Some extensions of a certain integral transform to a quotient space of generalized functions

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Abstract: In this paper, we establish certain spaces of generalized functions for a class of \( \mathcal{E}^{s}_{2,1} \) transforms. We give the definition and derive certain properties of the extended \( \mathcal{E}^{s}_{2,1} \) transform in a context of Boehmian spaces. The extended \( \mathcal{E}^{s}_{2,1} \) transform is therefore well defined, linear and consistent with the classical \( \mathcal{E}^{s}_{2,1} \) transforms. Certain results are also established in some detail.

Keywords: \( \mathcal{E}^{s}_{2,1} \) transform, Generalized function, Lebesgue space, Boehmian space

MSC: 44A15, 44A35

1 Introduction

As some physical situations are governed by differential equations whose boundary conditions are not smooth enough but are generalized functions, it is of great importance to extend the classical integral transforms to generalized functions. In pure mathematics, the concept of generalized integral transforms were invoked to study distributions and Boehmian spaces, which are found fruitful in obtaining solutions of differential and integral equations. In [9] various types of ordinary differential equations were given solutions in a generalized sense. In [24] authors introduce the solution of the Volterra and Abel integral equations by the distributional wavelet transform. In [23] an ordinary differential equation of \( n \)th order was treated by a Laplace transform method of right-side distributions. Indeed, if the differential equation \( \dot{u} = H \) where \( H \) is the heaviside step function, then the classical solution cannot satisfy the original differential equation at this point. But, on the other hand, if \( \mathcal{F}(\varphi; y) \) denotes the space of rapidly decreasing functions and \( \mathcal{F}(\varphi; y)^{\prime} \) denotes the strong dual of \( \mathcal{F}(\varphi; y) \) of distributions of slow growth, then for an arbitrary \( \varphi \in \mathcal{F}(\varphi; y) \) we have

\[
\dot{u}(\varphi) = -u(\dot{\varphi}) = -\int_{-\infty}^{\infty} u(x) \dot{\varphi}(x) \, dx = -\int_{0}^{\infty} a\varphi(x) \, dx - \int_{0}^{\infty} (x + a) \dot{\varphi}(x) \, dx
\]

\[
= -a\varphi(0) + a\varphi(0) - \int_{0}^{\infty} x\varphi(x) \, dx = \int_{0}^{\infty} \varphi(x) \, dx = \int_{-\infty}^{\infty} H(x) \, \varphi(x) \, dx = H(\varphi)
\]

where \( a \) is some suitable constant.

However, we deem it proper to recall in this article some of integral transforms that are represented in the space of Boehmians, but not all, such as: the Fourier transform [14]; the Radon transform [6, 7]; the Hilbert transform [11];
the Hartley-Hilbert and Fourier-Hilbert transforms [3]; the diffraction Fresnel transform [5]; the optical Fresnel wavelet transform [2]; the Hartley transform [1] to mention but a few.

In the sequence of these integrals, the classical theory of the transform \( s^{2;1} \) has recently been investigated by David Brawn, et al. in the survey article [8]. However, the generalized theory of this integral has not been explored yet. In this part of research, we discuss the definition and derive several properties of this integral in a class of Boehmians. The novelty of this extension fulfills when certain spaces of Boehmians are constructed to the given generalized definition. The classical \( s^{2;1} \) transform of an ordinary function \( \phi (x) \) is defined by [8]

\[
\mathcal{s}^{2;1} (\phi; y) = \int_{0}^{\infty} x \exp \left( x^2 y^2 \right) \mathcal{e}_1 \left( x^2 y^2 \right) \phi (x) \, dx.
\]  

(1)

where \( \mathcal{e}_1 (x) \) is an exponential integral function given by

\[
\mathcal{e}_1 (x) = \int_{1}^{\infty} e^{-xt} \, dt.
\]  

(2)

The transform \( \mathcal{s}^{2;1} \) has a close connection with some other related transforms, given as:

\[
\mathcal{s}^{2;1} (\phi; y) = 2p \, (I_2 (\phi; u); y) = 2I_2 (p (\phi; u); y),
\]

where \( I_2 \) and \( p \) are, respectively, the \( I_2 \) and Widder potential transforms.

