Coupled fixed point theorems for $(\alpha, \varphi)_g$-contractive type mappings in partially ordered $G$-metric spaces

Abstract: In this paper, we introduce a new concept of $(\alpha, \varphi)_g$-contractive type mappings and establish coupled coincidence and coupled common fixed point theorems for such mappings in partially ordered $G$-metric spaces. The results on fixed point theorems are generalizations of some existing results. We also give some examples to illustrate the usability of the obtained results.

Keywords: Partially ordered set, Coincidence point, Coupled fixed point, $(\alpha, \varphi)_g$-contractive type mappings, $G$-metric space

MSC: 47H10, 54H25

1 Introduction and preliminaries

Fixed point theory is one of the most powerful and fruitful tools in nonlinear analysis, differential equation, and economic theory and has been studied in many various metric spaces. Especially, in 2006, Mustafa and Sims [1] introduced a generalized metric spaces which are called $G$-metric space. Following Mustafa and Sims’ work, many authors developed and introduced various fixed point theorems in $G$-metric spaces (see [17–21]). Recently, some authors have been interested in partially ordered $G$-metric spaces and proved some fixed theorems. Simultaneously, fixed point theory has developed rapidly in partially ordered metric spaces [15, 16]. Fixed point theorems have also been considered in partially ordered probabilistic metric spaces [7], in partially ordered cone metric spaces [12, 13], and in partially ordered $G$-metric spaces [2–6, 8–11, 27, 28]. In particular, in [3], Bhaskar and Lakshmikantham introduced notions of a mixed monotone mapping and a coupled fixed point, proved some coupled fixed point theorems for mixed monotone mappings, and discussed the existence and uniqueness of solutions for periodic boundary value problems. Afterwards, some coupled fixed point and coupled coincidence point results and their applications have been established.

Some authors studied fixed point theorems for $\alpha$-$\psi$-contractive type mappings in various spaces. For example, in [24], Samet et al. introduced $\alpha$-$\psi$-contractive type mappings and proved some fixed point theorem for such mappings in metric spaces. In [25], Murseleen et al. introduced $\alpha$-$\psi$-contractive type mappings and proved some fixed point theorem for such mappings in partially metric spaces. Recently, in [26], Ali et al. has introduced $\alpha$-$\psi$-contractive type mappings and proved some fixed point theorem for such mappings in uniform spaces.

Throughout this paper, let $\mathbb{N}$ denote the set of nonnegative integers, and $\mathbb{R}^+$ be the set of positive real numbers.

Before giving our main results, we recall some basic concepts and results in $G$-metric spaces.

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Definition 1.1 ([1]). Let $X$ be a non-empty set, $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

\begin{enumerate}[(G1)]
  \item $G(x, y, z) = 0$ if $x = y = z$.
  \item $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$.
  \item $G(x, y, z) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$.
  \item $G(x, y, z) = G(y, z, x) = \ldots$ (symmetry in all three variables).
  \item $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).
\end{enumerate}

Then the function $G$ is called a generalized metric and the pair $(X, G)$ is called a G-metric space.

Definition 1.2 ([1]). Let $(X, G)$ be a G-metric space and let $\{x_n\}$ be a sequence of points of $X$. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n,m \to \infty} G(x_n, x_n, x_m) = 0$, and one says the sequence $\{x_n\}$ is $G$-convergent to $x$.

Thus, if $x_n \to x$ in G-metric space $(X, G)$ then, for any $\epsilon > 0$, there exists a positive integer $N$ such that $G(x, x_n, x_m) < \epsilon$ for all $n, m > N$.

On one hand, in [1], the authors have shown that the G-metric induces a Hausdorff topology, and the convergence described in the above definition is relative to this topology. The topology being Hausdorff, a sequence can converge to at most to a point. On the other hand, the authors achieve some conclusions. In case of being tedious, we will not list those conclusions which were obtained by [1].

Next, we begin with some definitions and conclusions which will be needed in the sequel.

