F \text{ for all } x \in B \}
= \{ [y] \in L/F \mid [y \vee x] = F \text{ for all } x \in B \}
= \{ [y] \in L/F \mid y \vee x \in F \text{ for all } x \in B \}
= \{ [y] \in L/F \mid y \in (F : B) \}
= (F : B)/F,
\]

(i) is true.

In (i) let $B = \{ x \}$ we obtain (ii).
If $F$ is a prime filter of $L$ and $[x] \neq F$, then $x \notin F$, it follows from Proposition 5.6 that $x^{-1}F = F$. By (ii), we have $\downarrow[x] = F$. (iii) holds.

By (i) we have
\[
\downarrow \downarrow (B/F) = \downarrow \downarrow (F : B)/F = (F : (F : B))/F.
\]
(iv) holds.

**Corollary 5.19.** Let $A, B$ be filters of $L$. If $B$ is an involutory filter relative $A$, then $B/A$ is an involutory filter of the quotient algebra $L/A$.

**Proof.** It follows immediately from Theorem 5.18(iv).

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**References**