The related Parseval-Goldstein type relations are given in terms of \( I_2 \) and \( p \) transforms as

\[
\int_{0}^{\infty} yI_2 (\phi; u) \ p (g; y) \, dy = \frac{1}{2} \int_{0}^{\infty} x\phi (x) \mathcal{s}^{2;1} (g; x) \, dx
\]

(3)

and

\[
\int_{0}^{\infty} yI_2 (\phi; u) \ p (g; y) \, dy = \frac{1}{2} \int_{0}^{\infty} u\phi (u) \mathcal{s}^{2;1} (\phi; u) \, du.
\]

(4)

Whereas, due to [8, Example 3, 4], we have

\[
\mathcal{s}^{2;1} (\sin (ax); y) = \frac{\pi}{2y^2} \left( 1 - \frac{\pi^2 a}{2y} \exp \left( \frac{a^2}{4y^2} \right) \text{Erfc} \left( \frac{a}{4y} \right) \right)
\]

and

\[
\mathcal{s}^{2;1} \left( \cos \left( \frac{ax}{x} \right); y \right) = \frac{\pi}{2y} \exp \text{Erfc} \left( \frac{a}{4y} \right).
\]

Properties of the \( \mathcal{s}^{2;1} \) transform and its relation with well known integrals are given in details in the above citation.

We divide this paper into three sections. In Section 2, we describe notations, definitions, and properties related to the abstract construction of Boehmian spaces and further we generate the desired spaces of Boehmians by certain convolution products. In Section 3, we give the definition and investigate the extended transform on the constructed spaces of generalized functions.

## 2 Construction of Boehmian spaces

Study of regular operators resulted into the theory of Boehmians, which is a generalization of Schwartz theory of distributions. Regular operators form a subalgebra of Mikusiński operators and they include only those functions whose support is bounded from the left and, at the same time, do not have restrictions on the support.

The minimal structure necessary for the construction of Boehmian spaces consists of the following axioms:

\( X (i) \): A nonempty set \( a \):

\( X (ii) \): A commutative semigroup \((b, \bullet)\):

\( X (iii) \): An operation \( \star : a \times b \to a \) such that for each \( x \in a \) and \( s_1, s_2, \in b \),

\[
x \star (s_1 \bullet s_2) = (x \star s_1) \bullet s_2;
\]
X (iv) : A collection \( \Delta \subset b \) such that

(a) If \( x, y \in a, (s_n) \in \Delta, x \ast s_n = y \ast s_n \) for all \( n \), then \( x = y \);
(b) If \( (s_n), (t_n) \in \Delta \), then \( (s_n \ast t_n) \in \Delta \).

Elements of \( \Delta \) are called delta sequences. Consider

\[ Q = \{ (x_n, s_n) : x_n \in a, (s_n) \in \Delta, x_n \ast s_m = x_m \ast s_n, \forall m, n \in \mathbb{N} \} . \]

If \( (x_n, s_n), (y_n, t_n) \in Q \), \( x_n \ast t_m = y_m \ast s_n, \forall m, n \in \mathbb{N} \), then we say \( (x_n, s_n) \sim (y_n, t_n) \). The relation \( \sim \) is an equivalence relation in \( Q \). The space of equivalence classes in \( Q \) is denoted by \( b \). Elements of \( b \) are called Boehmians.

Between \( a \) and \( b \) there is a canonical embedding expressed as

\[ x \rightarrow \frac{x \ast s_n}{s_n} . \]

The operation \( \ast \) can be extended to \( b \times a \) by

\[ \frac{x_n}{s_n} \ast t = \frac{x_n \ast t}{s_n} . \]

The relationship between the notion of convergence and the product \( \ast \) is given as:

(i) If \( f_n \rightarrow f \) as \( n \rightarrow \infty \) in \( a \) and, \( \phi \in b \) is any fixed element, then

\[ f_n \ast \phi \rightarrow f \ast \phi \text{ in } a \text{ (as } n \rightarrow \infty) . \]

(ii) If \( f_n \rightarrow f \) as \( n \rightarrow \infty \) in \( a \) and \( (\delta_n) \in \Delta \), then

\[ f_n \ast \delta_n \rightarrow f \text{ in } a \text{ (as } n \rightarrow \infty) . \]

The operation \( \ast \) is extended to \( b \times b \) as follows:

\[ \text{If } \left[ \frac{f_n}{s_n} \right] \in b \text{ and } \phi \in b, \text{ then } \left[ \frac{f_n \ast \phi}{s_n} \right] . \]

Convergence in \( b \) is defined as follows:

A sequence \( (h_n) \) in \( b \) is said to be \( \delta \) convergent to \( h \) in \( b \), \( h_n \delta \rightarrow h \), if there is a sequence \( (s_n) \in \Delta \) such that

\[ (h_n \ast s_n), (h \ast s_n) \in a, \forall k, n \in \mathbb{N}, \text{ and } (h_n \ast s_k) \rightarrow (h \ast s_k) \text{ as } n \rightarrow \infty \text{ in } a, \text{ for every } k \in \mathbb{N}. \]

A sequence \( (h_n) \) in \( b \) is said to be \( \Delta \) convergent to \( h \) in \( b \), \( h_n \Delta \rightarrow h \), if there is a sequence \( (s_n) \in \Delta \) such that

\[ (h_n - h) \ast s_n \in a, \forall n \in \mathbb{N}, \text{ and } (h_n - h) \ast s_n \rightarrow 0 \text{ as } n \rightarrow \infty \text{ in } a. \]

For a somehow much more detailed account of Boehmian spaces we refer to [1–7, 9–22].

Following are the necessary products we demand for our next investigation.

**Definition 2.1.** Denote by \( L^1(\mathbb{R}_+) \) the space of all Lebesgue integrable functions defined on \( \mathbb{R}_+ \). Let \( \phi \) and \( \psi \) be in \( L^1(\mathbb{R}_+) \). \( \mathbb{R}_+ = (0, \infty) \). Then, we have the following:

(i) The Mellin type convolution product of first kind is defined by [9]

\[ (\phi \ast \psi)(y) = \int_0^\infty t^{-1} \phi(yt^{-1}) \psi(t) dt, \quad (5) \]

when the integral exists.

(ii) The product \( \otimes \) between \( \phi \) and \( \psi \) is defined as

\[ (\phi \otimes \psi)(y) = \int_0^\infty z\phi(zy) \psi(z) dz, \quad (6) \]

provided the integral exists.
Denote by \(L^u_{bc}(\mathbb{R}_+^+)\) the space of all locally integrable functions whose support is contained in some compact subset of \(\mathbb{R}_+\). Hence, we say \(\phi \in L^u_{bc}(\mathbb{R}_+^+)\) if \(\phi(x) = 0\) outside some open interval and that integral

\[
\|\phi\|_K = \int_K |\phi(x)| \, dx
\]  

is finite for every compact subset \(K\) of \(\mathbb{R}_+^+\).

By \(C^{\infty}_c(\mathbb{R}_+^+)\), or \(C^{\infty}_C\), we denote the standard notation of the Schwartz’ space of test functions of compact supports defined on \(\mathbb{R}_+^+\).

Some general properties of the integral product \(\times\) that are worthwhile to be described here are:

(i) \(\psi_1 \times \psi_2 = \psi_2 \times \psi_1\);
(ii) \((\psi_1 \times \psi_2) \times \psi_3 = \psi_1 \times (\psi_2 \times \psi_3)\);
(iii) \((\alpha \psi_1) \times \psi_2 = \alpha(\psi_1 \times \psi_2)\);
(iv) \(\psi_1 \times (\psi_2 + \psi_3) = \psi_1 \times \psi_2 + \psi_1 \times \psi_3\).

**Definition 2.2.** By \(\Delta\) we denote the set of delta sequences \((\delta_n)\) satisfying the properties \(\Delta_1 - \Delta_4\) (see [3])

\[
\begin{align*}
\Delta_1 : & \int_0^\infty \delta_n(x) \, dx = 1; \\
\Delta_2 : & \int_0^\infty |\delta_n(x)| \, dx < M, M \in \mathbb{R}, M > 0; \\
\Delta_3 : & \sup \delta_n(x) \subseteq [a_n, b_n], a_n, b_n \to 0 \text{ as } n \to \infty; \\
\Delta_4 : & (\delta_n(x)) \in C^{\infty}_c.
\end{align*}
\]

To fulfill all requirements of the following extension, we generate the Boehmian spaces \(b(L^u_{bc}, C^{\infty}_C, \Delta, \times, \circ)\) and \(b(L^u_{loc}, C^{\infty}_C, \Delta, \times, \circ)\), respectively. Indeed, to get our extension accomplished, it is sufficient to generate the space \(b(L^u_{bc}, C^{\infty}_C, \Delta, \times, \circ)\) since the space \(b(L^u_{bc}, C^{\infty}_C, \Delta, \times, \circ)\) can be generated by a similar technique.

