Weak amenability for the second dual of Banach modules

1 Introduction

The concept of amenability for Banach algebras was introduced by B. E. Johnson in [14]. He showed that the group algebra $L^1(G)$ is amenable if and only if $G$ as a group is amenable. Subsequently, various generalizations of this notion have been studied by a number of authors (for instance, see [2, 6, 11, 19]). In other words, all authors investigated and obtained some results related to amenability of Banach algebras while many Banach algebras can be considered as a Banach module over another Banach algebras. Amini [1] used this fact and developed the concept of module amenability for a Banach algebra $A$ to the case that there is an extra $A$-module structure on $A$. He showed that for an inverse semigroup $S$ with the set of idempotents $E$, $l^1(S)$ is $l^1(E)$-module amenable if and only if $S$ is amenable.

There are many examples of Banach modules which do not have any natural algebra structure. In [7], Ebrahimi Bagha and Amini introduced the notion of $\Delta$-amenability for Banach modules and proved that $\Delta$-amenable Banach modules possess module virtual (approximate) diagonals in an appropriate sense. They also defined the notion of weak $\Delta$-amenability for Banach modules.

In this paper we modify the notion of weak $\Delta$-amenability for a Banach module $E$ and find some results about weak $\Delta$-amenability. Indeed, for a Banach algebra $A$ and a Banach $A$-module $E$, we obtain some sufficient conditions to be weakly $\Delta$-amenable of $E$. We also show (in some results) under certain conditions, the weak $\Delta''$-amenability of $E''$, the second dual space of $E$, implies the weak $\Delta$-amenability of $E$. These results can be regarded as the generalizations of the fact that, for a Banach algebra $A$, weak amenability of $A''$ implies weak amenability of $A$ under any of the following conditions:

- every derivation $D : A \rightarrow A'$ satisfies $D''(A'') \subseteq \text{WAP}(A)$ [8];
- $A$ is a left ideal in $A''$ [12];

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2 Preliminaries

Let $A$ be a Banach algebra and $E$ be a Banach $A$-bimodule. Let also $\Box$ and $\Diamond$ be the first and second Arens products of $A''$, respectively. It is well-known that $A''$ equipped with both Arens products is a Banach algebra (for more details, see [5]). The dual space $E'$ of $E$ is also a Banach $A$-module by the following module actions:

\[
(a' \cdot x', x) = (x', a \cdot x), \quad (x' \cdot a, x) = (x', a \cdot x), \quad (a \in A, \ x \in E, \ x' \in E').
\]

We put $E'' = (E')'$, and we regard $E$ as a subspace of $E''$ in the standard fashion. We denote the canonical images of $x$ and $E$ in $E''$ by $x$ and $E$, respectively.

One should remember that $E''$ is an $(A'', \Box)$-bimodule. Indeed, let $a'' \in A''$ and $e'' \in E''$. By Goldstine's theorem there are nets $(a_\alpha)$ in $A$ and $(e_\beta)$ in $E$ such that $a'' = \lim_\alpha a_\alpha$ and $e'' = \lim_\beta e_\beta$. Then

\[
e'' \cdot a'' = \lim_\alpha w^* - \lim_\beta e_\alpha a_\alpha \quad \text{and} \quad a'' \cdot e'' = w^* - \lim_\alpha \lim_\beta a_\alpha e_\beta.
\]

Also for any $e'' \in E''$ the map $a'' \mapsto a'' \cdot e''$ from $A''$ into $E''$ is $w^*$-continuous and for any $a'' \in A''$ the map $e'' \mapsto e'' \cdot a''$ from $E''$ into $E''$ is $w^*$-continuous.

Let $A$ be a Banach algebra, $E$ be a Banach $A$-bimodule and $\Delta : E \to A$ be a bounded Banach $A$-bimodule homomorphism, that is, a bounded linear map such that for any $a \in A, x \in E$,

\[
\Delta(a \cdot x) = a \cdot \Delta(x) \quad \text{and} \quad \Delta(x \cdot a) = \Delta(x) \cdot a.
\]

It is routine to check that $\Delta'' : E'' \to A''$, the second adjoint of $\Delta$, is a bounded $A''$-bimodule homomorphism.

Recall that for a Banach algebra $A$ and a Banach $A$-bimodule $X$, a derivation $D : A \to X$ is a linear map such that $D(ab) = D(a) \cdot b + a \cdot D(b)$ for all $a, b \in A$. Also, the derivation $D$ is said to be inner if there exists $x \in X$ such that $D(a) = a \cdot x - x \cdot a$ for every $a \in A$. The Banach algebra $A$ is called amenable if every continuous derivation $D : A \to X'$ is inner for every Banach $A$-bimodule $X$ and $A$ is called weakly amenable if every continuous derivation $D : A \to A'$ is inner.

