Abstract: The theory of Schur complement plays an important role in many fields, such as matrix theory and control theory. In this paper, applying the properties of Schur complement, some new estimates of diagonally dominant degree on the Schur complement of I(II)-block strictly diagonally dominant matrices and I(II)-block strictly doubly diagonally dominant matrices are obtained, which improve some relative results in Liu [Linear Algebra Appl. 435(2011) 3085-3100]. As an application, we present several new eigenvalue inclusion regions for the Schur complement of matrices. Finally, we give a numerical example to illustrate the advantages of our derived results.

Keywords: I(II)-Block strictly diagonally dominant matrix, I(II)-Block strictly doubly diagonally dominant matrix, Diagonally dominant degree, Eigenvalue

MSC: 15A45, 15A48

1 Introduction

Let $C^{n \times n}$ denote the set of all $n \times n$ complex matrices, $N = \{1, 2, \ldots, n\}$ and $A = (a_{ij}) \in C^{n \times n} (n \geq 2)$. We write

$$r_i(A) = \sum_{j=1, j \neq i}^{n} |a_{ij}|, \quad \forall i \in N.$$ 

We know that $A$ is called a diagonally dominant matrix if

$$|a_{ii}| \geq r_i(A), \quad \forall i \in N. \quad (1)$$

$A$ is called a strictly diagonally dominant matrix if all inequalities in (1) hold strictly.

$A$ is called a generalized Ostrowski matrix if

$$|a_{ii}| |a_{jj}| \geq r_i(A) r_j(A), \quad \forall i, j \in N, \quad i \neq j. \quad (2)$$

$A$ is called Ostrowski matrix if all inequalities in (2) hold strictly (see [1]).

$D_n$ ($SD_n$), and $OS_n$ ($GOS_n$) will be used to denote the sets of all $n \times n$ (strictly) diagonally dominant matrices and the sets of all $n \times n$ (generalized) Ostrowski matrices, respectively.

*Corresponding Author: Feng Wang: College of Science, Guizhou Minzu University, Guiyang, Guizhou 550025, P.R. China, E-mail: wangf991@163.com

Deshu Sun: College of Science, Guizhou Minzu University, Guiyang, Guizhou 550025, P.R. China, E-mail: sds198039@163.com
For $\alpha \subseteq N$, denote by $|\alpha|$ the cardinality of $\alpha$ and $\overline{\alpha} = N/\alpha$. If $\alpha, \beta \subseteq N$, then $A(\alpha, \beta)$ is the submatrix of $A$ with row indices in $\alpha$ and column indices in $\beta$. In particular, $A(\alpha, \alpha)$ is abbreviated to $A(\alpha)$. If $A(\alpha)$ is nonsingular,

$$A/\alpha = A/A(\alpha) = A(\overline{\alpha}) - A(\overline{\alpha}, \alpha)[A(\alpha)]^{-1} A(\alpha, \overline{\alpha}),$$

is called the Schur complement of $A$ with respect to $A(\alpha)$.

The Schur complement has been proved to be a useful tool in many fields, such as control theory, statistics and computational mathematics. A lot of work has been done on it (see [2-12]). It is well-known that the Schur complement of $SD_N$ and $OS_N$ are $SD_N$ and $OS_N$, respectively. These properties have been used for deriving matrix inequalities in matrix analysis and for the convergence of iterations in numerical analysis (see [13–16]).

**SD** and computational mathematics. A lot of work has been done on it (see [2-12]). It is well-known that the Schur

A matrix $A = (a_{ij}) \in C^{n \times n}$ is called an $M$-matrix, if there exist a nonnegative matrix $B$ and a real number $s > \rho(B)$, where $\rho(B)$ is the spectral radius of $B$, such that $A = sI - B$. We know that if $A$ is an $M$-matrix, then the Schur complement of $A$ is also an $M$-matrix and $det A > 0$ (see [17]). A matrix $A$ is an $H$-matrix if and only if $\mu(A)$ is an $M$-matrix. We denote by $H_n$ and $M_n$ the set of $H$-matrices and $M$-matrices, respectively.

Let $A \in C^{n \times n}$ be partitioned in the following form:

$$A = \begin{pmatrix}
A(\alpha_1, \alpha_1) & A(\alpha_1, \alpha_2) & \cdots & A(\alpha_1, \alpha_s) \\
A(\alpha_2, \alpha_1) & A(\alpha_2, \alpha_2) & \cdots & A(\alpha_2, \alpha_s) \\
\vdots & \vdots & \ddots & \vdots \\
A(\alpha_s, \alpha_1) & A(\alpha_s, \alpha_2) & \cdots & A(\alpha_s, \alpha_s)
\end{pmatrix},$$

(3)

where $A(\alpha_t, \alpha_t)$ is a $|\alpha_t| \times |\alpha_t|$ nonsingular principal submatrix of $A$, $t = 1, 2, \ldots, s$.

Let $C_s^{n \times n}$ denote the set of all $s \times s$ block matrices in $C^{n \times n}$ partitioned as (3). Suppose $A = (A(\alpha_t, \alpha_m))^{n \times n} \in C_s^{n \times n}$, and let $N(A) = (\|A(\alpha_t, \alpha_m)\|)_s$ be the norm matrix of $A$, where $\| \cdot \|$ is some consistent matrix norm.

Let $A = (A(\alpha_t, \alpha_m))^{n \times n} \in C_s^{n \times n}$. $A$ is called an I-block strictly diagonally dominant matrix if for all $1 \leq l \leq s$,

$$\|A(\alpha_l, \alpha_l)^{-1}^{-1} - \sum_{m=1, m \neq l}^s \|A(\alpha_l, \alpha_m)\| = P_l(A).$$

Let $A = (A(\alpha_t, \alpha_m))^{n \times n} \in C_s^{n \times n}$. $A$ is called an II-block strictly diagonally dominant matrix if for all $1 \leq l \leq s$,

$$1 > \sum_{m=1, m \neq l}^s \|A(\alpha_l, \alpha_l)^{-1} A(\alpha_l, \alpha_m)\| = P_l(A).$$

Let $A = (A(\alpha_t, \alpha_m))^{n \times n} \in C_s^{n \times n}$. $A$ is called an I-block strictly doubly diagonally dominant matrix if for all $1 \leq i, j \leq s, i \neq j$,

$$\|A(\alpha_i, \alpha_i)^{-1}^{-1} - \sum_{m=1, m \neq i}^s \|A(\alpha_i, \alpha_m)\| \sum_{m=1, m \neq j}^s \|A(\alpha_j, \alpha_m)\|. $$