**Theorem 2.3.** Let \(\phi\) and \(\psi\) be integrable functions defined on \(\mathbb{R}_+^+\). Then, we have

\[
\epsilon_{2,1}^x(\phi \times \psi)(y) = (\epsilon_{2,1}^x \phi \otimes \psi)(y).
\]  

**Proof.** Let the hypothesis of the theorem be satisfied for some integrable functions \(\phi\) and \(\psi\). Then, by aid of the integral equation (1) the lefthand side of equation (8) gives

\[
\epsilon_{2,1}^x(\phi \times \psi)(y) = \int_0^\infty xe^{x^2y^2} \epsilon_1 \left(x^2y^2\right) (\phi \times \psi)(x) \, dx.
\]

Therefore, after inserting equation (5), the preceding equation yields

\[
\epsilon_{2,1}^x(\phi \times \psi)(y) = \int_0^\infty xe^{x^2y^2} \epsilon_1 \left(x^2y^2\right) \int_0^z \phi(xz^{-1}) \, dx \int_0^\infty \psi(z) z^{-1} \, dz.
\]

By setting variables, we derive

\[
\epsilon_{2,1}^x(\phi \times \psi)(y) = \int_0^\infty we^{(z^2y^2)w^2} \epsilon_1 \left((z^2y^2)w^2\right) \phi(w) \int_0^\infty \psi(z) \, dz \, dw.
\]

On account of Fubini’s theorem, we can rearrange the above integrals as

\[
\epsilon_{2,1}^x(\phi \times \psi)(y) = \int_0^\infty \int_0^\infty we^{(zy)^2w^2} \epsilon_1 \left((zy)^2w^2\right) \phi(w) \psi(z) \, dz \, dw.
\]

Therefore, in view of equation (1), we put equation (9) into the form

\[
\epsilon_{2,1}^x(\phi \times \psi)(y) = \int_0^\infty z \left(\epsilon_{2,1}^x \phi\right)(zy) \psi(z) \, dz.
\]

Hence, the proof of the theorem is completed.
Theorem 2.4. Let \( \phi \in I^u_{loc}(\mathbb{R}^+) \) and \( \varphi, \psi \in C^\infty_c \). Then, we have
\[
(\phi \otimes (\varphi \times \psi))(y) = ((\phi \otimes \varphi) \otimes \psi)(y),
\]
for all \( y \in \mathbb{R}^+ \).

Proof. Let \( \phi \in I^u_{loc}(\mathbb{R}^+) \) and \( \varphi, \psi \in C^\infty_c \) be arbitrary. Then, in view of equation (5) and equation (6), the Fubini’s theorem yields
\[
(\phi \otimes (\varphi \times \psi))(y) = \int_0^\infty Z(\mathbb{R}^+) \int_0^\mathbb{R}^+ \psi(t)dt \, dz.
\]
Setting variables in equation (11) implies
\[
(\phi \otimes (\varphi \times \psi))(y) = \int_0^\infty Z(\mathbb{R}^+) \int_0^\mathbb{R}^+ \psi(t)dt \, dz.
\]
Therefore, by aid of Equation (6), equation (12) can be written as
\[
(\phi \otimes (\varphi \times \psi))(y) = \int_0^\infty Z(\mathbb{R}^+) \int_0^\mathbb{R}^+ \psi(t)dt \, dz.
\]
The proof of the theorem is therefore completed.

Now we establish the following result.

Theorem 2.5. Let \( \phi \in I^u_{loc}(\mathbb{R}^+) \). Then, we have \( \mathbf{e}^{2,1}_2 \phi \in I^u_{loc}(\mathbb{R}^+) \).

Proof. Let \( \phi \in I^u_{loc}(\mathbb{R}^+) \) and \( \varphi, \psi \in C^\infty_c \). Then, we have \( \mathbf{e}^{2,1}_2 \phi \in I^u_{loc}(\mathbb{R}^+) \).

Theorem 2.6. Let \( \phi \in I^u_{loc}(\mathbb{R}^+) \) and \( \varphi, \psi \in C^\infty_c \). Then, we have \( \mathbf{e}^{2,1}_2 \phi \in I^u_{loc}(\mathbb{R}^+) \).

Proof. Let \( \phi \in I^u_{loc}(\mathbb{R}^+) \) and \( \varphi, \psi \in C^\infty_c \). Then, we have \( \mathbf{e}^{2,1}_2 \phi \in I^u_{loc}(\mathbb{R}^+) \).
Theorem 2.7. Let $\phi_1, \phi_2 \in L^u_{\text{loc}}(\mathbb{R}_+)$ and $\phi_1, \phi_2 \in C^\infty_c$. Then, we have:

(i) $\alpha (\phi_1 \otimes \phi_2) (y) = (\alpha \phi_1 \otimes \phi_2) (y)$, $\alpha \in \mathbb{C}$.