Definition 1.3 ([3]). Let $(X, \leq)$ be a partially ordered set and let $F : X \times X \to X$. The mapping $F$ is said to have the mixed monotone property if $F(x, y)$ is monotone non-decreasing in $x$ and is monotone non-increasing in $y$; that is, for any $x, y \in X$,

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2).$$

Definition 1.4 ([3]). An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if

$$x = F(x, y) \text{ and } y = F(y, x).$$

Definition 1.5 ([8]). Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$ be two mappings. We say that $F$ has the mixed-g-monotone property if $F(x, y)$ is g-monotone nondecreasing in $x$ and it is g-monotone nonincreasing in $y$, that is, for any $x, y \in X$, we have:

$$x_1, x_2 \in X, \quad gx_1 \leq gx_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and, respectively,

$$y_1, y_2 \in X, \quad gy_1 \leq gy_2 \Rightarrow F(x, y_1) \geq F(x, y_2).$$

Definition 1.6 ([8]). An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$gx = F(x, y) \text{ and } gy = F(y, x).$$

Definition 1.7 ([8]). We say that the mappings $F : X \times X \to X$ and $g : X \to X$ are commutative if

$$g(F(x, y)) = F(gx, gy) \text{ for all } x, y \in X.$$
(b) \(\varphi(t) < t\) for all \(t > 0\).
(c) \(\lim_{r \to t^+} \varphi(r) < t\) for all \(t > 0\).

Hence, it concluded that \(\lim_{n \to \infty} \varphi^n(t) = 0\).

Choudhury et al. [5] proved the following theorems.

**Theorem 1.8.** Let \((X, \leq)\) be a partially ordered set and suppose there is a \(G\)-metric \(G\) on \(X\) such that \((X, G)\) is a complete \(G\)-metric space. Let \(F : X \times X \to X\) be continuous and has the mixed monotone property on \(X\). Assume there exists \(k \in [0, 1)\) such that

\[
G(F(x, y), F(u, v), F(w, z)) \leq \frac{k}{2} (G(x, u, w) + G(y, v, z))
\]

for all \(x, y, u, v, w \in X\) with \(w \leq u \leq x\) and \(y \leq v \leq z\) where either \(u \neq w\) or \(v \neq z\). If there exist \(x_0, y_0 \in X\) such that \(x_0 \leq F(x_0, y_0)\) and \(y_0 \geq F(y_0, x_0)\), then \(F\) and \(g\) have a coupled coincidence point, that is, there exists \((x, y) \in X \times X\) such that \(x = F(x, y)\) and \(y = F(y, x)\).

Aydi et al. [15] proved the following theorems.

**Theorem 1.9.** Let \((X, \leq)\) be a partially ordered set and suppose there is a \(G\)-metric \(G\) on \(X\) such that \((X, G)\) is a complete \(G\)-metric space. Let \(F : X \times X \to X\) and \(g : X \to X\) be such that \(F\) is continuous and has the mixed-\(g\)-monotone property. Assume there is a function \(\varphi \in \Phi\) such that

\[
G(F(x, y), F(u, v), F(w, z)) \leq \varphi \left( \frac{G(gx, gu, gw) + G(gy, gv, gz)}{2} \right)
\]

for all \(x, y, u, v, w \in X\) with \(gw \leq gu \leq gx\) and \(gy \leq gv \leq gz\). Suppose also that \(F(X \times X) \subseteq g(X)\) and \(g\) is continuous and commutes with \(F\). If there exist \(x_0, y_0 \in X\) such that \(gx_0 \leq F(x_0, y_0)\) and \(gy_0 \geq F(y_0, x_0)\), then \(F\) and \(g\) have a coupled coincidence point, that is, there exists \((x, y) \in X \times X\) such that \(gx = F(x, y)\) and \(gy = F(y, x)\).

The purpose of this paper is to introduce a new concept of \((\alpha, \varphi)_g\)-contractive type mappings and establish some fixed theorems for such mappings in partially ordered \(G\)-metric spaces. Our results extend and generalize some results obtained by [5] and [15]. Some examples are presented to support our main results.

## 2 Main results

In this section, we give coupled coincidence and coupled common fixed theorems for \((\alpha, \varphi)_g\)-type contractive mappings in partially ordered \(G\)-metric spaces. Our results extend some existing results in [5, 15]. Now, we give the following definitions.