Throughout this paper, $A$ is a Banach algebra, $E$ is a Banach $A$-bimodule and $\Delta : E \to A$ is a bounded Banach $A$-bimodule homomorphism unless otherwise stated explicitly.

Let $X$ be a Banach $A$-bimodule. A bounded linear map $D : A \to X$ is called a $\Delta$-derivation if

\[
D(\Delta(a \cdot x)) = a \cdot D(\Delta(x)) + D(a) \cdot \Delta(x)
\]

and

\[
D(\Delta(x \cdot a)) = D(\Delta(x)) \cdot a + \Delta(x) \cdot D(a)
\]

for any $a \in A$ and $x \in E$. Also $D$ is called $\Delta$-inner if there is $f \in X$, such that $D(\Delta(x)) = \Delta(x) \cdot f - f \cdot \Delta(x)$, for any $x \in E$. A Banach $A$-bimodule $E$ is called $\Delta$-amenable if for every $\Delta$-derivation $D : A \to X^*$ is $\Delta$-inner [7].

**Definition 2.1.** A Banach $A$-bimodule $E$ is called weakly $\Delta$-amenable (as an $A$-bimodule) if each $\Delta$-derivation from $A$ to $(\Delta(E))'$ is $\Delta$-inner.

**Remark 2.2.** Note that the definition of weak $\Delta$-amenability in [7] is an alternative definition from our definition ($(\Delta(E))'$ instead of $(\Delta(E))'$). Since $\Delta(E)$ is not necessarily a closed subspace of $A$, it may not be a Banach space, so we modified it. However, it seems that we are able to validate the results of [7] by applying this new definition.
3 Weak amenability of Banach modules

We start this section with a definition which is analogous to [13, Definition 1.1] for ker Δ.

**Definition 3.1.** Let A be a Banach algebra, E be a Banach A-bimodule and Δ : E → A be a bounded A-bimodule homomorphism. Then ker Δ has Δ-trace extension property if for every λ ∈ (ker Δ)' with λ · Δ(x) = Δ(x) · λ for every x ∈ E, there exists μ ∈ E' such that μ · Δ(x) = Δ(x) · μ for every x ∈ E and μ|ker Δ = λ.

**Theorem 3.2.** Suppose that every Δ-derivation D : A → E' is Δ-inner and ker Δ has Δ-trace extension property. Then E is weakly Δ-amenable.

*Proof.* Let D : A → \( \overline{\Delta(E)} \) be a Δ-derivation. Obviously, Δ' ◦ D : A → E' is a Δ-derivation. By assumption Δ' ◦ D is Δ-inner. So, there exists \( e' \in E' \) such that for every \( x \in E \), \( \Delta'(\Delta(x)) = \Delta(x) \cdot e' - e' \cdot \Delta(x) \). Thus for each \( x, y \in E \), we get

\[
\langle D(\Delta(x)), \Delta(y) \rangle = \langle \Delta' \circ D(\Delta(x)), y \rangle \\
= \langle \Delta(x) \cdot e' - e' \cdot \Delta(x), y \rangle \\
= \langle e', y \cdot \Delta(x) - \Delta(x) \cdot y \rangle.
\]

Thus, \( \langle \Delta(x) \cdot e' - e' \cdot \Delta(x), y \rangle = 0 \), for all \( y \in \ker \Delta \) and \( x \in E \). In other words, \( \langle \Delta(x) \cdot e' - e' \cdot \Delta(x) \rangle |_{\ker \Delta} = 0 \). Hence, \( (\Delta(x) \cdot e') |_{\ker \Delta} = (e' \cdot \Delta(x)) |_{\ker \Delta} \). Since ker Δ has Δ-trace extension property, there exists \( \lambda \in E' \) such that \( \lambda \cdot \Delta(x) = \Delta(x) \cdot \lambda \) for every \( x \in E \) and \( \lambda |_{\ker \Delta} = e' |_{\ker \Delta} \), and so \( (e' - \lambda) |_{\ker \Delta} = 0 \). Also

\[
\Delta'(\Delta(x)) = \Delta(x) \cdot (e' - \lambda) - (e' - \lambda) \cdot \Delta(x).
\]