(ii) $((\phi_1 + \phi_2) \otimes \varphi_1) (y) = (\phi_1 \otimes \varphi_1) (y) + (\phi_2 \otimes \varphi_1) (y)$.

(iii) Let $\phi_n \to \phi$ in $L^u_{\text{loc}}(\mathbb{R}_+)$ and $\varphi \in C^\infty_c$. Then, we have

$$
\phi_n \otimes \varphi \to \phi \otimes \varphi \quad \text{as} \quad n \to \infty.
$$

(iv) $(\phi_1 \otimes (\varphi_1 \times \varphi_2)) (y) = ((\phi_1 \otimes \varphi_1) \otimes \varphi_2) (y)$.

Proof. Proof of the equations (i), (ii) and (iii) follows from simple integral calculus. Proof of the identity (iv) is included in Theorem 2.4. Hence the theorem is fully proved.

Theorem 2.8. Let $\phi \in L^u_{\text{loc}}(\mathbb{R}_+)$ and $(\delta_n) \in \Delta$. Then, we have

$$(\phi \otimes \delta_n) (y) \to \phi (y),$$

as $n \to \infty$.

Proof. Let the hypothesis of the theorem be satisfied for some $\phi \in L^u_{\text{loc}}(\mathbb{R}_+)$ and $(\delta_n) \in \Delta$. Then, for a compact subset $K$ of $\mathbb{R}_+$ and by using the property $\Delta_1$ of $\Delta$ sequences, we write

$$
\| \phi \otimes \delta_n - \phi \|_K = \int_K \left| \int_0^\infty z \phi (zy) \delta_n (z) - \phi (y) \delta_n (z) \right| dy dz
$$

$$
\leq \int_K \int_0^\infty |z \phi (zy) - \phi (y)| |\delta_n (z)| dy dz
$$

$$
\leq \int_K \int_0^\infty (|z \phi (zy)| + |\phi (y)|) |\delta_n (z)| dz dy.
$$

Moreover, by virtue of $\Delta_3$ of $\Delta$ sequences, we assume $\text{supp} \delta_n \subseteq [a_n, b_n]$, where $a_n, b_n \to 0$ as $n \to \infty$. Therefore, appealing to equation (16) gives

$$
\| \phi \otimes \delta_n - \phi \|_K \leq \int_K \int_{a_n}^{b_n} (|z \phi (zy)| + |\phi (y)|) |\delta_n (z)| dz dy.
$$

Simple computation together with the fact that $a_n, b_n \to 0$ as $n \to \infty$ imply

$$
\| \phi \otimes \delta_n - \phi \|_K \to 0
$$

as $n \to \infty$. The theorem is therefore completely proved.

Therefore, the space $b (L^u_{\text{loc}}, C^\infty_c, \Delta, \times, \otimes)$ can be considered as a Boehmian space.

The sum and multiplication by a scalar in $b (L^u_{\text{loc}}, C^\infty_c, \Delta, \times, \otimes)$ are respectively defined in the natural way as

$$
\left[ \begin{array}{c} \phi_n \\ \varphi_n \end{array} \right] + \left[ \begin{array}{c} g_n \\ \varphi_n \end{array} \right] = \left[ \begin{array}{c} \phi_n \otimes \varphi_n \\ g_n \otimes \varphi_n \end{array} \right]
$$

and

$$
\rho \left[ \begin{array}{c} \phi_n \\ \delta_n \end{array} \right] = \left[ \begin{array}{c} \rho \phi_n \\ \delta_n \end{array} \right].
$$
where \( \rho \) is a complex number.

Between \( l^u_{\text{loc}} \) and \( b \left( l^u_{\text{loc}}, C^\infty_c, \Delta, \times, \otimes \right) \) there is a canonical embedding expressed as

\[
\mathbf{x} \rightarrow \frac{x \otimes \delta_n}{\delta_n}
\]

as \( n \rightarrow \infty \).

