Three open problems and a conjecture

Abstract: In this paper we solve three open problems and a conjecture related to the calculations of some classes of multiple series posed by Furdui in [1].

Keywords: Multiple series, Polylogarithm function, Stirling numbers of the first kind, Riemann zeta function

1 Introduction and the main results

In [1, Section 3.7, p.163] Furdui posed four open problems related to the calculation of some classes of multiple series and a conjecture involving a series containing a factorial term. These problems were motivated by the exercises and problems collected in [1]. In this paper, we solve these problems and prove the conjecture is true. Throughout this paper $H_n$ denotes the $n$th harmonic number defined by $H_n = \sum_{k=1}^{n} k^{-1}$. For the sake of completeness we record below the problems we solve. First we give the solution of problem 3.134.

3.134. Let $k \geq 3$ be an integer. Calculate

$$S_k = \sum_{n_1, \ldots, n_k = 1}^{\infty} (-1)^{n_1 + n_2 + \ldots + n_k} \frac{H_{n_1 + n_2 + \ldots + n_k}}{n_1 n_2 \cdots n_k}.$$ 

We record the solution of this problem in the next theorem.

Theorem 1.1. Let $k \geq 1$ be an integer. Then,

$$S_k = (-1)^{k-1} \ln^{k-1} 2 + k! \ln_{k+1} \left(\frac{1}{2}\right) - k! \sum_{l=1}^{k} \frac{(-1)^{k-l} \zeta(l+1) \ln^{k-l} 2}{(k-l)!},$$

where $\ln_k$ is the polylogarithm function defined, for $|z| \leq 1$, by $\ln_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$ and $\zeta$ is the Riemann zeta function defined by $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $\Re s > 1$.

Proof. We use the following formula for the $n$th harmonic number (see [2, p.176])

$$\int_{0}^{1} \frac{1-t^n}{1-t} \, dt = H_n, \quad n \geq 1.$$
We have

\[ S_k = \sum_{n_1, \ldots, n_k=1}^{\infty} \frac{(-1)^{n_1+n_2+\cdots+n_k}}{n_1 n_2 \cdots n_k} \int_0^1 \frac{1 - t^{n_1+n_2+\cdots+n_k}}{1 - t} \, dt \]

\[ = \frac{1}{1-t} \left( \sum_{n_1, \ldots, n_k=1}^{\infty} \frac{(-1)^{n_1+n_2+\cdots+n_k}}{n_1 n_2 \cdots n_k} - \sum_{n_1, \ldots, n_k=1}^{\infty} \frac{(-t)^{n_1+n_2+\cdots+n_k}}{n_1 n_2 \cdots n_k} \right) \, dt \]

\[ = (-1)^k \frac{1}{1-t} \left( \ln^k 2 - \ln^k (1+t) \right) \, dt = (-1)^{k-1}k \frac{1}{1+t} \ln(1-t) \, dt. \]

With the variable \( t = 2u - 1 \), the integral is easily evaluated with iterated integration by parts. We have

\[ \int_0^{1/2} \frac{\ln^{k-1}(1+t) \ln(1-t)}{1+t} \, dt = \int_0^{1/2} \frac{\ln^{k-1}(2u) \ln(1-u)}{u} \, du \]

\[ = \frac{\ln^{k+1} 2}{k} - \frac{1}{1/2} \int_0^{1/2} \ln^{k-1}(2u) \frac{\text{Li}_1(u)}{u} \, du \]

\[ = \frac{\ln^{k+1} 2}{k} - \zeta(2) \ln^{k-1} 2 + (k-1) \int_0^{1/2} \ln^{k-2}(2u) \frac{\text{Li}_2(u)}{u} \, du \]

\[ \vdots \]

\[ = \frac{\ln^{k+1} 2}{k} + (k-1)! \sum_{l=1}^{k} \frac{(-1)^l \zeta(l+1) \ln^{k-l} 2}{(k-l)!} + (-1)^{k-1}(k-1)! \text{Li}_{k+1} \left( \frac{1}{2} \right). \]

When combining (1) and (2) Theorem 1.1 is proved. \( \square \)

The second author thanks Cornel I. Vălean for communicating to him, without proof, the result in Theorem 1.1.

The second open problem we discuss is about calculating an alternating sum involving a product of \( k \) harmonic numbers.