Let $A = (A(\alpha_t, \alpha_m))^{n \times n} \in C_s^{n \times n}$. $A$ is called an II-block strictly doubly diagonally dominant matrix if for all $1 \leq i, j \leq s, i \neq j$,

$$\sum_{m=1, m \neq i}^s \|A(\alpha_i, \alpha_i)^{-1} A(\alpha_i, \alpha_m)\| \sum_{m=1, m \neq j}^s \|A(\alpha_j, \alpha_m)\| < 1.$$
We know that if $A \in \text{I-BSDD}_s$ but $A \notin \text{I-BSD}_s$, then there exists a unique index $i_0$ such that
\[
\| [A(\alpha_{i_0}, \alpha_{i_0})]^{-1} \|^{-1} \leq \sum_{m=1, m \neq i_0}^s \| A(\alpha_{i_0}, \alpha_m) \| = P_{i_0}(A). \tag{4}
\]
And if $A \in \text{II-BSDD}_s$ but $A \notin \text{II-BSD}_s$, then there exists a unique index $i_0$ such that
\[
1 \leq \sum_{m=1, m \neq i_0}^s \| [A(\alpha_{i_0}, \alpha_{i_0})]^{-1} A(\alpha_{i_0}, \alpha_m) \| = \tilde{P}_{i_0}(A). \tag{5}
\]
As shown in [4], for $1 \leq i \leq s$, we call
\[
\| [A(\alpha_i, \alpha_j)]^{-1} \|^{-1} - \sum_{m \neq i}^s \| A(\alpha_i, \alpha_m) \|, \quad 1 - \sum_{m=1, m \neq i}^s \| [A(\alpha_i, \alpha_i)]^{-1} A(\alpha_i, \alpha_m) \|,
\]
the $i$-th I-block and II-block diagonally dominant degree of $A$, respectively.

A is called an I-block $H$-matrix and II-block $H$-matrix, respectively, if the comparison matrices of block matrix $A \mu I(A) = (\alpha_{im}) \in M_s$ and $\tilde{A} \mu I(A) = (\tilde{\alpha}_{im}) \in M_s$, where
\[
\alpha_{im} = \begin{cases} 
\| [A(\alpha_i, \alpha_i)]^{-1} \|^{-1}, & \text{if } l = m, \\
-\| A(\alpha_i, \alpha_m) \|, & \text{if } l \neq m.
\end{cases}
\]
\[
\tilde{\alpha}_{im} = \begin{cases} 
1, & \text{if } l = m, \\
-\| [A(\alpha_i, \alpha_i)]^{-1} A(\alpha_i, \alpha_m) \|, & \text{if } l \neq m.
\end{cases}
\]

The paper is organized as follows. In Section 2, we give several new estimates of block diagonally dominant degree on the Schur complement of I(II)-BSDD$_s$ and I(II)-BSDD$_s$, which improve some related results in [4]. In Section 3, based on these derived results in the Section 2, new inclusion regions for eigenvalues of the Schur complement are obtained. In Section 4, we present a numerical example to illustrate the advantages of our derived results.

2 The block diagonally dominant degree for the Schur complement

In this section, we present several new estimates of block diagonally dominant degree on the Schur complement of I(II)-BSDD$_s$ and I(II)-BSDD$_s$.

Lemma 2.1 ([18]). If $A \in H_n$, then $[\mu(A)]^{-1} \geq |A^{-1}|$.

Lemma 2.2 ([17]). If $A \in SD_n$ or $A \in OS_n$, then $A \in H_n$, i.e., $\mu(A) \in M_n$.

Lemma 2.3 ([11]). Let $A$ be an I(II)-block $H$-matrix. Then $A$ is nonsingular.

Lemma 2.4 ([11]). Let $A \in I$-BSDD$_s$, then $[\mu I(A)]^{-1} \geq N(A^{-1})$.

Lemma 2.5 ([11]). Let $A \in II$-BSDD$_s$, then $[\mu II(A)]^{-1} \geq N(A^{-1} D)$, where
\[
D = \text{diag}(A(\beta_1, \beta_1), A(\beta_2, \beta_2), \ldots, A(\beta_s, \beta_s)).
\]

Lemma 2.6 ([11]). Let $A \in I(II)$-BSDD$_s$, $\alpha = \sum_{u=1}^k \alpha_i \subset N$, $\overline{\alpha} = \sum_{i=1}^l \alpha_j$, and $k + l = s$. Then for any $t = 1, 2, \ldots, l$,
\[
\begin{align*}
\psi_t &= 1 - \left| [A(\alpha_{ji}, \alpha_{ji})]^{-1} [A(\alpha_{ji}, \alpha_{ji}), \ldots, A(\alpha_{ji}, \alpha_{ji})] [A(\alpha_{ji}, \alpha_{ji})]^{-1}
\begin{pmatrix}
A(\alpha_{i_1}, \alpha_{j_1}) \\
\vdots \\
A(\alpha_{i_k}, \alpha_{j_l})
\end{pmatrix}
\right| > 0.
\end{align*}
\]
Lemma 2.7 ([19]). Let $A \in C^{n \times n}$. If $\|A\| < 1$, then $I_n - A$ is nonsingular and
\[
\|(I_n - A)^{-1}\| \leq \frac{1}{1 - \|A\|},
\]
where $I_n$ is an identity matrix.

Lemma 2.8 ([4]). Let $A \in I\cdot BSD_s$, $\alpha = \sum_{r=1}^{k} \alpha_{j_r} \in N$, $\bar{\alpha} = \sum_{v=1}^{l} \alpha_{j_v}, k + l = s$ and $A/\alpha = (\bar{\alpha}(\alpha_1, \alpha_r))$. Then
\[
\| \bar{A}(\alpha_t, \alpha_t) \|^{-1} - P_t(A/\alpha) \geq \|[A(\alpha_t, \alpha_t)]^{-1} - P_j(A) + \omega_{j_t} \| \geq \|[A(\alpha_t, \alpha_t)]^{-1} - P_j(A) > 0,
\]
and
\[
\| \bar{A}(\alpha_t, \alpha_t) \|^{-1} + P_t(A/\alpha) \leq \|[A(\alpha_t, \alpha_t)]^{-1} + P_j(A) - \omega_{j_t} \| \leq \|[A(\alpha_t, \alpha_t)]^{-1} + P_j(A),
\]
where
\[
\omega_{j_t} = \min_{1 \leq u \leq k} \frac{\|[A(\alpha_{j_u}, \alpha_{j_u})]^{-1} - P_{j_u}(A)\|}{\|[A(\alpha_{j_u}, \alpha_{j_u})]^{-1}\|} \sum_{v=1}^{k} \|A(\alpha_{j_t}, \alpha_{j_u})\|.
\]

Lemma 2.9 ([4]). Let $A \in II\cdot BSD_s$, $\alpha = \sum_{r=1}^{k} \alpha_{j_r} \in N$, $\bar{\alpha} = \sum_{v=1}^{l} \alpha_{j_v}, k + l = s$ and $A/\alpha = (\bar{\alpha}(\alpha_t, \alpha_t))$. Then
\[
1 - \bar{P}_t(A/\alpha) \geq 1 - \bar{P}_j(A) + \bar{\omega}_{j_t} \geq 1 - \bar{P}_j(A) > 0,
\]
where
\[
\bar{\omega}_{j_t} = \min_{1 \leq u \leq k} (1 - \bar{P}_{j_u}(A)) \sum_{v=1}^{k} \|A(\alpha_{j_t}, \alpha_{j_u})\|.
\]

Now we give our main results, which are more accurate than those in Lemma 2.8 and Lemma 2.9.