The operation \( \otimes \) can be extended to \( b \left( l^u_{\text{loc}}, C^\infty_c, \Delta, \times, \otimes \right) \times l^u_{\text{loc}} \) by the equation

\[
\frac{x_n \otimes l}{\delta_n} = \frac{x_n \otimes l}{\delta_n}.
\]

Convergence in \( b \left( l^u_{\text{loc}}, C^\infty_c, \Delta, \times, \otimes \right) \) is defined as follows:

A sequence \( (\beta_n) \in b \left( l^u_{\text{loc}}, C^\infty_c, \Delta, \times, \otimes \right) \) is said to be \( \delta \) convergent to \( \beta \in b \left( l^u_{\text{loc}}, C^\infty_c, \Delta, \times, \otimes \right) \) if there can be a delta sequence \( (\delta_n) \) such that \( (\beta_n \otimes \delta_n), (\beta \otimes \delta_n) \in l^u_{\text{loc}}, \forall k, n \in \mathbb{N} \), and \( (\beta_n \otimes \delta_n) \rightarrow (\beta \otimes \delta_n) \) as \( n \rightarrow \infty \), in \( l^u_{\text{loc}} \) for every \( k \in \mathbb{N} \).

A sequence \( (\beta_n) \in b \left( l^u_{\text{loc}}, C^\infty_c, \Delta, \times, \otimes \right) \) is said to be \( \Delta \) convergent to \( \beta \in b \left( l^u_{\text{loc}}, C^\infty_c, \Delta, \times, \otimes \right) \) if there can be a sequence \( (\delta_n) \in \Delta \) such that \( (\beta_n - \beta) \otimes \delta_n \in l^u_{\text{loc}}, \forall n \in \mathbb{N} \), and \( (\beta_n - \beta) \otimes \delta_n \rightarrow 0 \) as \( n \rightarrow \infty \) in \( l^u_{\text{loc}} \).

Similarly, the space \( b \left( l^u_{\text{loc}}, C^\infty_c, \Delta, \times, \otimes \right) \) can be established by aid of the properties of the product \( \times \) and following analysis similar to that of \( b \left( l^u_{\text{loc}}, C^\infty_c, \Delta, \times, \otimes \right) \).

Addition, scalar multiplications and \( \times \) in \( b \left( l^u_{\text{loc}}, C^\infty_c, \Delta, \times, \otimes \right) \) are, respectively, defined as:

\[
\left[ \begin{array}{c}
(\phi_n) \\
(\delta_n)
\end{array} \right] + \left[ \begin{array}{c}
(g_n) \\
(\psi_n)
\end{array} \right] = \left[ \begin{array}{c}
(\phi_n) \times (\psi_n) + (g_n) \times (\delta_n) \\
(\delta_n) \times (\psi_n)
\end{array} \right]
\]

and

\[
\rho \left[ \begin{array}{c}
(\phi_n) \\
(\delta_n)
\end{array} \right] = \left[ \begin{array}{c}
(\rho \phi_n) \\
(\delta_n)
\end{array} \right],
\]

\[
\frac{x_n \times l}{\delta_n} = \frac{x_n \times l}{\delta_n}.
\]

\( \rho \) is a complex number.

A sequence \( (\beta_n) \in b \left( l^u_{\text{loc}}, C^\infty_c, \Delta, \times, \otimes \right) \) is said to be \( \delta \) convergent to \( \beta \in b \left( l^u_{\text{loc}}, C^\infty_c, \Delta, \times, \otimes \right) \) if there can be a delta sequence \( (\delta_n) \) such that \( (\beta_n \times \delta_n), (\beta \otimes \delta_n) \in l^u_{\text{loc}}, \forall k, n \in \mathbb{N} \), and \( (\beta_n \times \delta_n) \rightarrow (\beta \otimes \delta_n) \) as \( n \rightarrow \infty \), in \( l^u_{\text{loc}} \) for every \( k \in \mathbb{N} \).

A sequence \( (\beta_n) \in b \left( l^u_{\text{loc}}, C^\infty_c, \Delta, \times, \otimes \right) \) is said to be \( \Delta \) convergent to \( \beta \in b \left( l^u_{\text{loc}}, C^\infty_c, \Delta, \times, \otimes \right) \) if there can be a \( (\delta_n) \in \Delta \) such that \( (\beta_n - \beta) \times \delta_n \in l^u_{\text{loc}}, \forall n \in \mathbb{N} \), and \( (\beta_n - \beta) \times \delta_n \rightarrow 0 \) as \( n \rightarrow \infty \) in \( l^u_{\text{loc}} \).

### 3 The extended \( e_{2,1}^x \) transform of \( b \left( l^u_{\text{loc}}, C^\infty_c, \Delta, \times, \otimes \right) \)

We devote this section to give the definition and the properties of the extended \( e_{2,1}^x \) transform in the generalized spaces. We further obtain some general results.

In view of our previous results, we introduce the following definition.