**Definition 2.1.** Let \((X, G, \leq)\) be a partially ordered \(G\)-metric space and \(F : X \times X \to X\) and \(g : X \to X\) be two mappings. Then a map \(F\) is said to be \((\alpha, \varphi)_g\)-contractive if there exist two functions \(\alpha : X^2 \times X^2 \times X^2 \to [0, +\infty)\) and \(\varphi \in \Phi\) such that

\[
\alpha((gx, gy), (gu, gv), (gz, gw)) G(F(x, y), F(u, v), F(z, w)) \leq \varphi \left( \frac{G(gx, gu, gz) + G(gy, gv, gw)}{2} \right)
\]

for all \(x, y, u, v, z, w \in X\) with \(gx \geq gu \geq gz\) and \(gy \leq gv \leq gw\).

**Remark 2.2.**

(1) If \(F : X \times X \to X\) satisfies (1), then \(F\) is a \((\alpha, \varphi)_g\)-contractive mapping, where \(\alpha((gx, gy), (gu, gv), (gz, gw)) = 1\) for all \(x, y, u, v, z, w \in X\), \(g = I_X\) and \(\varphi(t) = kt\) for all \(t \geq 0\) and some \(k \in [0, 1)\).
(2) If \( \varphi \) satisfies (2), then \( F \) is a \((\alpha, \varphi)_{e}\)-contractive mapping, where \( \alpha((gx, gy), (gu, gv), (gz, gw)) = 1 \) for all \( x, y, u, v, z, w \in X \).

**Definition 2.3.** Let \( F : X \times X \to X \), \( g : X \to X \) and \( \alpha : X^2 \times X^2 \to [0, +\infty) \) be three mappings. Then \( F \) is said to be \( g-\alpha \)-admissible if

\[
\alpha((gx, gy), (gu, gv), (gz, gw)) \geq 1 \Rightarrow \alpha((F(x, y), F(u, v), (F(z, w))) \geq 1,
\]
for all \( x, y, u, v, z, w \in X \).

Before presenting our main fixed point results for \((\alpha, \varphi)_{e}\)-contractive type mappings, we show some examples to illustrate the potential applicability of Definition 2.3.

**Example 2.4.** Let \( X = [0, +\infty) \). Define \( F : X \times X \to X \), \( g : X \to X \) and \( \alpha : X^2 \times X^2 \to [0, +\infty) \) by \( gx = \frac{1}{2}x \), \( F(x, y) = x \) for all \( x, y \in X \) and

\[
\alpha((gx, gy), (gu, gv), (gz, gw)) = \begin{cases} 2 & \text{if } gx \geq gy, \\ 0 & \text{others}. \end{cases}
\]

Then \( F \) is \( g-\alpha \)-admissible.

**Example 2.5.** Let \( X = [0, +\infty) \). Define \( F : X \times X \to X \), \( g : X \to X \) and \( \alpha : X^2 \times X^2 \to [0, +\infty) \) by \( gx = \frac{1}{2}x \), \( F(x, y) = x^2 \) for all \( x, y \in X \) and

\[
\alpha((gx, gy), (gu, gv), (gz, gw)) = \begin{cases} e^{x-y} & \text{if } gx \geq gy, \\ 0 & \text{others}. \end{cases}
\]

Then \( F \) is \( g-\alpha \)-admissible.

**Example 2.6.** Let \( X = [0, +\infty) \). Define \( F : X \times X \to X \), \( g : X \to X \) and \( \alpha : X^2 \times X^2 \to [0, +\infty) \) by \( gx = e^x \), \( F(x, y) = x^2 + y^2 \) for all \( x, y \in X \) and

\[
\alpha((gx, gy), (gu, gv), (gz, gw)) = e^x
\]

Then \( F \) is \( g-\alpha \)-admissible.

Next, we prove our main results under \((\alpha, \varphi)_{e}\)-contractive conditions.