Theorem 2.10. Let $A \in I\cdot BSD_s$, $\alpha = \sum_{w=1}^{k} \alpha_{j_w} \in N$, $\bar{\alpha} = \sum_{v=1}^{l} \alpha_{j_v}, k + l = s$ and $A/\alpha = (\bar{\alpha}(\alpha_t, \alpha_t))$. Then
\[
\| \bar{A}(\alpha_t, \alpha_t) \|^{-1} - P_t(A/\alpha) \geq \|[A(\alpha_t, \alpha_t)]^{-1} - P_j(A) + \delta_{j_t} \| \geq \|[A(\alpha_t, \alpha_t)]^{-1} - P_j(A),
\]
and
\[
\| \bar{A}(\alpha_t, \alpha_t) \|^{-1} + P_t(A/\alpha) \leq \|[A(\alpha_t, \alpha_t)]^{-1} + P_j(A) - \delta_{j_t} \| \leq \|[A(\alpha_t, \alpha_t)]^{-1} + P_j(A),
\]
where $1 \leq w \leq k$,
\[
\delta_{j_t} = \sum_{w=1}^{k} \|A(\alpha_{j_t}, \alpha_{j_w})\| \frac{\|[A(\alpha_{j_w}, \alpha_{j_w})]^{-1} - R_{j_w}(A)\|}{\|[A(\alpha_{j_w}, \alpha_{j_w})]^{-1}\|},
\]
\[
\eta = \max_{1 \leq w \leq k} \frac{\sum_{v=1}^{l} \|A(\alpha_{j_w}, \alpha_{j_v})\|}{\|[A(\alpha_{j_w}, \alpha_{j_w})]^{-1}\| - \sum_{v=1}^{k} \|A(\alpha_{j_w}, \alpha_{j_v})\|},
\]
\[
R_{j_w}(A) = \eta \sum_{v=1}^{k} \|A(\alpha_{j_w}, \alpha_{j_v})\| + \sum_{v=1}^{l} \|A(\alpha_{j_w}, \alpha_{j_v})\|.\]
Proof. Let

$$
\Omega_{ts} = (A(\alpha_j, \alpha_{i_1}), \cdots, A(\alpha_j, \alpha_{i_k}))[A(\alpha)]^{-1} \left( \begin{array}{c} A(\alpha_{i_1}, \alpha_j) \\ \vdots \\ A(\alpha_{i_k}, \alpha_j) \end{array} \right),
$$

$$
G_t = (\| A(\alpha_j, \alpha_{i_1}) \|, \cdots, \| A(\alpha_j, \alpha_{i_k}) \|)^T,
$$

$$
H_t^T = \left( \sum_{s=1}^l \| A(\alpha_{i_1}, \alpha_{j_s}) \|, \cdots, \sum_{s=1}^l \| A(\alpha_{i_k}, \alpha_{j_s}) \| \right)^T, \quad 1 \leq t, s \leq l.
$$

Let $J_t = \{ \alpha_{j_t} \}$ and $I_{J_t}$ be an identity matrix. From Lemma 2.6, we have \[ \| [A(\alpha_j, \alpha_{j_t})]^{-1} \Omega_{tt} \|^{-1} < 1. \] And by Lemma 2.7, we have

$$
\| \{ I_{J_t} - [A(\alpha_j, \alpha_{j_t})]^{-1} \Omega_{tt} \}^{-1} \| \leq \frac{1}{1 - \| [A(\alpha_j, \alpha_{j_t})]^{-1} \Omega_{tt} \|},
$$

that is

$$
\| \{ I_{J_t} - [A(\alpha_j, \alpha_{j_t})]^{-1} \Omega_{tt} \}^{-1} \|^{-1} \geq 1 - \| [A(\alpha_j, \alpha_{j_t})]^{-1} \Omega_{tt} \|.
$$

Thus, by Lemma 2.1, Lemma 2.3 and Lemma 2.4, for any $\epsilon > 0$ and $1 \leq t \leq l$,

$$
\| A(\alpha_t, \alpha_j) \|^2 - P_t (A/\alpha)
$$

$$
= \| A(\alpha_j, \alpha_{j_t}) - \Omega_{tt} \|^2 - \sum_{s \neq t}^l \| A(\alpha_j, \alpha_{j_s}) - \Omega_{ts} \|
$$

$$
\geq \| A(\alpha_j, \alpha_{j_t}) \|^{-1} \| \{ I_{J_t} - [A(\alpha_j, \alpha_{j_t})]^{-1} \Omega_{tt} \}^{-1} \|^{-1} - \sum_{s=1, s \neq t}^l \| A(\alpha_j, \alpha_{j_s}) - \Omega_{ts} \|
$$

$$
\geq \| A(\alpha_j, \alpha_{j_t}) \|^{-1} - \sum_{s=1, s \neq t}^l \| A(\alpha_{j_s}, \alpha_{j_t}) \| - G_t^T \mu_t (A(\alpha))^{-1} H_t^T
$$

$$
= \| A(\alpha_j, \alpha_{j_t}) \|^2 - P_{J_t} (A) + \sum_{w=1}^k \| A(\alpha_j, \alpha_{i_w}) \| + (\delta_j_t - \epsilon) - (\delta_j_t - \epsilon) - G_t^T \mu_t (A(\alpha))^{-1} H_t^T
$$

$$
= \| A(\alpha_j, \alpha_{j_t}) \|^2 - P_{J_t} (A) + \delta_j_t - \epsilon + \frac{1}{\| \mu_t (A(\alpha)) \|} \left( \sum_{w=1}^k \| A(\alpha_j, \alpha_{i_w}) \| - \delta_j_t + \epsilon - G_t^T H_t \mu_t (A(\alpha)) \right).
$$

Let

$$
B = \left( \begin{array}{cc} x & -G_t^T \\
-H_t & \mu_t (A(\alpha)) \end{array} \right).
$$

If

$$
x > \sum_{w=1}^k \| A(\alpha_j, \alpha_{i_w}) \| \frac{R_{i_w} (A)}{\| [A(\alpha_{i_w}, \alpha_{i_w})]^{-1} \|^{-1}}.
$$

then there exists sufficiently small positive number $\epsilon_0$ such that

$$
x > \sum_{w=1}^k \| A(\alpha_j, \alpha_{i_w}) \| \left( \frac{R_{i_w} (A)}{\| [A(\alpha_{i_w}, \alpha_{i_w})]^{-1} \|^{-1}} + \epsilon_0 \right).
$$

Construct a positive diagonal matrix $X = diag(x_1, x_2, \ldots, x_{k+1})$, where

$$
x_t = \begin{cases} 
1, & \text{if } t = 1, \\
\frac{R_{i_t} (A)}{\| [A(\alpha_{i_t-1}, \alpha_{i_t-1})]^{-1} \|^{-1}} + \epsilon_0, & \text{if } t = 2, 3, \ldots, k + 1.
\end{cases}
$$
Let $\tilde{B} = BX = (\tilde{b}_{pq})$. For $p = 1$, by (8), we have

$$|\tilde{b}_{pp} - r_p(\tilde{B})| = |\tilde{b}_{11} - \sum_{j=2}^{k+1} \tilde{b}_{1j}| = x - \sum_{w=1}^{k} \|A(\alpha_j, \alpha_i)\| \left( \frac{R_{iw}(A)}{\|A(\alpha_i, \alpha_i)\|^{-1} - 1} + \epsilon_0 \right) > 0.$$ 