**Definition 3.1.** Let \( \left[ \begin{array}{c}
(\phi_n) \\
(\delta_n)
\end{array} \right] \in b \left( l^u_{\text{loc}}, C^\infty_c, \Delta, \times, \otimes \right) \). Then, we define the extension of \( e^x_{2,1} \) as

\[
e_{2,1}^x \left[ \begin{array}{c}
(\phi_n) \\
(\delta_n)
\end{array} \right] = \left[ \begin{array}{c}
e_{2,1}^x (\phi_n) \\
(\delta_n)
\end{array} \right], \quad (17)
\]

in the space \( b \left( l^u_{\text{loc}}, C^\infty_c, \Delta, \times, \otimes \right) \).

Definition 3.1 is indeed well-defined by Theorem 2.3 and Theorem 2.5. We prefer to omit the details.

We recite some properties of the transform \( e_{2,1}^x \) in the course of the following theorems.
Theorem 3.2. Let \( \beta_1, \beta_2 \in b (l^u_{\text{loc}}, C_c^\infty, \Delta, \times, \times) \). Then, we have
\[
\varepsilon_{2,1}^\varepsilon (\beta_1 \times \beta_2) = \varepsilon_{2,1}^\varepsilon \beta_1 \otimes \beta_2.
\]

Proof. Under the assumption that the requirements of the theorem are satisfied for some \( \beta_1, \beta_2 \in b (l^u_{\text{loc}}, C_c^\infty, \Delta, \times, \times) \), we find \( (\phi_n), (\kappa_n) \in l^u_{\text{loc}} (\mathbb{R}^+) \) and \( (\phi_n), (\delta_n) \in \Delta \) such that \( \beta_1 = \frac{(\phi_n)}{(\kappa_n)} \) and \( \beta_2 = \frac{(\kappa_n)}{(\delta_n)} \). Therefore, on account of Theorem 2.3, we write
\[
\varepsilon_{2,1}^\varepsilon (\beta_1 \times \beta_2) = \varepsilon_{2,1}^\varepsilon \left( \frac{(\phi_n) \times (\kappa_n)}{(\kappa_n) \times (\delta_n)} \right) = \left[ \frac{\varepsilon_{2,1}^\varepsilon ((\phi_n) \times (\kappa_n))}{(\kappa_n) \times (\delta_n)} \right] = \left[ \frac{\varepsilon_{2,1}^\varepsilon (\phi_n) \otimes (\kappa_n)}{(\kappa_n) \otimes (\delta_n)} \right] = \left[ \frac{(\phi_n)}{(\delta_n)} \right] = \varepsilon_{2,1}^\varepsilon (\beta_1) \otimes \beta_2.
\]
Thus, we have obtained
\[
\varepsilon_{2,1}^\varepsilon (\beta_1 \times \beta_2) = \varepsilon_{2,1}^\varepsilon (\beta_1) \otimes \beta_2.
\]
This completes the proof of the theorem. \( \square \)

Theorem 3.3. The mapping \( \varepsilon_{2,1}^\varepsilon : b (l^u_{\text{loc}}, C_c^\infty, \Delta, \times, \times) \) into \( b (l^u_{\text{loc}}, C_c^\infty, \Delta, \times, \otimes) \) is linear.

Proof is straightforward from definitions. Hence, we omit the details.

Theorem 3.4. Let \( \frac{(\phi_n)}{(\delta_n)} \in b (l^u_{\text{loc}}, C_c^\infty, \Delta, \times, \times) \) and \( \delta \in C_c^\infty \). Then, we have
\[
\varepsilon_{2,1}^\varepsilon \left( \frac{(\phi_n)}{(\delta_n)} \right) \times \delta = \left[ \frac{\varepsilon_{2,1}^\varepsilon (\phi_n)}{(\delta_n)} \right] \otimes \delta.
\]

Proof. By applying (17) for some \( \frac{(\phi_n)}{(\delta_n)} \in b (l^u_{\text{loc}}, C_c^\infty, \Delta, \times, \otimes) \) and \( \delta \in C_c^\infty \), we write
\[
\varepsilon_{2,1}^\varepsilon \left( \frac{(\phi_n)}{(\delta_n)} \right) \times \delta = \left[ \frac{\varepsilon_{2,1}^\varepsilon ((\phi_n) \times \delta)}{(\delta_n)} \right].
\]
By employing Theorem 2.3 we get
\[
\varepsilon_{2,1}^\varepsilon \left( \frac{(\phi_n)}{(\delta_n)} \right) \times \delta = \left[ \frac{\varepsilon_{2,1}^\varepsilon (\phi_n)}{(\delta_n)} \right] \otimes \delta.
\]
This completes the proof of the theorem. \( \square \)

Theorem 3.5. The transform \( \varepsilon_{2,1}^\varepsilon \) is consistent with \( \varepsilon_{2,1}^\varepsilon : l^u_{\text{loc}} (\mathbb{R}^+) \rightarrow l^u_{\text{loc}} (\mathbb{R}^+) \).