Define \( \mu \in \overline{\Delta(E)} \) by \( \mu(\Delta(x)) = (e' - \lambda)(x) \), for every \( x \in E \). We have \( (e' - \lambda) |_{\ker \Delta} = 0 \). This shows that \( \mu \) is well-defined. Now, for each \( x, y \in E \), we have

\[
\langle D(\Delta(x)), \Delta(y) \rangle = \langle e', y \cdot \Delta(x) - \Delta(x) \cdot y \rangle = \langle e' - \lambda, y \cdot \Delta(x) - \Delta(x) \cdot y \rangle \\
= \langle \mu, \Delta(y) \cdot \Delta(x) - \Delta(x) \cdot y \rangle = \langle \mu, \Delta(y) \cdot \Delta(x) - \Delta(x) \cdot \Delta(y) \rangle \\
= \langle \Delta(x) \cdot \mu - \mu \cdot \Delta(x), y \rangle.
\]

Thus, \( \langle D(\Delta(x)), a \rangle = \langle \Delta(x) \cdot \mu - \mu \cdot \Delta(x), a \rangle \), for all \( a \in \overline{\Delta(E)} \) and \( x \in E \). Therefore \( D \) is Δ-inner. \( \square \)

**Definition 3.3.** Let A be a Banach algebra, E be a Banach A-bimodule and Δ : E → A be a bounded A-bimodule homomorphism. Then ker Δ has weak Δ-trace extension property if for every λ ∈ (ker Δ)' with λ · a = a · λ for every \( a \in A \), there exists \( \mu \in E' \) such that \( \mu \cdot \Delta(x) = \Delta(x) \cdot \mu \) for every \( x \in E \), and \( \mu |_{\ker \Delta} = \lambda \).

**Proposition 3.4.** If E is weakly Δ-amenable and \( \Delta(E) \) is norm closed in A, then ker Δ has weak Δ-trace extension property.

*Proof.* Suppose that \( \lambda \in (\ker \Delta)' \) such that \( a \cdot \lambda = \lambda \cdot a \), for all \( a \in A \). By the Hahn-Banach theorem, there exists \( \Lambda \in E' \) such that \( \Lambda |_{\ker \Delta} = \lambda \). Define \( D : A \to E' \) by \( D(a) = a \cdot \Lambda - \Lambda \cdot a \) for all \( a \in A \). So, \( D(a) |_{\ker \Delta} = 0 \), for every \( a \in A \). Therefore, \( D : A \to (\ker \Delta)^{\perp} \) is a derivation. We know that there is an isometric isomorphism between \( (\ker \Delta)^{\perp} \) and \( \overline{\Delta(E)} \) and there is a homeomorphism between \( \overline{\Delta(E)} \) and \( \Delta(E) \) which is also an A-bimodule isomorphism. By weak Δ-amenability of E, there exists \( \mu \in (\ker \Delta)^{\perp} \) such that

\[
D(\Delta(x)) = \Delta(x) \cdot \mu - \mu \cdot \Delta(x) = \Delta(x) \cdot \Lambda - \Lambda \cdot \Delta(x).
\]

Putting \( T = \Lambda - \mu \), we have \( T |_{\ker \Delta} = \lambda \) and \( T \cdot \Delta(x) = \Delta(x) \cdot T \) for any \( x \in E \). \( \square \)

In analogy with Proposition 3.4, we have the following result. The proof is similar, but we include it.

**Proposition 3.5.** If E is weakly Δ-amenable and \( \Delta(E) \) is norm closed in A and has a bounded approximate identity, then ker Δ has Δ-trace extension property.
Proof. Assume that \( \lambda \in (\ker \Delta)' \) such that \( \Delta(x) \cdot \lambda = \lambda \cdot \Delta(x) \), for all \( x \in E \). By the Hahn-Banach theorem, there is \( \Lambda \in E' \) such that \( \Lambda|_{\ker \Delta} = \lambda \). Define \( D : \Delta(E) \rightarrow E' \) by \( D(\Delta(x)) = \Delta(x) \cdot \Lambda - \Lambda \cdot \Delta(x) \), for all \( x \in E \).

So for any \( x \in E \), \( D(\Delta(x))|_{\ker \Delta} = 0 \). Hence, \( D : \Delta(E) \rightarrow (\ker \Delta)^\perp \) is a derivation. Since \( \Delta(E) \) has a bounded approximate identity, by [18, Proposition 2.1.6], we can extend \( D \) to a derivation \( \tilde{D} : A \rightarrow (\ker \Delta)^\perp \). Similar to the proof of Proposition 3.4, there exists \( \mu \in (\ker \Delta)^\perp \), such that

\[
D(\Delta(x)) = \tilde{D}(\Delta(x)) = \Delta(x) \cdot \mu - \mu \cdot \Delta(x) = \Delta(x) \cdot \Lambda - \Lambda \cdot \Delta(x).
\]

Take \( T = \Lambda - \mu \). So, \( T|_{\ker \Delta} = \lambda \) and \( T \cdot \Delta(x) = \Delta(x) \cdot T \) for all \( x \in E \).

\[
\square
\]

4 Weak amenability for second dual

In this section, for a Banach module \( E \), we prove under what conditions the weak \( \Delta'' \)-amenability of \( E'' \) implies the weak \( \Delta \)-amenability of \( E \). From now on, we consider \( A' \) with the first Arens product as a Banach algebra.