**Theorem 2.7.** Let \((X, \preceq)\) be a partially ordered set and suppose there is a \( G \)-metric \( G \) on \( X \) such that \((X, G)\) is a complete \( G \)-metric space. Let \( F : X \times X \to X \) and \( g : X \to X \) be such that \( F \) has the mixed-\( g \)-monotone property. Assume there is a function \( \varphi \in \Phi \) and \( \alpha : X^2 \times X^2 \to [0, +\infty) \) for all \( x, y, z, u, v, w \in X \), the following hold:

\[
\alpha((gx, gy), (gu, gv), (gz, gw))G(F(x, y), F(u, v), F(z, w)) \leq \varphi \left( G(gx, gu, gz) + G(gy, gv, gw) \right) / 2
\]

for all \( gw \preceq gu \preceq gx \) and \( gy \preceq gv \preceq gz \). Suppose also that \( F(X \times X) \subseteq g(X) \), \( g \) is continuous and commutes with \( F \) and

(i) \( F \) is \( g-\alpha \)-admissible,

(ii) there exist \( x_0, y_0 \in X \) such that

\[
\alpha((gx_0, gy_0), (F(x_0, y_0), F(y_0, x_0))) \geq 1
\]

and

\[
\alpha((gy_0, gx_0), (F(y_0, x_0), (F(x_0, y_0))) \geq 1,
\]

(iii) \( F \) is continuous.
If there exist $x_0, y_0 \in X$ such that $g x_0 \leq F(x_0, y_0)$ and $g y_0 \geq F(y_0, x_0)$, then $F$ and $g$ have a coupled coincidence point, that is, there exists $(x, y) \in X \times X$ such that $g x = F(x, y)$ and $g y = F(y, x)$.

Proof. Let $x_0, y_0 \in X$ be such that $g x_0 \leq F(x_0, y_0)$ and $g y_0 \geq F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $g x_1 = F(x_0, y_0)$ and $g y_1 = F(y_0, x_0)$. Again since $F(X \times X) \subseteq g(X)$, we can choose $x_2, y_2 \in X$ such that $g x_2 = F(x_1, y_1)$ and $g y_2 = F(y_1, x_1)$. Since $F$ has the mixed g-monotone property, we have $g x_0 \leq g x_1 \leq g x_2$ and $g y_2 \leq g y_1 \leq g y_0$. Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$g x_n = F(x_{n-1}, y_{n-1}) \leq g x_{n+1} = F(x_n, y_n)$$

and

$$g y_{n+1} = F(y_{n-1}, x_n) \leq g y_n = F(y_{n-1}, x_{n-1}).$$

If for some $n$, we have $(g x_{n+1}, g y_{n+1}) = (g x_n, g y_n)$, then $F(x_n, y_n) = g x_n$ and $F(y_n, x_n) = g y_n$, that is, $F$ and $g$ have a coincidence point. So from now on, we assume $(g x_{n+1}, g y_{n+1}) \neq (g x_n, g y_n)$ for all $n \in \mathbb{N}$, that is, we assume that either $g x_{n+1} = F(x_n, y_n) \neq g x_n$ or $g y_{n+1} = F(y_n, x_n) \neq g y_n$. Since $F$ is g-$\alpha$ admissible, we have

$$\alpha((g x_0, g y_0), (g x_1, g y_1), (g x_1, g y_1)) = \alpha((g x_0, g y_0), (F(x_0, y_0), (F(x_0, y_0), (F(x_0, y_0), F(y_0, x_0)))) \geq 1$$

$$\Rightarrow \alpha((F(x_0, y_0), F(y_0, x_0)), (F(x_1, y_1), (F(y_1, x_1)), (F(x_1, y_1), F(y_1, x_1)))$$

$$= \alpha((g x_1, g y_1), (g x_2, g y_2), (g x_2, g y_2)) \geq 1.$$

Thus, by mathematical induction, we have

$$\alpha((g x_n, g y_n), (g x_{n+1}, g y_{n+1}), (g x_{n+1}, g y_{n+1})) \geq 1$$

and similarly,

$$\alpha((g y_n, g x_n), (g y_{n+1}, g x_{n+1}), (g y_{n+1}, g x_{n+1})) \geq 1$$

for all $n \in \mathbb{N}$. Using (4) and (7), we obtain

$$G(g x_n, g x_{n+1}, g x_{n+1}) = G(F(x_{n-1}, y_{n-1}), F(x_n, y_n), F(x_n, y_n))$$

$$\leq \varphi \left( \frac{G(g x_{n-1}, g x_n, g x_n) + G(g y_{n-1}, g y_n, g y_n)}{2} \right).$$