3.135. Let \( k \geq 2 \) be an integer. Evaluate, in closed form, the multiple series

\[ T_k = \sum_{n_1, \ldots, n_k=1}^{\infty} \frac{H_{n_1} H_{n_2} \cdots H_{n_k}}{n_1 n_2 \cdots n_k (n_1 + n_2 + \cdots + n_k)}. \]

We give only a partial result to this problem which we record in the next theorem.

**Theorem 1.2.** Let \( k \geq 2 \) be an integer. The following equality holds

\[ T_k = \int_0^{1/2} \left( \text{Li}_2(1-t) + \frac{1}{2} \ln^2(1+t) \right) \frac{dt}{t}. \]

**Proof.** We have

\[ T_k = \sum_{n_1, \ldots, n_k=1}^{\infty} \frac{(-1)^{n_1+\cdots+n_k} H_{n_1} H_{n_2} \cdots H_{n_k}}{n_1 n_2 \cdots n_k (n_1 + n_2 + \cdots + n_k)} \int_0^1 t^{n_1+\cdots+n_k-1} \, dt \]

\[ = \frac{1}{1} \sum_{n_1=1}^{\infty} \frac{(-t)^{n_1} H_{n_1}}{n_1} \sum_{n_2=1}^{\infty} \frac{(-t)^{n_2} H_{n_2}}{n_2} \cdots \sum_{n_k=1}^{\infty} \frac{(-t)^{n_k} H_{n_k}}{n_k} \, dt = \frac{1}{1} \left( \sum_{n=1}^{\infty} \frac{(-t)^n H_n}{n} \right)^k \, dt. \]
Using the identity (with the convention that \( H_0 = 0 \))
\[
\sum_{n=1}^{\infty} \frac{(-t)^n H_n}{n} = \sum_{n=1}^{\infty} \frac{(-t)^n H_{n-1}}{n} + \sum_{n=1}^{\infty} \frac{(-t)^n}{n^2} = \frac{1}{2} \ln^2(1 + t) + \text{Li}_2(-t), \quad t \in (-1, 1],
\]
the theorem is proved.

**Remark 1.3.** It is worth mentioning that the problem cannot be solved in the sense that in general, alternating sums appear which are (conjecturally) not expressible as polynomials in \( \zeta(s), \ln 2 \) and \( \text{Li}_s(1/2) \). Only for the first integral, the case when \( k = 2 \), one has such a simple form
\[
\int_0^1 \left( \text{Li}_2(-t) + \frac{1}{2} \ln^2(1 + t) \right)^2 \frac{dt}{t} = \frac{33}{32} \zeta(5) + \frac{5}{8} \zeta(2)\zeta(3) - 2 \ln 2 \text{Li}_4 \left( \frac{1}{2} \right) - 2 \text{Li}_5 \left( \frac{1}{2} \right) - \frac{1}{15} \ln^5 2 + \frac{1}{3} \zeta(2) \ln^3 2 - \frac{7}{8} \zeta(3) \ln^2 2.
\]

However, the next integral is (in the data mine basis [3])
\[
\int_0^1 \left( \text{Li}_2(-t) + \frac{1}{2} \ln^2(1 + t) \right)^3 \frac{dt}{t} = \frac{213}{17} \zeta(2)\zeta(5) - \frac{3}{2} \zeta(2)\zeta(1, 1, -3) + \frac{30009}{2720} \zeta^2(2)\zeta(3)
\]
\[
= \frac{247695}{4352}\zeta(7) - \frac{555}{17}\zeta(1, 1, -5) + \frac{186}{17}\zeta(1, 3, -3) - 9\zeta(1, 1, 1, 1, -3),
\]
in terms of the alternating multiple sums
\[
\zeta(n_1, n_2, \ldots, n_d) = \sum_{0 < k_1 < \ldots < k_d} \frac{(\text{sign} n_1)^{k_1} \ldots (\text{sign} n_d)^{k_d}}{k_1! \ldots k_d!}.
\]
Note that the last three sums in (3) are assumed to be linearly independent over \( \mathbb{Q} \), also modulo zeta values and any products of sums of lower weight. Hence, since on this basis,
\[
\text{Li}_7 \left( \frac{1}{2} \right) = \frac{159}{128}\zeta(7) + \frac{1}{4} \zeta(1, 1, -5) - \frac{1}{2} \zeta(1, 1, 1, -3) + \text{products}
\]
does not even involve \( \zeta(1, 3, -3) \) at all, it is impossible to rewrite (3) as a polynomial in \( \ln 2 \) and single zetas \( \zeta(n) \) and the values \( \text{Li}_s(1/2) \). True (non-products), nested multiple (alternating) sums are required here.