A functional \( a' \in A' \) is said to be WAP (weakly almost periodic) on \( A \) if the mapping \( a \mapsto a'a \) from \( A \) into \( A' \) is weakly compact. In [17], Pym showed that this definition is equivalent to the following condition: For any two nets \( (a_j) \) and \( (b_k) \) in \( \{a \in A : \parallel a \parallel \leq 1\} \) we have \( \lim_j \lim_k (a_j'a_jb_k) = \lim_k \lim_j (a'_jb'_a) \), whenever both iterated limits exist. The collection of all WAP functionals on \( A \) is denoted by WAP(A). Also we have \( a' \in \text{WAP}(A) \) if and only if \( (a'' \boxtimes b'', a') = (a'' \circ b'', a') \) is \( w^* \)-continuous on \( A'' \) for every \( b'' \in A'' \).

Following the above, for a Banach \( A \)-bimodule \( X \), we consider the set \( \text{WAP}_r(X) = \{x' \in X' : a'' \mapsto \langle x'', a'' \rangle \} \) is \( w^* \)-continuous on \( A'' \) for every \( x'' \in X'' \).

The idea of the following theorem is taken from [8, Theorem 2.1].

Theorem 4.1. Suppose that \( E'' \) is weakly \( \Delta'' \)-amenable and \( \Delta(E) \) is norm closed in \( A \). If every \( \Delta \)-derivation \( D : A \rightarrow (\Delta(E))' \) satisfies \( D''(A'') \subseteq \text{WAP}_r(\Delta(E)) \), then \( E \) is weakly \( \Delta \)-amenable.

Proof. By hypothesis, \( \Delta''(E'') \) is norm and thus \( w^* \)-closed in \( A'' \). So

\[
\Delta''(E'') = \Delta''(E'')|_{\ker \Delta'} = (\ker \Delta')^\perp = (\text{Im} \Delta)^{1\perp} \cong (A(E))''.
\]

Assume that \( D : A \rightarrow (\Delta(E))' \) is a \( \Delta \)-derivation. We show that \( D'' : A'' \rightarrow (\Delta(E))'' = (\Delta''(E''))' \) is a \( \Delta'' \)-derivation. Take the nets \( \{a_{\alpha}\} \subseteq A \) and \( \{x_{\beta}\} \subseteq E \) so that converge to \( a'' \) and \( x'' \) in the \( w^* \)-topology of \( A'' \) and \( E'' \), respectively. Using the \( w^* \)-\( w^* \)-continuity of \( D'' \) and \( \Delta'' \), we obtain

\[
D''(a''(\Delta''(x''))) = w^* - \lim_{\alpha} w^* - \lim_{\beta} (D(a_{\alpha}) \Delta(x_{\beta}))
= w^* - \lim_{\alpha} w^* - \lim_{\beta} (D(a_{\alpha}) \cdot \Delta(x_{\beta}) + a_{\alpha} \cdot D(\Delta(x_{\beta})))
\]

For every \( c'' \in (\Delta(E))'' \), we get

\[
\lim_{\alpha} \lim_{\beta} (D(a_{\alpha}) \cdot \Delta(x_{\beta}), c'') \lim_{\alpha} (D(c''(a_{\alpha}), a_{\alpha} \cdot \Delta(x_{\beta})) \lim_{\alpha} (D(a_{\alpha}) \cdot \Delta(x_{\beta}), c'') = \lim_{\alpha} (\Delta''(a'') \cdot \Delta''(x'')) = \langle D''(a''), \Delta''(x'') \rangle.
\]

Now, for each \( c'' \in (\Delta(E))'' \), we have

\[
\lim_{\alpha} \lim_{\beta} (D(a_{\alpha}), c'') = \lim_{\alpha} (c''(a_{\alpha}), a_{\alpha} \cdot \Delta(x_{\beta})) = \lim_{\alpha} (c'' a_{\alpha}, D(\Delta(x_{\beta})))
= \lim_{\alpha} \lim_{\beta} (D(\Delta(x_{\beta})), c'' a_{\alpha}) \lim_{\alpha} (D''(\Delta''(x''))). \]