Similarly, we have

$$G(g y_n, g y_{n+1}, g y_{n+1}) = G(F(y_{n-1}, x_{n-1}), F(y_n, x_n), F(y_n, x_n))$$

$$\leq \varphi \left( \frac{G(g y_{n-1}, g y_n, g y_n) + G(g x_{n-1}, g x_n, g x_n)}{2} \right).$$

Adding (9) and (10), we get

$$\frac{G(g y_n, g y_{n+1}, g y_{n+1}) + G(g x_n, g x_{n+1}, g x_{n+1})}{2} \leq \varphi \left( \frac{G(g y_{n-1}, g y_n, g y_n) + G(g x_{n-1}, g x_n, g x_n)}{2} \right).$$

Repeating the above process, since $\varphi$ is nondecreasing, we get

$$\frac{G(g x_n, g x_{n+1}, g x_{n+1}) + G(g y_n, g y_{n+1}, g y_{n+1})}{2} \leq \varphi^n \left( \frac{G(g x_0, g x_1, g x_1) + G(g y_0, g y_1, g y_1)}{2} \right)$$

for all $n \in \mathbb{N}$. Hence, for any $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{n \geq n(\varepsilon)} \varphi^n \left( \frac{G(g x_0, g x_1, g x_1) + G(g y_0, g y_1, g y_1)}{2} \right) < \frac{\varepsilon}{2}.$$
Let \( n, m \in \mathbb{N} \) be such that \( m > n > n(\varepsilon) \). Then by using the triangle inequality, we have
\[
\frac{G(gx_n, gx_m, gx_m) + G(gy_n, gy_m, gy_m)}{2} \leq \sum_{k=n}^{m} \frac{G(gx_k, gx_{k+1}, gx_{k+1}) + G(gy_k, gy_{k+1}, gy_{k+1})}{2} \\
\leq \sum_{k=n}^{m-1} \phi^k \left( \frac{G(gx_0, gx_1, gx_1) + G(gy_0, gy_1, gy_1)}{2} \right) \\
\leq \sum_{n \geq n(\varepsilon)} \phi^n \left( \frac{G(gx_0, gx_1, gx_1) + G(gy_0, gy_1, gy_1)}{2} \right) < \frac{\varepsilon}{2}.
\]

This implies that \( G(gx_n, gx_m, gx_m) + G(gy_n, gy_m, gy_m) < \frac{\varepsilon}{2} \). Since
\[
G(gx_n, gx_m, gx_m) \leq G(gx_n, gx_m, gx_m) + G(gy_n, gy_m, gy_m) < \frac{\varepsilon}{2},
\]
and
\[
G(gy_n, gy_m, gy_m) \leq G(gx_n, gx_m, gx_m) + G(gy_n, gy_m, gy_m) < \frac{\varepsilon}{2},
\]
and hence \( \{gx_n\} \) and \( \{gy_n\} \) are \( G \)-Cauchy sequences in the \( G \)-metric space \((X, G)\). Now, since \((X, G)\) is \( G \)-complete, there are \( x, y \in X \) such that \( \{gx_n\} \) and \( \{gy_n\} \) are respectively \( G \)-convergent to \( x \) and \( y \), that is, from the definition of limit, we have
\[
\lim_{n \to +\infty} F(x_n, y_n) = \lim_{n \to +\infty} g(x_n) = x, \quad \lim_{n \to +\infty} F(y_n, x_n) = \lim_{n \to +\infty} g(y_n) = y. \tag{11}
\]

Since \( g \) is continuous and commutes with \( F \), hence we have
\[
gx = \lim_{n \to \infty} gF(x_n, y_n) = \lim_{n \to \infty} F(gx_n, gy_n) = F(x, y), \tag{12}
\]
and
\[
gy = \lim_{n \to \infty} gF(y_n, x_n) = \lim_{n \to \infty} F(gy_n, gx_n) = F(y, x). \tag{13}
\]
The proof is completed.

\( \square \)

Remark 2.8.

1. In Theorem 2.7, taking \( \alpha((gx, gy), (gu, gv), (gz, gw)) = 1 \), we can get Theorem 1.9 (Theorem 3.1 of [15]).

2. If we take \( \phi(t) = kt \) and \( \alpha((gx, gy), (gu, gv), (gz, gw)) = 1 \) for all \( k \in [0, 1) \) in Theorem 2.7, we can get the following corollary.