And for $p = 2, 3, \ldots, k+1$, from $\frac{R_{iw}(A)}{\|A(\alpha_i, \alpha_i)\|^{-1} - 1} \leq \eta$, $1 \leq w \leq k$, we obtain

$$|\tilde{b}_{pp} - r_p(\tilde{B})| = \sum_{w=1, w \neq p-1}^{k} \|A(\alpha_{p-1}, \alpha_i)\| \left( \frac{R_{iw}(A)}{\|A(\alpha_i, \alpha_i)\|^{-1} - 1} + \epsilon_0 \right) - \sum_{w=1}^{k} \|A(\alpha_{p-1}, \alpha_i)\| + \eta \sum_{w=1, w \neq p-1}^{k} \|A(\alpha_{p-1}, \alpha_i)\| - \sum_{w=1}^{k} \|A(\alpha_{p-1}, \alpha_i)\| = \epsilon_0(\|A(\alpha_{p-1}, \alpha_i)\|^{-1} - 1) + \eta \sum_{w=1, w \neq p-1}^{k} \|A(\alpha_{p-1}, \alpha_i)\| > 0.$$ 

Thus, $\tilde{B} \in SD_{k+1}$, and so $B \in H_{k+1}$. Note that $B = \mu(B) \in M_{k+1}$, then

$$\det B > 0. \tag{9}$$

Let $x$ be $\sum_{w=1}^{k} \|A(\alpha_j, \alpha_i)\| + \delta_{j_1} + \epsilon$ in $B$. Then

$$\sum_{w=1}^{k} \|A(\alpha_j, \alpha_i)\| - \delta_{j_1} + \epsilon - \sum_{w=1}^{k} \|A(\alpha_j, \alpha_i)\| = \sum_{w=1}^{k} \|A(\alpha_j, \alpha_i)\| - \sum_{w=1}^{k} \|A(\alpha_j, \alpha_i)\| + \frac{R_{iw}(A)}{\|A(\alpha_j, \alpha_i)\|^{-1} - 1} > 0.$$ 

By Lemma 2.2, we have $\mu_1(A(\alpha)) \in M_k$, thus $\det[\mu_1(A(\alpha))] > 0$. Further, by (9), we obtain

$$\|A(\alpha_j, \alpha_i)\|^{-1} - P_t(A(\alpha)) > \|A(\alpha_j, \alpha_i)\|^{-1} - P_j(A) + \delta_{j_1} - \epsilon.$$ 

Let $\epsilon \to 0$. Then we obtain (6). Similarly, we can prove (7).

**Remark 2.11.** Note that

$$P_{iw}(A) \leq P_{iw}(A), \quad 1 \leq w \leq k; \quad \omega_j \leq \delta_{j_1}, \quad 1 \leq t \leq l.$$ 

This shows that Theorem 2.10 improves Lemma 2.8.

**Theorem 2.12.** Let $A \in I-BSD_{ks}$ with the index $i_d$ such as in (4), $\alpha = \bigcup_{w=1}^{k} \alpha_i, \alpha_j \subset \alpha, \overline{\alpha} = \bigcup_{w=1}^{l} \alpha_j$, $k+l = s$ and $A(\alpha) = \tilde{A}(\alpha, \alpha)$. Then

$$\|A(\alpha_j, \alpha_i)\|^{-1} - P_t(A(\alpha))$$
Let similarly as in the proof of Theorem 2.10, we have

\[ \| [A(\alpha_j, \alpha_{j_i})]^{-1} - P_{j_i}(A) + \left( 1 - \frac{Q_{id}(A)}{\| [A(\alpha_{id}, \alpha_{id})]^{-1} \|^{-1}} \right) \sum_{v=1}^{k} \| A(\alpha_{j_i}, \alpha_{i_v}) \| \geq \| [A(\alpha_j, \alpha_{j_i})]^{-1} - P_{j_i}(A) \|, \]  

(10)

and

\[ \| \widetilde{A}(\alpha, \alpha) \|^{-1} - 1 + P_{t}(A/\alpha) \leq \| [A(\alpha_j, \alpha_{j_i})]^{-1} - 1 + P_{j_i}(A) - \left( 1 - \frac{Q_{id}(A)}{\| [A(\alpha_{id}, \alpha_{id})]^{-1} \|^{-1}} \right) \sum_{v=1}^{k} \| A(\alpha_{j_i}, \alpha_{i_v}) \| \leq \| [A(\alpha_j, \alpha_{j_i})]^{-1} - 1 + P_{j_i}(A) \|, \]  

(11)

where

\[ r = \max \left\{ \max_{u \neq \alpha} \frac{\| A(\alpha_u, \alpha_{id}) \|}{\| A(\alpha_u, \alpha_{id}) \|^{-1} - 1 - \sum \| A(\alpha_u, \alpha_{id}) \| \| A(\alpha_{id}, \alpha_{i_v}) \| P_{id}(A)} \right\}, \]

\[ Q_{id}(A) = r P_{id}(A). \]

Proof. Similarly as in the proof of Theorem 2.10, for any \( \epsilon > 0 \) and \( 1 \leq t \leq l \),

\[ \| \widetilde{A}(\alpha, \alpha) \|^{-1} - 1 - P_{t}(A/\alpha) \geq \| [A(\alpha_j, \alpha_{j_i})]^{-1} - 1 - P_{j_i}(A) + \left( 1 - \frac{Q_{id}(A)}{\| [A(\alpha_{id}, \alpha_{id})]^{-1} \|^{-1}} \right) \sum_{v=1}^{k} \| A(\alpha_{j_i}, \alpha_{i_v}) \| - \epsilon \]

\[ + \frac{1}{\text{det} [\mu_f(A(\alpha))] } \text{det} \left( \frac{Q_{id}(A)}{\| [A(\alpha_{id}, \alpha_{id})]^{-1} \|^{-1} - 1 - \sum_{v=1}^{k} \| A(\alpha_{j_i}, \alpha_{i_v}) \| + \epsilon - G_T} \mu_f(A(\alpha) \right), \]  

where \( G_T, H^T \) are such as in the proof of Theorem 2.10.