\[
\square
\]
Since $D''(\Delta''(x'')) \in \text{WAP}_r(\Delta(E))$, there exists $y' \in \text{WAP}_r(\Delta(E))$ such that $D''(\Delta''(x'')) = \tilde{y}'$. Hence,
\[
\lim_{\alpha}(D''(\Delta''(x'')), c''(a_{\alpha})) = \lim_{\alpha}(\tilde{y}', c''(a_{\alpha})) = \lim_{\alpha}(c''(\tilde{a}_{\alpha}), y') = \langle c''\tilde{a}', y' \rangle
\]
\[
= \langle \tilde{y}', c''\tilde{a}' \rangle = \langle D''(\Delta''(x'')), c''\tilde{a}' \rangle = \langle a'' \cdot D''(\Delta''(x'')), c'' \rangle. \tag{4}
\]
It follows from (1), (2) and (4) that
\[
D''(a''(\Delta''(x''))) = D''(a''(\Delta''(x''))) + a'' \cdot D''(\Delta''(x'')).
\]
Similarly, one can show that
\[
D''(\Delta''(x''))a'' = D''(\Delta''(x''))a'' + \Delta''(x'') \cdot D''(a'').
\]
Therefore $D''$ is a $\Delta''$-derivation. Since $E''$ is weakly $\Delta''$-amenable, $D''$ is $\Delta''$-inner. So there is $a''' \in (\Delta(E))''$ such that $D''(\Delta''(x'')) = \Delta''(x'') \cdot a''' - a'''. \Delta''(x'')$, for all $x'' \in E''$. Now for any $x \in E$,
\[
D(\Delta(\bar{x})) = D''(\Delta(\bar{x})) = \Delta(\bar{x}) \cdot a''' - a'''. \Delta(\bar{x})
\]
and we obtain $D(\Delta(x)) = \Delta(x) \cdot a' - a' \cdot \Delta(x)$ where $a' = a'''|_{\Delta(E)}$. \hfill \square

**Remark 4.2.** Let $A$ be a Banach algebra and $E$ be a Banach $A$-bimodule and $\Lambda : E \to A$ be a bounded $A$-bimodule homomorphism with norm closed range. Suppose that $R : \Delta(E)'' \to \Delta(E)'$ is the restriction map. Also assume that $\Lambda : (\Delta(E))'' \to \Delta(E)''$ is defined by $\Lambda(\psi) = \overline{R(\psi)}$ for every $\psi \in (\Delta(E))''$. So, for any $x \in \Delta(E)$, we have
\[
\langle \Lambda(\psi), \bar{x} \rangle = \langle \overline{R(\psi)}, \bar{x} \rangle = \langle \bar{x}, R(\psi) \rangle = \langle R(\psi), x \rangle = \langle \psi, \bar{x} \rangle.
\]
If $x'' = w^* - \lim_i \bar{x}_i$ where $x'' \in \Delta(E)'$ and $x_i \in \Delta(E)$, then
\[
\langle \Lambda(\psi), x'' \rangle = \langle \overline{R(\psi)}, x'' \rangle = \langle x'', R(\psi) \rangle = \langle w^* - \lim_i \bar{x}_i, R(\psi) \rangle
\]
\[
= \lim_i \langle \bar{x}_i, R(\psi) \rangle = \lim_i \langle R(\psi), x_i \rangle = \lim_i \langle \psi, \bar{x}_i \rangle = \lim_i \langle \Lambda(\psi), \bar{x}_i \rangle.
\]
The proof of the following result is close to the proof of [12, Theorem 2.3], but we indicate its proof for the sake of completeness.

**Theorem 4.3.** Let $A$ be a Banach algebra, let $E$ be a Banach $A$-bimodule and $\Delta : E \to A$ be a bounded $A$-bimodule homomorphism. If $\Delta(E)$ is norm closed in $A$ and $A', \Delta(E) \subseteq \Delta(E)$, then weak $\Delta''$-amenability of $E''$ (as an $A''$-bimodule) implies the weak $\Delta''$-amenability of $E'$ (as an $A$-bimodule).