Corollary 2.9. Let \((X, \preceq)\) be a partially ordered set and suppose there is a \( G \)-metric \( G \) on \( X \) such that \((X, G)\) is a complete \( G \)-metric space. Let \( F : X \times X \to X \) and \( g : X \to X \) be such that \( F \) has the mixed-\( g \)-monotone property. Assume there is \( k \in [0, 1) \) such that
\[
G(F(x, y), F(u, v), F(w, z)) \leq \frac{k}{2} (G(gx, gx, gw) + G(gy, gv, gz))
\]
for all \( x, y, z, u, v, w \in X \) with \( gw \preceq gu \preceq gx \) and \( gy \preceq gv \preceq gz \). Suppose that \( F(X \times X) \subseteq g(X) \) and \( g \) is continuous and commutes with \( F \) and \( F \) is continuous. If there exist \( x_0, y_0 \in X \) such that \( gx_0 \preceq F(x_0, y_0) \) and \( gy_0 \preceq F(y_0, x_0) \), then \( F \) and \( g \) have a coupled coincidence point, that is, there exists \((x, y) \in X \times X \) such that \( gx = F(x, y) \) and \( gy = F(y, x) \).

Remark 2.10.

1. In Corollary 2.9, taking \( g = 1_X \), we can get Theorem 1.8 (Theorem 3.1 of [5]).

2. If taking \( \alpha((gx, gy), (gu, gv), (gz, gw)) = \frac{1}{k} \) for all \( k \in (0, 1] \) in Theorem 2.7, we can get the following corollary.
Corollary 2.11. Let $(X, \preceq)$ be a partially ordered set and suppose there is a $G$-metric $G$ on $X$ such that $(X, G)$ is a complete $G$-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be such that $F$ has the mixed-g-monotone property. Assume there exists $k \in (0, 1]$ and $\varphi \in \Phi$ such that
\[ G(F(x, y), F(u, v), F(w, z)) \leq k \varphi \left( \frac{G(gx, gu, gw) + G(gy, gv, gz)}{2} \right) \]
for all $x, y, z, u, v, w \in X$ with $g w \preceq g u \preceq gx$ and $gy \preceq gv \preceq gz$. Suppose that $F(X \times X) \subseteq g(X)$ and $g$ is continuous and commutes with $F$ and $F$ is continuous. If there exist $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $gy_0 \geq F(y_0, x_0)$, then $F$ and $g$ have a coupled coincidence point, that is, there exists $(x, y) \in X \times X$ such that $gx = F(x, y)$ and $gy = F(y, x)$.

Let $g = I_X$ in Corollary 2.11, we can get the following corollary.

Corollary 2.12. Let $(X, \preceq)$ be a partially ordered set and suppose there is a $G$-metric $G$ on $X$ such that $(X, G)$ is a complete $G$-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be continuous. Assume there exists $k \in (0, 1]$ such that
\[ G(F(x, y), F(u, v), F(w, z)) \leq k \varphi \left( \frac{G(x, u, w) + G(y, v, z)}{2} \right) \]
for all $x, y, z, u, v, w \in X$ with $w \preceq u \preceq x$ and $y \preceq v \preceq z$. If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then $F$ has a coupled fixed point, that is, there exists $(x, y) \in X \times X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Let $\Omega$ denote all continuous functions $\psi : [0, \infty) \to [0, \infty)$ satisfying $\lim_{t \to r} \psi(t) > 0$ for each $r > 0$. Using the definition of $\Omega$, we can get the following corollary.

Corollary 2.13. Let $(X, \preceq)$ be a partially ordered set and suppose there is a $G$-metric $G$ on $X$ such that $(X, G)$ is a complete $G$-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be such that $F$ has the mixed-g-monotone property. Assume there exists $\psi \in \Omega$ such that
\[ G(F(x, y), F(u, v), F(w, z)) \leq \frac{G(gx, gu, gw) + G(gy, gv, gz)}{2} - \psi \left( \frac{G(gx, gu, gw) + G(gy, gv, gz)}{2} \right) \]
for all $x, y, z, u, v, w \in X$ with $g w \preceq g u \preceq gx$ and $gy \preceq gv \preceq gz$. If there exist $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0, y_0)$ and $gy_0 \geq F(y_0, x_0)$, then $F$ and $g$ have a coupled coincidence point, that is, there exists $(x, y) \in X \times X$ such that $gx = F(x, y)$ and $gy = F(y, x)$.