Let

\[ B_1 = \left( \frac{Q_{id}(A)}{\| [A(\alpha_{id}, \alpha_{id})]^{-1} \|^{-1} - 1 - \sum_{v=1}^{k} \| A(\alpha_{j_i}, \alpha_{i_v}) \| + \epsilon - G_T, \mu_f(A(\alpha) \right). \]

Since

\[ \frac{Q_{id}(A)}{\| [A(\alpha_{id}, \alpha_{id})]^{-1} \|^{-1} - 1 - \sum_{v=1}^{k} \| A(\alpha_{j_i}, \alpha_{i_v}) \| - G_T \mu_f(A(\alpha) \right), \]

similarly as in the proof of Theorem 2.10, we have \( B_1 \in H_{k+1} \). Note that \( B_1 = \mu(B_1) \in M_{k+1} \), then

\[ \text{det} B_1 > 0. \]  

Let \( \epsilon \to 0 \). Since \( \text{det} [\mu_f(A(\alpha))] > 0 \), by (12), we obtain (10). Similarly, we can prove (11). \( \square \)

**Theorem 2.13.** Let \( A \in II-BSDS, \alpha = \bigcup_{u=1}^{k} \alpha_{i_u} \subset N, \alpha = \bigcup_{u=1}^{l} \alpha_{j_u} \), \( k + l = s \) and \( A/\alpha = (\widetilde{A}(\alpha, \alpha)) \). Then

\[ 1 - \widetilde{P}_t(A/\alpha) \geq 1 - \widetilde{P}_{j_i}(A) + \sigma_{j_i} \geq 1 - \widetilde{P}_{j_i}(A) > 0, \]  

(13)

and

\[ 1 + \widetilde{P}_t(A/\alpha) \leq 1 + \widetilde{P}_{j_i}(A) - \sigma_{j_i} \leq 1 + \widetilde{P}_{j_i}(A), \]  

(14)
where \( 1 \leq w \leq k, \)

\[
\sigma_j = \sum_{w=1}^{k} \| [A(\alpha_{j_1}, \alpha_{j_2})^{-1} A(\alpha_{j_3}, \alpha_{j_4})] (1 - \overline{R}_{i_w}(A)) \|.
\]

\[
h = \max_{1 \leq w \leq k} \frac{\sum_{v=1}^{l} \| [A(\alpha_{i_w}, \alpha_{i_v})^{-1} A(\alpha_{i_w}, \alpha_{i_v})] \|}{1 - \sum_{v=1, v \neq w}^{k} \| [A(\alpha_{i_w}, \alpha_{i_v})^{-1} A(\alpha_{i_w}, \alpha_{i_v})] \|}.\]

\[
\overline{R}_{i_w}(A) = h \sum_{v=1, v \neq w}^{k} \| [A(\alpha_{i_w}, \alpha_{i_v})^{-1} A(\alpha_{i_w}, \alpha_{i_v})] \| + \sum_{v=1}^{l} \| [A(\alpha_{i_w}, \alpha_{i_v})^{-1} A(\alpha_{i_w}, \alpha_{i_v})] \|.
\]

**Proof.** For \( 1 \leq t, s \leq l, \) let

\[
D = \text{diag}(A(\alpha_{i_1}, \alpha_{i_2}), \ldots, A(\alpha_{i_k}, \alpha_{i_k})).
\]

\[
\Omega_{ts} = (A(\alpha_{i_1}, \alpha_{i_1}), \ldots, A(\alpha_{i_k}, \alpha_{i_k}))[A(\alpha)]^{-1} \left( \begin{array}{c} A(\alpha_{i_1}, \alpha_{i_2}) \\ \vdots \\ A(\alpha_{i_k}, \alpha_{i_k}) \end{array} \right).
\]

\[
\Gamma_t = (A(\alpha_{i_1}, \alpha_{i_1})^{-1} A(\alpha_{i_1}, \alpha_{i_1}), \ldots, A(\alpha_{i_k}, \alpha_{i_k})^{-1} A(\alpha_{i_k}, \alpha_{i_k})).
\]

\[
\Gamma_s = (A(\alpha_{i_1}, \alpha_{i_1})^{-1} A(\alpha_{i_1}, \alpha_{i_1}), \ldots, A(\alpha_{i_k}, \alpha_{i_k})^{-1} A(\alpha_{i_k}, \alpha_{i_k}))^T.
\]

\[
L_t = (||A(\alpha_{i_1}, \alpha_{i_1})^{-1} A(\alpha_{i_1}, \alpha_{i_1})||, \ldots, ||A(\alpha_{i_k}, \alpha_{i_k})^{-1} A(\alpha_{i_k}, \alpha_{i_k})||)^T.
\]

\[
H^T = (\sum_{s=1}^{k} ||A(\alpha_{i_1}, \alpha_{i_1})^{-1} A(\alpha_{i_1}, \alpha_{i_1})||, \ldots, \sum_{s=1}^{l} ||A(\alpha_{i_k}, \alpha_{i_k})^{-1} A(\alpha_{i_k}, \alpha_{i_k})||)^T.
\]

Let \( J_t = |\alpha_{j_t}| \) and \( I_t \) be an identity matrix. From Lemma 2.6, we have

\[
\| [A(\alpha_{j_t}, \alpha_{j_t})^{-1} \Omega_{tt}^{-1} < 1.
\]

And by Lemma 2.7, we know

\[
\| [I_t - [A(\alpha_{j_t}, \alpha_{j_t})^{-1} \Omega_{tt}]^{-1} \| \leq \frac{1}{1 - \||A(\alpha_{j_t}, \alpha_{j_t})^{-1} \Omega_{tt}]^{-1}} = \frac{1}{\Psi_t}.
\]

Thus, by Lemma 2.1, Lemma 2.3 and Lemma 2.5, for any \( \epsilon > 0 \) and \( 1 \leq t \leq l, \)

\[
1 - \tilde{P}_t(A/\alpha) = 1 - \sum_{s=1, s \neq t}^{l} \| [I_t - [A(\alpha_{j_t}, \alpha_{j_t})^{-1} \Omega_{tt}]^{-1}] [A(\alpha_{j_t}, \alpha_{j_t})^{-1} A(\alpha_{j_t}, \alpha_{j_t}) - [A(\alpha_{j_t}, \alpha_{j_t})^{-1} \Omega_{tt}] \|
\]

\[
\geq \frac{1}{\Psi_t} \left[ 1 - \sum_{s=1, s \neq t}^{l} \| [A(\alpha_{j_t}, \alpha_{j_t})^{-1} A(\alpha_{j_t}, \alpha_{j_t})] - \sum_{s=1}^{l} \| \Gamma_t \| N([A(\alpha)]^{-1} D) \| \| \Gamma_s \| \|
\right]
\]

\[
\geq 1 - \sum_{s=1, s \neq t}^{l} \| [A(\alpha_{j_t}, \alpha_{j_t})^{-1} A(\alpha_{j_t}, \alpha_{j_t})] - L_t^T [\mu_{I_1}(A)(\alpha)]^{-1} H^T
\]

\[
= 1 - \tilde{p}_{j_t} - \frac{1}{\mu_{I_1}(A)(\alpha)} \mu_{I_1}(A)(\alpha) \left( \sum_{w=1}^{k} \| [A(\alpha_{j_t}, \alpha_{j_t})^{-1} A(\alpha_{j_t}, \alpha_{j_t})] - \sigma_{j_t} + \epsilon - \sigma_{j_t} - \epsilon - L_t^T [\mu_{I_1}(A)(\alpha)]^{-1} H^T
\]

\[
= 1 - \tilde{p}_{j_t} - \sigma_{j_t} + \frac{1}{\det[\mu_{I_1}(A)(\alpha)]} \det \left( \sum_{w=1}^{k} \| [A(\alpha_{j_t}, \alpha_{j_t})^{-1} A(\alpha_{j_t}, \alpha_{j_t})] - \sigma_{j_t} + \epsilon - L_t^T [\mu_{I_1}(A)(\alpha)]^{-1} H^T
\]
Let
\[
B_2 = \begin{pmatrix}
x & -L_1^T \\
-H^T & \mu_{II}(A)\end{pmatrix}.
\]