**Proof.** Let $D : A \to \Delta(E)'$ be a $\Delta$-derivation and $R$ and $\Lambda$ be as in Remark 4.2. It suffices to show that $\Lambda \circ D'' : A'' \to \Delta(E)'''$ is a $\Delta''$-derivation. So, by $\Delta''$-weak amenability of $E''$, there is $F \in \Delta(E)'''$ such that for any $y'' \in (E'')'$, we have
\[
\Lambda \circ D''(\Delta''(y'')) = \Delta''(y'') \cdot F - F \cdot (\Delta''(y'')).
\]
Now, by putting $f = R(F)$, for any $y \in E$, we have $D(\Delta(y)) = \Delta(y) \cdot f - f \cdot \Delta(y)$. It shows that $D$ is a $\Delta$-inner derivation. Let $a'' \in A''$ and $b'' \in E''$. Take the nets $(a_i) \subseteq A$ and $(b_j) \subseteq E$ with $a'' = w^* - \lim_i a_i$ and $b'' = w^* - \lim_j b_j$. We have
\[
D''(a'' \Delta''(b'')) = w^* - \lim_i w^* - \lim_j D''(a_i \Delta(b_j))
\]
\[
= w^* - \lim_i w^* - \lim_j (D(a_i \cdot \Delta(b_j)) + a_i \cdot D(\Delta(b_j)))
\]
\[
= D''(a'') \cdot \Delta''(b'') + w^* - \lim_i a_i \cdot D''(\Delta''(b'')).
\]
So $\Lambda \circ D''(a'' \Delta''(b'')) = \Lambda(D''(a'') \cdot \Delta''(b'')) + \Lambda(w^* - \lim_i a_i \cdot D''(\Delta''(b''))).$ Applying Remark 4.2, we get
\[
\langle \Lambda(D''(a'') \cdot \Delta''(b'')), \bar{x} \rangle = \langle D''(a'') \cdot \Delta''(b''), \bar{x} \rangle = \langle D''(a''), \Delta''(b'') \cdot \bar{x} \rangle \quad (x \in \Delta(E)).
\]
Since \( A'' \Delta(E) \subseteq \Delta(E) \), we have \( \Delta''(b'') \cdot \mathfrak{x} \in \Delta(E) \). Thus

\[
\{ D''(a''), \Delta''(b'') \cdot \mathfrak{x} \} = \{ \Lambda \circ D''(a''), \Delta''(b'') \cdot \mathfrak{x} \} = \{ \Lambda \circ D''(a'') \cdot \Delta''(b''), \mathfrak{x} \}.
\]

For each \( x'' \in \Delta(E)'' \), we deduce that

\[
\{ \Lambda(D''(a'') \cdot \Delta''(b'')), x'' \} = \{ \Lambda \circ D''(a'') \cdot \Delta''(b''), x'' \}.
\]

Hence \( \Lambda(D''(a'') \cdot \Delta''(b'')) = (\Lambda \circ D''(a'')) \cdot \Delta''(b''). \) For each \( x \in \Delta(E) \), we have

\[
\{ \Lambda(w^* - \lim_{\mathcal{I}}(\Delta_i \cdot D''(\Delta''(b'))), \mathfrak{x} \} = \{ (w^* - \lim_{\mathcal{I}}(\Delta_i \cdot D''(\Delta''(b'))), \mathfrak{x} \} = \lim_{\mathcal{I}}(\Delta_i \cdot D''(\Delta''(b')))
\]

Thus

\[
\Lambda(\Delta''(b'')) = (\Lambda \circ D''(a'')) \cdot \Delta''(b'') + \Lambda''(\Delta''(b))).
\]

Similarly we can show that

\[
\Lambda \circ D''(\Delta''(b''))a'' = (\Lambda \circ D''(\Delta''(b'')) \cdot a'' + \Delta''(b'') \cdot (\Lambda \circ D''(a'')).
\]

Therefore \( \Lambda \circ D'' \) is a \( \Delta'' \)-derivation. This completes the proof.

The idea of the following theorem is taken from \[10, Theorem 2.2\]. We bring its proof.

**Theorem 4.4.** Let \( A \) be a Banach algebra, let \( E \) be a Banach \( A \)-bimodule and \( \Delta : E \to A \) be a bounded \( A \)-bimodule homomorphism with norm closed range. Suppose that \( \Delta(E) \) is a dual Banach algebra. If \( E'' \) is weakly \( \Delta'' \)-amenable (as an \( A'' \)-bimodule) and \( \Delta(E)'' \) has a bounded approximate identity, then \( E \) is weakly \( \Delta \)-amenable (as an \( A \)-bimodule).

**Proof.** By assumptions, \( \Delta(E) \) is a norm closed ideal of \( A \) and hence \( \Delta(E) \) is a Banach algebra. Assume that \( \Delta(E) = B' \) for a Banach space \( B \) such that \( \tilde{B} \) is a submodule of the dual module \( \Delta(E)' \). Suppose that \( i : B \to B'' = \Delta(E)' \) is the embedding map and \( i' : \Delta(E)'' \to B'' = \Delta(E) \) is the adjoint of \( i \). Let \( D : A \to \Delta(E)' \) be a \( \Delta \)-derivation. Then

\[
\tilde{D} = i''' \circ D \circ i' : \Delta(E)'' \to \Delta(E)'''\]

is a derivation. Since \( \Delta(E)''' = \Delta''(E'') \) is a closed ideal of \( A''' \) and \( \Delta(E)'' \) has a bounded approximate identity, by \[18, Proposition 2.1.6\], we can extend \( \tilde{D} \) to a derivation \( \tilde{D} : A'' \to \Delta(E)'\). It follows from the weak \( \Delta'' \)-amenability of \( E'' \) that there exists \( F \in \Delta(E)''' \) such that

\[
\tilde{D}(a'') = a'' \cdot F - F \cdot a'' \quad (a'' \in \Delta(E)'').
\]

Let \( j : \Delta(E) \to \Delta(E)'' \) be the canonical mapping and let \( f = j'(F) \). Then \( D(a) = a \cdot f - f \cdot a \) for all \( a \in \Delta(E) \).