The next theorem is about calculating, in closed form, a multiple series that resembles the harmonic sum \( T_k \).

**Theorem 1.4.** Let \( k \geq 1 \) be an integer. The following equality holds
\[
U_k = \sum_{n_1, \ldots, n_k = 1}^{\infty} (-1)^{n_1 + \ldots + n_k} \frac{H_{n_1-1}H_{n_2-1} \cdots H_{n_k-1}}{n_1 n_2 \cdots n_k (n_1 + n_2 + \cdots + n_k)}
\]
\[
= \ln^{2k+1} 2 \cdot \frac{(2k)!}{2^k (2k + 1)} + \frac{(2k)!}{2^k} \left( \zeta(2k + 1) - \sum_{l=1}^{2k+1-1} \frac{\ln^{2k+1-l} 2}{(2k + 1 - l)!} \text{Li}_l \left( \frac{1}{2} \right) \right),
\]
with the convention that \( H_0 = 0 \).
Theorem. We have

\[ U_k = \sum_{n_1, \ldots, n_k=1}^{\infty} \frac{(-1)^{n_1+n_2+\cdots+n_k}}{n_1 n_2 \cdots n_k} \frac{H_{n_1-1} H_{n_2-1} \cdots H_{n_k-1}}{H_{n-1}} \int_0^1 t^{n_1+n_2+\cdots+n_k-1} dt \]

\[ = \int_0^1 \frac{1}{t} \sum_{n_1=1}^{\infty} \frac{(-1)^n H_{n-1}}{n} \frac{H_{n_1-1}}{n_1} \cdots \frac{H_{n_k-1}}{n_k} dt \]

\[ = \int_0^1 \left( \sum_{n=1}^{\infty} \frac{(-1)^n H_{n-1}}{n} \right)^k dt. \]

Since

\[ \sum_{n=1}^{\infty} \frac{(-1)^n H_{n-1}}{n} = \frac{1}{2} \ln^2(1+t), \quad t \in (-1,1], \]

it follows, based on (4), that

\[ U_k = \frac{1}{2^k} \int_0^1 \ln^k (1+t) dt = \frac{\ln^{2k+2}(2k+1)}{2k+1} \frac{(2k)!}{2^k} \left( \frac{\ln^{k+1/2}}{2} - \sum_{j=1}^{2k+1} \frac{\ln^{k+1/2-j}}{(2k+1-j)!} \frac{1}{2} \right), \]

where the last equality is based on the following formula

\[ \int_0^1 \frac{\ln^k (1+t)}{t} dt = \frac{\ln^{k+1/2}}{k+1} + k! \xi(k+1) - k! \sum_{j=1}^{k+1} \frac{\ln^{k+1/2-j}}{(k+1-j)!} \frac{1}{2}, \]

where \( k \geq 1 \) is an integer.

Formula (5) can be proved either by direct computation or it can be seen it follows from the identity [4, equations (7.48) and (7.49), p.178]

\[ \sum_{k=0}^{n-1} \frac{\ln^k (1/z)}{k!} \frac{1}{1-u} du, \]

via the substitution \( t = z^{-1} - 1 \):

\[ \int_0^1 \frac{\ln^k (1+t)}{t} dt = \int_0^{1/2} \frac{\ln^k (1/z)}{z(1-z)} dz = \frac{\ln^{k+1/2}}{k+1} + (-1)^{k-1} \int_0^{1/2} \frac{\ln^k z}{1-z} dz. \]

The theorem is proved.

Remark 1.5. Using the generating function for the Stirling numbers of the first kind ([5, p.56]) \( \ln^k (1+t) = k! \sum_{n=k}^{\infty} s(n,k) \frac{t^n}{n!}, |t| < 1 \), and then integrating \( \frac{\ln^k (1+t)}{t} \) from 0 to 1 term-by-term we have, based on formula (5), that

\[ \sum_{n=k}^{\infty} \frac{s(n,k)}{n \cdot n!} = \frac{\ln^{k+1/2}}{k+1} + \xi(k+1) - \sum_{j=1}^{k+1} \frac{\ln^{k+1/2-j}}{(k+1-j)!} \frac{1}{2} \frac{1}{2}, \]

This formula resembles the surprising series involving \( (-1)^{n+k} s(n,k) \) due to Li-Chien Shen [6]

\[ \sum_{n=k}^{\infty} \frac{(-1)^n s(n,k)}{n \cdot n!} = (-1)^{k} \xi(k+1). \]