If
\[
x > \sum_{i=1}^{k} \| [A(\alpha_j, \alpha_j)]^{-1} A(\alpha_j, \alpha_i) \| \tilde{R}_w(A),
\]
then there exists sufficiently small positive number \( \epsilon_0 \) such that
\[
x > \sum_{i=1}^{k} \| [A(\alpha_j, \alpha_j)]^{-1} A(\alpha_j, \alpha_i) \| (\tilde{R}_w(A) + \epsilon_0).
\]

Construct a positive diagonal matrix \( X = \text{diag}(x_1, x_2, \ldots, x_{k+1}) \), where
\[
x_t = \begin{cases}
1, & \text{if } t = 1, \\
\tilde{R}_{i-1}(A) + \epsilon_0, & \text{if } t = 2, 3, \ldots, k + 1.
\end{cases}
\]

Let \( \tilde{B} = B_2 X = (\tilde{b}_{pq}) \). For \( p = 1 \), by (15), we have
\[
| \tilde{b}_{1p} | - r_p(\tilde{B}) = | \tilde{b}_{11} | - \sum_{j=2}^{k+1} | \tilde{b}_{1j} | = x - \sum_{i=1}^{k} \| [A(\alpha_j, \alpha_j)]^{-1} A(\alpha_j, \alpha_i) \| (\tilde{R}_w(A) + \epsilon_0) > 0.
\]

And for \( p = 2, 3, \ldots, k + 1 \), from \( \tilde{R}_w(A) \leq h, 1 \leq w \leq k \), we obtain
\[
| \tilde{b}_{1p} | - r_p(\tilde{B}) = (\tilde{R}_{i-1}(A) + \epsilon_0) - \sum_{w=1, w \neq p-1} \| [A(\alpha_j, \alpha_j)]^{-1} A(\alpha_j, \alpha_i) \| (\tilde{R}_w(A) + \epsilon_0)
\]
\[
- \sum_{i=1, w \neq p-1} \| [A(\alpha_j, \alpha_j)]^{-1} A(\alpha_j, \alpha_i) \|
\]
\[
= h \sum_{i=1, w \neq p-1} \| [A(\alpha_j, \alpha_j)]^{-1} A(\alpha_j, \alpha_i) \|
\]
\[
- \sum_{i=1, w \neq p-1} \| [A(\alpha_j, \alpha_j)]^{-1} A(\alpha_j, \alpha_i) \| \tilde{R}_w(A)
\]
\[
+ \epsilon_0 \left( 1 - \sum_{i=1, w \neq p-1} \| [A(\alpha_j, \alpha_j)]^{-1} A(\alpha_j, \alpha_i) \| \right)
\]
\[
> 0.
\]

Thus, \( \tilde{B} \in SD_{k+1} \), and so \( B_2 \in H_{k+1} \). Note that \( B_2 = \mu(B_2) \in M_{k+1} \), then
\[
det B_2 > 0.
\]

Let \( x \) be \( \sum_{i=1}^{k} \| [A(\alpha_j, \alpha_j)]^{-1} A(\alpha_j, \alpha_i) \| - \sigma_j + \epsilon \) in \( B_2 \). Then
\[
\sum_{i=1}^{k} \| [A(\alpha_j, \alpha_j)]^{-1} A(\alpha_j, \alpha_i) \| - \sigma_j + \epsilon - \sum_{i=1}^{k} \| [A(\alpha_j, \alpha_j)]^{-1} A(\alpha_j, \alpha_i) \| \tilde{R}_w(A)
\]
\[
= \sum_{i=1}^{k} \| [A(\alpha_j, \alpha_j)]^{-1} A(\alpha_j, \alpha_i) \| - \sum_{i=1}^{k} \| [A(\alpha_j, \alpha_j)]^{-1} A(\alpha_j, \alpha_i) \| (1 - \tilde{R}_w(A))
\]
\[ + \epsilon - \sum_{w=1}^{k} \| [A(\alpha_j, \alpha_{j_2})]^{-1} A(\alpha_j, \alpha_{i_w}) \| \widetilde{R}_{i_w}(A) \]
\[ > 0. \]

By Lemma 2.2, we have \( \mu_{II}(A)(\alpha) \in M_k \), thus \( \det[\mu_{II}(A)(\alpha)] > 0 \). Further, by (16), we obtain
\[ 1 - \widetilde{P}_i(A/\alpha) > 1 - \widetilde{P}_{j_i}(A) + \sigma_{j_i} - \epsilon. \]

Let \( \epsilon \to 0 \). Then we obtain (13). Similarly, we can prove (14). \( \square \)

**Remark 2.14.** Note that
\[ \widetilde{R}_{i_w}(A) \leq \widetilde{P}_{i_w}(A). \quad 1 \leq w \leq k; \quad \alpha_j \leq \sigma_j; \quad 1 \leq t \leq l. \]

This shows that Theorem 2.13 improves Lemma 2.9.

**Theorem 2.15.** Let \( A \in II-BSDD_S \) with the index \( i_d \) such as in (5), \( \alpha = \bigcup_{w=1}^{k} \alpha_{i_w} \subset N, \bar{\alpha} = \bigcup_{v=1}^{l} \alpha_{j_i}, k + l = s \)
and \( A/\alpha = (\bar{\alpha}(\alpha, \alpha_j)) \). Then
\[ 1 - \widetilde{P}_i(A/\alpha) \geq 1 - \widetilde{P}_{j_i}(A) + (1 - \widetilde{Q}_{i_d}(A)) \sum_{v=1}^{k} \| [A(\alpha_j, \alpha_{j_2})]^{-1} A(\alpha_j, \alpha_{i_v}) \| \]
\[ \geq 1 - \widetilde{Q}_{i_d}(A) \widetilde{P}_{j_i}(A) > 0, \tag{17} \]
and
\[ 1 + \widetilde{P}_i(A/\alpha) \leq 1 + \widetilde{P}_{j_i}(A) - (1 - \widetilde{Q}_{i_d}(A)) \sum_{v=1}^{k} \| [A(\alpha_j, \alpha_{j_2})]^{-1} A(\alpha_j, \alpha_{i_v}) \| \]
\[ \leq 1 + \widetilde{Q}_{i_d}(A) \widetilde{P}_{j_i}(A), \tag{18} \]
where
\[ \xi = \max \left\{ \max_{w \neq i_d} \frac{\max_{v \in N/\{i_d, i_w\}} \| [A(\alpha_{i_w}, \alpha_{i_v})]^{-1} A(\alpha_{i_w}, \alpha_{i_v}) \| \| [A(\alpha_{i_w}, \alpha_{i_v})]^{-1} A(\alpha_{i_w}, \alpha_{i_v}) \|}{\widetilde{P}_{i_d}(A)}, \frac{1}{\widetilde{P}_{i_d}(A)} \right\}. \]