Let \( A \) be a Banach algebra and \( X \) be a Banach \( A \)-bimodule. Then \( X''' \) can have two \( A'' \)-bimodule structures (see \[9\]). First we regard \( X''' \), as the dual space of \( X'' \). Since \( X \) is an \( A \)-bimodule, \( X'' \) is an \( A'' \)-bimodule, and thus \( X''' = (X'')' \) is also an \( A''' \)-bimodule. In fact, for each \( x'''' \in X''' \), \( a''' \in A''' \), \( x'' \in X'' \),

\[
\langle a''', x'''' \rangle = \langle x''', x'' \cdot a'' \rangle \quad \text{and} \quad \langle x'''' \cdot a''', x'' \rangle = \langle x''''', a'' \cdot x'' \rangle.
\]
Assume that \( x''' = w^* - \lim_k \tilde{x}_k, x'' = w^* - \lim_i \tilde{x}_i, a'' = w^* - \lim_j \tilde{a}_j \) for nets \((x'_k)\) in \(X'\) and \((x_i)\) in \(X\) and \((a_j)\) in \(A\). It is easily verified that
\[
\langle a'' \cdot x''' \cdot x'' \rangle = \lim_k \lim_i \langle x'_k \cdot x_i \cdot a_j \rangle, \quad \langle x'' \cdot a'' \cdot x'' \rangle = \lim_k \lim_i \langle x'_k \cdot a_j \cdot x_i \rangle.
\]

In the second way, since \(X\) is an \(A\)-bimodule, \(X'\) is also an \(A\)-bimodule and hence \(X'' = (X')''\) is an \(A''\)-bimodule.

Considering the above nets, we have
\[
a'' \cdot x''' = w^* - \lim_j w^* - \lim_k a_j \cdot x'_k \quad \text{and} \quad x'' \cdot a'' = w^* - \lim_j w^* - \lim_k x'_k \cdot a_j.
\]
By a routine computation, we get
\[
\langle a'' \cdot x''' \cdot x'' \rangle = \lim_k \lim_i \langle x'_k \cdot x_i \cdot a_j \rangle \quad \text{and} \quad \langle x'' \cdot a'' \cdot x'' \rangle = \lim_k \lim_i \langle x'_k \cdot a_j \cdot x_i \rangle.
\]

**Theorem 4.5.** Let \(A\) be a Banach algebra, \(E\) be a Banach \(A\)-bimodule and \(\Delta: E \rightarrow A\) be a bounded \(A\)-bimodule homomorphism with norm closed range. If two \(A''\)-bimodule structures on \(X'' = (X')''\) coincide, then weak \(\Delta''\)-amenability of \(E'' = (\Delta(E)''\) implies the weak \(\Delta\)-amenability of \(E\). (as an \(A\)-bimodule).

**Proof.** Let \(D: A \rightarrow (\Delta(E)')'\) be a \(\Delta\)-derivation. Then by [6, Proposition 1.7] \(D'' : A'' \rightarrow (\Delta(E)'') = (\Delta(E)')''\) is a \(\Delta''\)-derivation. (Here the action of \(A''\) on \(\Delta(E)''\) is \(\bullet\)) By hypothesis,
\[
(\Delta(E)')'' = (\Delta(E)')'' = (\Delta(E)')' = (\Delta''(E)'').
\]
Since \(E''\) is weakly \(\Delta''\)-amenable, \(D''\) is a \(\Delta''\)-inner derivation and hence \(D\) is a \(\Delta\)-inner derivation. \(\square\)

The next definition was introduced by Medghalchi and Yazdanpanah [16].

**Definition 4.6.** The Banach algebra \(A\) has strongly double limit property (SDLP) if for each bounded net \((a_i)\) in \(A\) and each bounded net \((f_\alpha)\) in \(A'\)
\[
\lim_{\alpha} \lim_i \langle f_\alpha, a_i \rangle = \lim_i \lim_{\alpha} \langle f_\alpha, a_i \rangle.
\]

Some results about (SDLP) can be found in [4].

**Proposition 4.7.** Let \(A\) be a Banach algebra and \(X\) be a Banach \(A\)-bimodule. If \(A\) has (SDLP), then two \(A''\)-bimodule structures on \(X''\) coincide.