Proof. Let \( \phi \in l^u_{\text{loc}} (\mathbb{R}^+) \) and \( \beta \) be its representative in \( b (l^u_{\text{loc}}, C_c^\infty, \Delta, \times, \times) \). Then, we have, \( (\delta_n) \in \Delta, \beta = \frac{(\phi \times (\delta_n))}{(\delta_n)} \), \( \forall n \in \mathbb{N} \). Its clear that \( (\delta_n) \) is independent from the representative for every \( n \in \mathbb{N} \). We also have
\[
\varepsilon_{2,1}^\varepsilon \left( \frac{\phi \times (\delta_n)}{(\delta_n)} \right) = \left[ \frac{\varepsilon_{2,1}^\varepsilon (\phi \times (\delta_n))}{(\delta_n)} \right] = \left[ \frac{\varepsilon_{2,1}^\varepsilon \phi \times (\delta_n)}{(\delta_n)} \right],
\]
which is the representative of \( \varepsilon_{2,1}^\varepsilon \phi \) in the space \( l^u_{\text{loc}} (\mathbb{R}^+) \). Hence the proof is completed. \( \square \)

Theorem 3.6. The necessary and sufficient condition for \( \frac{(\phi_n)}{(\delta_n)} \in b (l^u_{\text{loc}}, C_c^\infty, \Delta, \times, \otimes) \) to be in the range of \( \varepsilon_{2,1}^\varepsilon \) is that \( g_n \) belongs to the range of \( \varepsilon_{2,1}^\varepsilon \), for every \( n \in \mathbb{N} \).

Proof. Let \( \frac{(\phi_n)}{(\delta_n)} \) be in the range of \( \varepsilon_{2,1}^\varepsilon \). Then of course \( g_n \) belongs to the range of \( \varepsilon_{2,1}^\varepsilon \), \( \forall n \in \mathbb{N} \).
To establish the converse, let \( g_n \) be in the range of \( \mathcal{E}^{s}_{2,1} \). Then, there is \( \phi_n \in l^u_{\text{loc}}(\mathbb{R}_+) \) such that 
\[
\mathcal{E}^{s}_{2,1} \phi_n = g_n, \quad \forall n \in \mathbb{N}.
\]
Since \( \frac{(\phi_n)}{(\psi_n)} \in b \left((l^u_{\text{loc}},\mathbb{R}_+),\bigtriangleup,\times,\otimes\right) \), we get
\[
g_n \otimes \psi_m = g_m \otimes \psi_n, \quad \forall m,n \in \mathbb{N}.
\]
Therefore, applying Theorem 2.3 yields
\[
\mathcal{E}^{s}_{2,1} (\phi_n \times \delta_m) = \mathcal{E}^{s}_{2,1} (\phi_m \times \delta_n), \quad \forall m,n \in \mathbb{N},
\]
where \( \phi_n \in l^u_{\text{loc}}(\mathbb{R}_+) \) and \( (\delta_n) \in \bigtriangleup, \forall n \in \mathbb{N} \).

Thus \( \phi_n \times \delta_m = \phi_m \times \delta_n, \quad m,n \in \mathbb{N} \). Hence, the Boehmian \( \left[\frac{(\phi_n)}{(\delta_n)}\right] \in b \left((l^u_{\text{loc}},\mathbb{R}_+),\bigtriangleup,\times,\otimes\right) \) satisfies the condition
\[
\mathcal{E}^{s}_{2,1} \left[\frac{(\phi_n)}{(\delta_n)}\right] = \left[\frac{(g_n)}{(\psi_n)}\right].
\]
The theorem is therefore completely proved.

**Theorem 3.7.** The mappings \( \mathcal{E}^{s}_{2,1} \) are continuous with respect to \( \delta \) and \( \bigtriangleup \) convergence.

Similar proof of this theorem is available in the cited papers of the first author. We prefer to omit the details.

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**References**

Some extensions of a certain integral transform to a quotient space of generalized functions