**Proof.** Similarly as in the proof of Theorem 2.13, for any \( \epsilon > 0 \), we obtain
\[ 1 - \widetilde{P}_i(A/\alpha) \geq 1 - \widetilde{P}_{j_i}(A) + (1 - \widetilde{Q}_{i_d}(A)) \sum_{v=1}^{k} \| [A(\alpha_j, \alpha_{j_2})]^{-1} A(\alpha_j, \alpha_{i_v}) \| - \epsilon \]
\[ + \frac{1}{\det[\mu_{II}(A)(\alpha)]} \det \begin{pmatrix} \widetilde{Q}_{i_d}(A) \sum_{w=1}^{k} \| [A(\alpha_j, \alpha_{j_2})]^{-1} A(\alpha_j, \alpha_{i_w}) \| + \epsilon & -L_i^T \\ -H^T & -H^T & \mu_{II}(A)(\alpha) \end{pmatrix}, \]
where \( L_i, H^T \) are such as in the proof of Theorem 2.13.

Let
\[ B_3 = \begin{pmatrix} \widetilde{Q}_{i_d}(A) \sum_{w=1}^{k} \| [A(\alpha_j, \alpha_{j_2})]^{-1} A(\alpha_j, \alpha_{i_w}) \| + \epsilon & -L_i^T \\ -H^T & \mu_{II}(A)(\alpha) \end{pmatrix}. \]

Since
\[ \widetilde{Q}_{i_d}(A) \sum_{w=1}^{k} \| [A(\alpha_j, \alpha_{j_2})]^{-1} A(\alpha_j, \alpha_{i_w}) \| \geq \max_{1 \leq u \leq k} \widetilde{R}_{i_w}(A) \sum_{w=1}^{k} \| [A(\alpha_j, \alpha_{j_2})]^{-1} A(\alpha_j, \alpha_{i_w}) \|, \]
Similarly as in the proof of Theorem 2.13, we can prove \( B_3 \in H_{k+1} \). Note that \( B_3 = \mu(B_3) \in M_{k+1} \), then
\[ \det B_3 > 0. \tag{19} \]

Let \( \epsilon \to 0 \). Since \( \det[\mu_{II}(A)(\alpha)] > 0 \), by (19), we obtain (17). Similarly, we can prove (18). \( \square \)
3 Eigenvalue inclusion regions for the Schur complement

In this section, based on the results in Section 2, we present new eigenvalue inclusion regions for the Schur complement of I(II)-BSD. In the following, we assume that \( \alpha = \bigcup_{i=1}^{k} \alpha_i \subseteq N, \overline{\alpha} = \bigcup_{i=1}^{l} \alpha_j, k + l = s \).

Let \( A/\alpha = (\overline{A}) \), \( |\alpha_i| = t \) and \( I_t \) be an identity matrix. \( \lambda(A/\alpha) \) and \( \lambda(A) \) denote the set of eigenvalues of \( A/\alpha \) and \( A \), respectively.

**Lemma 3.1 ([4]).** If \( A \in \mathbb{C}^{n \times n} \) is nonsingular, then

\[
\|A^{-1}\|^{-1} = \left\{ \sup_{x \neq 0} \frac{\|A^{-1}x\|}{\|x\|} \right\}^{-1} = \left\{ \sup_{y \neq 0} \frac{\|y\|}{\|Ay\|} \right\}^{-1} = \inf_{x \neq 0} \frac{\|Ax\|}{\|x\|}.
\]

**Lemma 3.2 ([20]).** Let \( A \in I-BSD \). Then

\[
\lambda(A) \subseteq G = \bigcup_{i=1}^{s} [G_i \cup \lambda(A(\alpha_i, \alpha_i))],
\]

where

\[
G_i = \left\{ \lambda \mid \lambda \notin \lambda(A(\alpha_i, \alpha_i)) \text{ and } \| [A(\alpha_i, \alpha_i) - \lambda I(\alpha_i)]^{-1} \|^{-1} \leq \sum_{k=1, k \neq i}^{s} \|A(\alpha_i, \alpha_k)\| \right\}.
\]

**Lemma 3.3 ([4]).** Let \( A \in I-BSD \). Then

\[
\lambda(A/\alpha) \subseteq G = \bigcup_{t=1}^{s} \left\{ G_t \cup \{ \lambda \mid \lambda \in \lambda(A(\alpha_j, \alpha_j)) \} \right\},
\]

where

\[
G_t = \left\{ \lambda \mid \lambda \notin \lambda(A(\alpha_j, \alpha_j)) \text{ and } \| [\lambda I - A(\alpha_j, \alpha_j)]^{-1} \|^{-1} \leq P_j(A) - \omega_j \right\}.
\]

**Lemma 3.4 ([4]).** Let \( A \in II-BSD \). Then

\[
\lambda(A/\alpha) \subseteq G = \bigcup_{t=1}^{s} \left\{ G_t \cup \{ \lambda \mid \lambda \in \lambda(A(\alpha_j, \alpha_j)) \} \right\},
\]

where

\[
G_t = \left\{ \lambda \mid \lambda \notin \lambda(A(\alpha_j, \alpha_j)) \text{ and } \| [\lambda I - A(\alpha_j, \alpha_j)]^{-1} \|^{-1} \leq \|A(\alpha_j, \alpha_j)\| (P_j(A) - \tilde{\omega}_j) \right\}.
\]

Now we present our main results, which are more accurate than those in Lemma 3.3 and Lemma 3.4.

**Theorem 3.5.** Let \( A \in I-BSD \). Then

\[
\lambda(A/\alpha) \subseteq \Gamma = \bigcup_{t=1}^{s} \{ \Gamma_t \cup \{ \lambda \mid \lambda \in \lambda(A(\alpha_j, \alpha_j)) \} \},
\]

where

\[
\Gamma_t = \left\{ \lambda \mid \lambda \notin \lambda(A(\alpha_j, \alpha_j)) \text{ and } \| [\lambda I - A(\alpha_j, \alpha_j)]^{-1} \|^{-1} \leq P_j(A) - \delta_j \right\},
\]

and \( \delta_j \) is such as in Theorem 2.10.
Proof. Let $\Omega_{ts}$ be such as in Theorem 2.10. If $\lambda \notin \lambda(A_1, A_2)$ and $\lambda \notin \lambda[A(\alpha_j, \alpha_j)]$, by Lemma 3.1, we have
\[
\| [\lambda I_t - A(\alpha_j, \alpha_j)]^{-1} \|^{-1} = \left\{ \sup_{x \in C^m} \left\{ \frac{\| [\lambda I_t - A(\alpha_j, \alpha_j)]^{-1} x \|}{\| x \|} \right\} \right\}^{-1}
\]
\[
= \inf_{x \in C^m} \left\{ \frac{\| [\lambda I_t - A(\alpha_j, \alpha_j)] x \|}{\| x \|} \right\}^{-1}
\]
\[
\geq \inf_{x \in C^m} \left\{ \frac{\| [\lambda I_t - A(\alpha_j, \alpha_j)] x \|}{\| x \|} - \sup_{x \in C^m} \| \Omega_{ts} x \| \right\}^{-1}
\]
\[
= \| [\lambda I_t - A(\alpha_j, \alpha_j)]^{-1} \|^{-1} - \| \Omega_{ts} \|.
\]