The third problem we solve is about calculating, in closed form, a multiple alternating Hardy series.
3.136. Let \( k \geq 3 \) be an integer. Calculate, in closed form, the series

\[
A_k = \sum_{n_1, \ldots, n_k=1}^{\infty} (-1)^{n_1+n_2+\cdots+n_k} \left( H_{n_1+n_2+\cdots+n_k} - \ln(n_1+n_2+\cdots+n_k) - \gamma \right). 
\]

This problem was motivated by the alternating series

\[
A_1 = \sum_{n=1}^{\infty} (-1)^n \left( H_n - \ln n - \gamma \right) = \frac{\gamma - \ln \pi}{2}
\]
due to Hardy [7, p.277]. The double series

\[
A_2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{n+m} \left( H_{n+m} - \ln(n+m) - \gamma \right) = -3 \ln A + \frac{7}{12} \ln 2 + \frac{1}{2} \ln \pi \frac{\gamma}{4},
\]

where \( A \) denotes the Glaisher–Kinkelin constant, can be found in [1, Problem 3.106, p.157]. Thus, the problem was open only for the case when \( k \geq 3 \).

**Theorem 1.6 (A multiple alternating Hardy series).** The following equality holds

\[
A_k = (-1)^k \left[ \frac{2^k - 1}{2^{k-1}} \ln 2 - \sum_{n=1}^{k} \frac{1}{2n^2} + \frac{H_k}{2^k} + \frac{1}{2} \int_0^1 \frac{y^{k-1} \ln \left( \frac{1}{y} \right)^{k+1}}{(1+y)^{k+1}} \, dy \right].
\]

**Proof.** First we observe that, if \( k \geq 0 \) is an integer, then using summation by parts we have

\[
\sum_{n=1}^{\infty} (-1)^n (H_{n+k} - \ln(n+k) - \gamma) = \sum_{n=1}^{\infty} \left[ \frac{1}{2n+k} + \ln \left( 1 - \frac{1}{2n+k} \right) \right].
\]

It follows that,

\[
A_k = \sum_{n_1, \ldots, n_k=1}^{\infty} (-1)^{n_1+\cdots+n_k} \sum_{n_1=1}^{\infty} (-1)^{n_1} \left( H_{n_1+n_2+\cdots+n_k} - \ln(n_1+n_2+\cdots+n_k) - \gamma \right)
\]

\[
= \sum_{n_1, \ldots, n_k=1}^{\infty} (-1)^{n_1+\cdots+n_k-1} \sum_{n_1=1}^{\infty} \left( \frac{1}{n_1+\cdots+n_k-1+2n} + \ln \left( 1 - \frac{1}{n_1+\cdots+n_k-1+2n} \right) \right)
\]

\[
= \sum_{n_1, \ldots, n_k=1}^{\infty} (-1)^{n_1+\cdots+n_k-1} \sum_{n=1}^{\infty} \sum_{m=2}^{\infty} \frac{1}{m^{2m} \left( n_1+\cdots+n_k-1+2n \right)^m}
\]

\[
= \sum_{m=2}^{\infty} \frac{1}{m^{2m}} \int_0^1 \frac{t^{m-1} e^{-\frac{k+1}{t}}}{1 - e^{-t}} \, dt
\]

\[
= (-1)^k \sum_{m=2}^{\infty} \frac{1}{m!} \int_0^1 \frac{t^{m-1} e^{-(k+1)t}}{(1-e^{-t}) (1+e^{-t})^k} \, dt (t \to 2t)
\]

\[
= (-1)^k \sum_{m=2}^{\infty} \frac{1}{m!} \int_0^1 \frac{t^{m-1} e^{-(k+1)t}}{(1-e^{-t}) (1+e^{-t})^k} \, dt.
\]
We calculate the preceding integral by parts, with \( f(t) = \frac{1}{(e^{t+1})^k} \), \( f'(t) = -\frac{ke^t}{(e^{t+1})^{k+1}} \), \( g(t) = \frac{e^{t-t-1}}{t(e^{t}-1)} \).