**Proof.** Let \(x''' \in X'''\) and \(a'' \in A''\). Choose the nets \((x'_k)\) in \(X'\), \((x_i)\) in \(X\) and \((a_j)\) in \(A\) such that \(x''' = w^* - \lim_k \tilde{x}_k, x'' = w^* - \lim_i \tilde{x}_i, a'' = w^* - \lim_j \tilde{a}_j\). Then
\[
\langle a'' \cdot x''' \cdot x'' \rangle = \lim_k \lim_i \langle x'_k \cdot x_i \cdot a_j \rangle = \lim_k \lim_i \langle x'_k \cdot x_i \cdot a_j \rangle = \lim_k \lim_i \langle x'_k \cdot a_j \cdot x_i \rangle = \langle a'' \cdot x''' \cdot x'' \rangle.
\]

**Corollary 4.8.** Let \(A\) be a Banach algebra, \(E\) be a Banach \(A\)-bimodule and \(\Delta: E \rightarrow A\) be a bounded \(A\)-bimodule homomorphism with norm closed range. If \(A\) has (SDLP), then weak \(\Delta''\)-amenability of \(E''\) as an \(A''\)-bimodule implies the weak \(\Delta\)-amenability of \(E\) as an \(A\)-bimodule.

**Proof.** If \(A\) has (SDLP), then two \(A''\)-bimodule structures on \(\Delta(E)''\) coincide. Now apply Theorem 4.5. \(\square\)

We close this section by two examples.

**Example 4.9.** Let \(G\) be a locally compact group. We know that \(L^1(G)\) is a closed ideal in \(M(G)\) and thus \((L^1(G))''\) is a closed ideal in \((M(G))''\). Let \(\Delta: L^1(G) \rightarrow M(G)\) be the inclusion map. We consider \(L^1(G)\) as an \(M(G)\)-module with the convolution module actions, then \(L^1(G)\) is weakly \(\Delta\)-amenable as an \(M(G)\)-module. To see this, if \(D: M(G) \rightarrow (L^1(G))' = (\Delta(L^1(G)))'\) is a \(\Delta\)-derivation, then \(D|_{L^1(G)}: L^1(G) \rightarrow L^1(G)\) is
$L^1(G)'$ is a derivation. Since $L^1(G)$ as a Banach algebra is weakly amenable [15], $D|_{L^1(G)}$ is inner and thus $D$ is $\Delta$-inner. On the other hand, if $G$ is a non-discrete abelian group, then $L^1(G)''$ is not weakly amenable as a Banach algebra. So, we have a non-inner derivation $D_1 : L^1(G)'' \rightarrow L^1(G)'''$ which by [18, Proposition 2.1.6], this derivation can be extended to a derivation $\tilde{D}_1 : M(G)'' \rightarrow L^1(G)'''$. Since $D_1$ is not inner, $\tilde{D}_1$ is not $\Delta''$-inner. This shows that $L^1(G)'''$ is not weakly $\Delta''$-amenable as a $M(G)'''$-bimodule.

Example 4.10. Let $G$ be a locally compact group and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then $L^p(G)$ is a Banach $L^1(G)$-bimodule. Suppose that $f \in L^q(G)$ and $\Delta_f : L^p(G) \rightarrow L^1(G)$ is defined by $\Delta_f(g) = f \ast g$. Clearly, $\Delta_f$ is a bounded Banach $L^1(G)$-bimodule homomorphism. If $G$ is compact and $(L^p(G))''$ is weakly $\Delta''_f$-amenable as an $L^1(G)''$-bimodule, we show that $L^p(G)$ is weakly $\Delta_f$-amenable as an $L^1(G)$-bimodule.

Let $D : L^1(G) \rightarrow \Delta_f(L^p(G))$ be a $\Delta_f$-derivation. It is known that $L^1(G)$ has a bounded approximate identity and $L^1(G)$ is an ideal in $L^1(G)''$ [20]. So by [18, Proposition 2.1.6], we can extend $D$ to a $\Delta''_f$-derivation $\tilde{D} : L^1(G)'' \rightarrow \Delta_f(L^p(G))$. Due to the reflexivity of $L^p(G)$ we have $\text{Im}(\Delta_f) = \text{Im}(\Delta''_f)$. So $\tilde{D}$ can be replaced by $\tilde{D} : L^1(G)'' \rightarrow (\Delta''_f(L^p(G)))'$. Since $L^p(G)''$ is weakly $\Delta''_f$-amenable as an $L^1(G)''$-bimodule, $\tilde{D}$ is a $\Delta''_f$-inner derivation. Consequently, $D$ is a $\Delta_f$-inner derivation.

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References