Thus, by Lemma 3.2, for any $\epsilon > 0$, we obtain
\[
\| [\lambda I_t - A(\alpha_j, \alpha_j)]^{-1} \|^{-1} \leq P_t(A/\alpha) + \| \Omega_{ts} \|
\]
\[
\leq \sum_{t=1, t \neq t}^l \| A(\alpha_j, \alpha_j) \| + \sum_{t=1}^l \| \Omega_{ts} \|
\]
\[
= P_j(A) - \delta_j + \frac{\det B}{\det \mu_I(A)},
\]
where $B$ and $\mu_I(A)$ are defined as in the proof of Theorem 2.10. Since $\det \mu_I(A) > 0$ and $\det B > 0$, then
\[
\| [\lambda I_t - A(\alpha_j, \alpha_j)]^{-1} \|^{-1} \leq P_j(A) - \delta_j + \epsilon.
\]

Let $\epsilon \to 0$. Then we obtain (20).

If $\lambda \in \lambda(A_1, A_2)$ and $\lambda \notin \lambda[A(\alpha_j, \alpha_j)]$, assume that $\tilde{x} \neq 0$ is the eigenvector of $\tilde{A}(\alpha_j, \alpha_j)$ corresponding to $\lambda$, by Lemma 3.1, we have
\[
0 = \| [\lambda I_t - \tilde{A}(\alpha_j, \alpha_j)] \tilde{x} \|
\]
\[
\geq \inf_{x \in C^m} \left\{ \frac{\| [\lambda I_t - A(\alpha_j, \alpha_j)] + \Omega_{ts} \|}{\| x \|} \right\}^{-1}
\]
\[
\geq \inf_{x \in C^m} \left\{ \frac{\| [\lambda I_t - A(\alpha_j, \alpha_j)] x \|}{\| x \|} - \sup_{x \in C^m} \| \Omega_{ts} x \| \right\}^{-1}
\]
\[
= \| [\lambda I_t - A(\alpha_j, \alpha_j)]^{-1} \|^{-1} - \| \Omega_{ts} \|.
\]

Further,
\[
\| [\lambda I_t - A(\alpha_j, \alpha_j)]^{-1} \|^{-1} \leq \| \Omega_{ts} \| \leq P_t(A) - \delta_j.
\]

Thus, we complete the proof of Theorem 3.5. \qed

Similarly as in the proof of Theorem 3.5, we can prove the following theorem according to Theorem 2.13.

**Theorem 3.6.** Let $A \in I1$-BSD$_S$. Then
\[
\lambda(A/\alpha) \subset \Gamma = \bigcup_{t=1}^s \{ \lambda \in \lambda[A(\alpha_j, \alpha_j)] \},
\]
where
\[
\Gamma_t = \{ \lambda \notin \lambda[A(\alpha_j, \alpha_j)] \} \cdot \| [\lambda I_j - A(\alpha_j, \alpha_j)]^{-1} \|^{-1} \leq \| A(\alpha_j, \alpha_j) \| \| \tilde{P}_j(A) - \sigma_j \|.
\]

and $\sigma_j$ is such as in Theorem 2.13.
Remark 3.7. By Remark 2.11 and Remark 2.14, we know that Theorem 3.5 and Theorem 3.6 improve Lemma 3.3 and Lemma 3.4, respectively.

4 A numerical example

In this section, we present a numerical example to illustrate the advantages of our derived results.

Example 4.1. Let

$$A = \begin{bmatrix} 6 & 3 & 0.01 & -0.03 & 0.02 & 0.04 & 0.3 & 0.9 & 0.6 & 0.3 \\ 4 & 5 & 0.02 & 0.02 & 0.05 & 0.01 & 0.2 & -0.4 & 0.9 & 0.1 \\ -0.02 & 0.01 & 6 & 3 & 0.03 & 0.02 & 0.6 & 0.5 & 0.45 & -0.3 \\ 0.04 & -0.01 & 0.01 & 0.04 & 6 & 3 & 0.4 & 0.3 & 0.74 & 0.2 \\ 0.01 & 0.03 & 0.02 & 0.03 & 4 & 5 & 0.2 & 0.1 & 0.3 & 0.6 \\ 0.01 & 0.03 & 0.04 & 0.02 & 6 & 3 & -0.5 & 0.8 \\ 0.02 & 0.02 & 0.05 & 0.01 & 6 & 3 & 0.3 & 0.6 \\ 0.04 & -0.05 & 0.01 & 0.03 & 0.05 & 0.02 & 0.3 & 1 & 6 & 3 \\ 0.06 & 0.03 & 0.02 & -0.02 & 0.03 & 0.04 & 0.4 & 0.3 & 4 & 5 \end{bmatrix}.$$

In the following, we choose \( \alpha = \{4, 5\} \), \( \| \cdot \| = \| \cdot \|_\infty \), \( \alpha = \bigcup_{w=1}^2 \alpha_{i_w}, \overline{\alpha} = \bigcup_{v=1}^3 \alpha_{j_v}, i_w = w + 3 (w = 1, 2), j_v = v (v = 1, 2, 3) \) and \( A/\alpha = (\bar{A}(\alpha, \alpha_v)) \).

By calculation with Matlab 7.1, we have \( A \in I - BSD_s \). From Theorem 3.5, the eigenvalue inclusion set of \( A/\beta \) is

$$\Gamma = \{z|z^2 - 11z + 18| \leq 0.9800(|z - 5| + 3)\} \cup \{z|z^2 - 11z + 18| \leq 0.9800(|z - 6| + 4)\}.$$

By Theorem 4.1 of [4], the eigenvalue inclusion set of \( A/\beta \) is

$$\Gamma' = \{z|z^2 - 11z + 18| \leq 1.9333(|z - 5| + 3)\} \cup \{z|z^2 - 11z + 18| \leq 1.9333(|z - 6| + 4)\}.$$

The sets \( \Gamma \) and \( \Gamma' \) are shown in Fig. 1. It is apparent that \( \Gamma \subset \Gamma' \) from both (22), (23) and Fig. 1. And the eigenvalues of \( A/\alpha \) are denoted by ‘+’ in Fig. 1.

**Fig. 1.** The red dotted line and blue dashed line denote the corresponding discs \( \Gamma \) and \( \Gamma' \), respectively.

In Fig. 1, notice that the sets \( \Gamma \) and \( \Gamma' \) consist of two disjoint components, respectively, where the red circular-like sets in the right consist of two different circular-like sets.
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