\[
\int_0^\infty \frac{e^t - 1 - t}{t (e^t - 1)(1 + e^t)^k} \, dt = -k \int_0^\infty \frac{e^t - 1}{te^t} \, dt \\
= k \int_0^\infty \frac{te^t}{(e^t + 1)^{k+1}} \, dt - k \int_0^\infty \frac{e^t}{(e^t + 1)^{k+1}} \ln \frac{e^t - 1}{t} \, dt \\
= \int_0^\infty \frac{1}{(1 + e^t)^k} \, dt - k \int_0^\infty \frac{e^t (\ln (e^t - 1) - \ln t)}{(1 + e^t)^{k+1}} \, dt \\
= \int_0^\infty \frac{dt}{(1 + e^t)^k} - k \int_0^\infty \frac{\ln u}{(2 + u)^{k+1}} \, du + k \int_0^\infty \frac{e^t \ln t}{(1 + e^t)^{k+1}} \, dt. \\
\]

(7)

On the other hand,

\[
\int_0^\infty \frac{dt}{(1 + e^t)^k} = \ln 2 - \sum_{n=1}^{k-1} \frac{1}{n 2^n} \quad \text{and} \quad \int_0^\infty \frac{\ln u}{(2 + u)^{k+1}} \, du = \frac{\ln 2}{k 2^k} + \frac{1}{k 2^k} = \frac{H_k}{k 2^k}. \\
\]

(8)

Combining (7) and (8) we get that

\[
\int_0^\infty \frac{e^t - 1 - t}{t (e^t - 1)(1 + e^t)^k} \, dt = \frac{2^k - 1}{2^k} \ln 2 - \sum_{n=1}^k \frac{1}{n 2^n} + \frac{H_k}{k 2^k} + k \int_0^\infty \frac{e^t \ln t}{(1 + e^t)^{k+1}} \, dt \\
= \frac{2^k - 1}{2^k} \ln 2 - \sum_{n=1}^k \frac{1}{n 2^n} + \frac{H_k}{k 2^k} + k \int_0^\infty \frac{y^{k-1} \ln \left(\frac{1}{y}\right)}{(1 + y)^{k+1}} \, dy. \\
\]

We mention that, integrals of the form \( \int_0^1 Q(y) \ln \left(\frac{1}{y}\right) \, dy \), where \( Q \) is a rational function, have been calculated by Medina and Moll in [8]. The theorem is proved.

The last open problem we solve is about calculating in closed form a multiple factorial series.

3.137. Does the sum equal a rational multiple of \( e \)?

Let \( k \geq 4 \) be an integer. Calculate the sum

\[
\sum_{n_1, \ldots, n_k=1}^\infty \frac{n_1 n_2 \cdots n_k}{(n_1 + n_2 + \cdots + n_k)!}. \\
\]

Conjecture. It is reasonable to conjecture that

\[
\sum_{n_1, \ldots, n_k=1}^\infty \frac{n_1 n_2 \cdots n_k}{(n_1 + n_2 + \cdots + n_k)!} = a_k e, \\
\]

where \( a_k \) is a rational number. We have that \( a_1 = 1, a_2 = 2/3, \) and \( a_3 = 31/120 \) (see Problems 3.114 and 3.118 in [1]). If the conjecture holds true, an open problem would be to study the properties of the sequence \((a_k)_{k \in \mathbb{N}}\).

In the next theorem we solve this problem and show the conjecture holds true. We also give the formula for the sequence \((a_k)_{k \in \mathbb{N}}\) as a rational linear combination of binomial coefficients.
Let \( k, l \) be integers such that \( k \geq 1 \) and \( l = 1, \ldots, k \), let
\[
f_{k,l}(z) = \sum_{n_1, \ldots, n_k = 1}^{\infty} \frac{n_1 n_2 \cdots n_l z^{n_1 + n_2 + \cdots + n_k}}{(n_1 + n_2 + \cdots + n_k)!}, \quad z \in \mathbb{C},
\]
and let
\[
f_{k,0}(z) = \sum_{n_1, \ldots, n_k = 1}^{\infty} \frac{z^{n_1 + n_2 + \cdots + n_k}}{(n_1 + n_2 + \cdots + n_k)!}, \quad z \in \mathbb{C}.
\]

**Theorem 1.7** (A multiple factorial sum). Let \( k, l \) be integers such that \( k \geq 1 \) and \( l = 1, \ldots, k \), let \( f_{k,l} \) be as in (9) and \( f_{k,0} \) be as in (10). The following equalities hold
\[
f_{k,0}(z) = e^z \sum_{i=0}^{k-1} \frac{(-1)^{k-1-l_i}z^l}{l!} + (-1)^k
\]
and
\[
f_{k,l}(z) = e^z \sum_{i=0}^{l-1} \binom{l-1}{i} \frac{z^{k+i}}{(k+i)!}.
\]
Thus,
\[
f_{k,k}(1) = \sum_{n_1, \ldots, n_k = 1}^{\infty} \frac{n_1 n_2 \cdots n_k}{(n_1 + n_2 + \cdots + n_k)!} = e \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{1}{(k+i)!}
\]
and hence
\[
a_k = \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{1}{(k+i)!}.
\]
In particular,
\[
f_{2,2}(1) = \frac{2e}{3}, \quad f_{3,3}(1) = \frac{31e}{120}, \quad f_{4,4}(1) = \frac{179e}{2520}, \quad f_{5,5}(1) = \frac{787e}{51840}, \quad f_{6,6}(1) = \frac{6631e}{2494800}.
\]

**Proof.** First we observe that
\[
\sum_{n_1, n_2, \ldots, n_k = 1}^{\infty} n_1 n_2 \cdots n_l z^{n_1 + n_2 + \cdots + n_k} = \frac{z^k}{(1-z)^{k+l}}, \quad |z| < 1.
\]
We compare the coefficients of \( z^N \) of both sides, for \( N \geq k \), and we have
\[
\sum_{n_1, n_2, \ldots, n_k \geq 1} n_1 n_2 \cdots n_l = \binom{N + l - 1}{k + l - 1} = \sum_{i=0}^{l-1} \binom{N}{k+i} \binom{l-1}{i}.
\]
Thus (note that we can sum \( N \) from zero, because the binomial coefficient vanishes for \( N < k \)):
\[
f_{k,l}(z) = \sum_{N=0}^{\infty} \frac{z^N}{N!} \binom{N + l - 1}{k + l - 1} = \sum_{i=0}^{l-1} \binom{l-1}{i} \sum_{k+i}^{\infty} \frac{z^N}{(N-k-i)!} = e^z \sum_{i=0}^{l-1} \frac{1}{(k+i)!} \frac{z^{k+i}}{(k+i)!}.
\]
For calculating \( f_{k,0}(z) \) we apply the same technique as above. We have
\[
f_{k,0}(z) = \sum_{N=k}^{\infty} \frac{z^N}{N!} \binom{N-1}{k-1} = \sum_{N=k}^{\infty} \frac{z^N}{N(N-k)(k-1)!} = \frac{1}{(k-1)!} \int_0^z e^{u} u^{k-1} du
\]
and the result follows by using iterated integration by parts. \( \square \)
Remark 1.8. We proved the conjecture that $f_{k,k}(1) \in e^Q$ is a rational multiple of $e$. However, the problem holds much more generally. Namely, for any polynomial $P \in \mathbb{Q}[n_1, n_2, \ldots, n_k]$ in $k$ variables the associated series

$$A(k, P) = \sum_{n_1, n_2, \ldots, n_k=1}^{\infty} \frac{P(n_1, \ldots, n_k)}{(n_1 + \cdots + n_k)!} \in e^Q$$

is a rational multiple of $e$. Theorem 1.7 is a special case when $P = n_1 n_2 \cdots n_k$. This follows from

$$F(z_1, z_2, \ldots, z_k) = \sum_{n_1, n_2, \ldots, n_k=1}^{\infty} \frac{z_1^{n_1} \cdots z_k^{n_k}}{(n_1 + \cdots + n_k)!} = \sum_{i=1}^{k} \frac{e^{z_i} - 1}{\prod_{j \neq i} (z_i/z_j - 1)},$$

(11)

because we can build up the numerator $P$ in the summand via derivatives. We end up with some rational (in the $z_i$) linear combination of the $e^{z_i}$. Taking limits as $z_i \to 1$, it is clear that we get rational multiples of $e$:

$$A(k, P) = \lim_{z_1 \to 1} \cdots \lim_{z_k \to 1} [P(z_1 \partial z_1, \ldots, z_k \partial z_k)F(z_1, \ldots, z_k)] \in e^Q.$$ 

Formula (11) can be proved inductively from $F(z_1) = e^{z_1} - 1$ via

$$F(z_1, z_2, \ldots, z_k) = \frac{1}{z_1/z_2 - 1} F(z_1, z_3, z_4, \ldots, z_k) + \frac{1}{z_2/z_1 - 1} F(z_2, z_3, z_4, \ldots, z_k),$$

which follows from $\sum_{n_1 + n_2 = n} z_1^{n_1} z_2^{n_2} = \frac{z_1^n}{z_1/z_2 - 1} + \frac{z_2^n}{z_2/z_1 - 1}$.

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References