Proof. Let $\varphi(t) = t - \psi(t)$. Obviously, $\varphi \in \Phi$. Hence, Corollary 2.13 satisfies all conditions of Theorem 2.7. The proof is completed.

In the next theorem, we use another condition instead of the continuity hypothesis of $F$ so that we can illustrate that the introduction of the function $\alpha$ is useful.

Theorem 2.14. Let $(X, \preceq)$ be a partially ordered set and suppose there is a $G$-metric $G$ on $X$ such that $(X, G)$ is a complete $G$-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be such that $F$ has the mixed-g-monotone property. Assume there is a function $\varphi \in \Phi$ and $\alpha : X \times X \times X \to [0, \infty)$ for all $x, y, z, u, v, w \in X$, the following hold:
\[ \alpha((gx, gy), (gu, gv), (gz, gw))G(F(x, y), F(u, v), F(z, w)) \leq \varphi \left( \frac{G(gx, gu, gz) + G(gy, gv, gw)}{2} \right) \]
for all $g w \preceq g u \preceq gx$ and $gy \preceq gv \preceq gz$. Suppose also that $F(X \times X) \subseteq g(X)$, $g$ is continuous and
(i) conditions (i) and (ii) of Theorem 2.7 hold,
(ii) if \( \{g_{xn}\} \) and \( \{g_{yn}\} \) are sequences in \( X \) such that
\[
\alpha((g_{xn}, g_{yn}), (g_{xn+1}, g_{yn+1})), (g_{xn+1}, g_{yn+1})) \geq 1
\]
and
\[
\alpha((g_{yn}, g_{xn}), (g_{yn+1}, g_{xn+1})), (g_{yn+1}, g_{xn+1})) \geq 1.
\]
for all \( n \) and \( \lim_{n \to \infty} g_{xn} = x \in X \) and \( \lim_{n \to \infty} g_{yn} = y \in X \), then
\[
\alpha((g_{xn}, g_{yn}), (g_{x}, g_{y})), (g_{x}, g_{y})) \geq 1
\]
and
\[
\alpha((g_{yn}, g_{xn}), (g_{y}, g_{x})), (g_{y}, g_{x})) \geq 1.
\]
If there exist \( x_0, y_0 \in X \) such that \( g_{xn} \leq F(x_0, y_0) \) and \( g_{yn} \geq F(y_0, x_0) \), then \( F \) and \( g \) have a coupled coincidence point, that is, there exists \( (x, y) \in X \times X \) such that \( g_{x} = F(x, y) \) and \( g_{y} = F(y, x) \).

**Proof.** Proceeding along the same lines as in the proof of Theorem 2.7, we know that \( \{g_{xn}\} \) and \( \{g_{yn}\} \) are Cauchy sequences in the complete \( G \)-metric space \( (X, G) \). Then there exist \( x, y \in X \) such that \( \lim_{n \to \infty} g_{xn} = x \) and \( \lim_{n \to \infty} g_{yn} = y \). On the other hand, from (7) and hypothesis (ii), we obtain
\[
\alpha((g_{xn}, g_{yn}), (g_{x}, g_{y})), (g_{x}, g_{y})) \geq 1 \quad (14)
\]
and similarly,
\[
\alpha((g_{yn}, g_{xn}), (g_{y}, g_{x})), (g_{y}, g_{x})) \geq 1 \quad (15)
\]
for all \( n \in \mathbb{N} \). Using the triangle inequality, (14) and the property of \( \varphi(t) < t \) for all \( t > 0 \), we get
\[
G(g_{x}, F(x, y), F(x, y)) \leq G(g_{x}, g_{(xn+1)}), g_{(yn+1)}) + G(g_{yn+1}), F(x, y), F(x, y))
\]
\[
\leq G(g_{x}, g_{(xn+1)}), g_{(yn+1)}) + G(F(g_{xn}, g_{yn}), F(x, y), F(x, y))
\]
\[
\leq G(g_{x}, g_{(xn+1)}), g_{(yn+1)}) + \varphi(G(g_{xn}, g_{x}, g_{x}) + G(g_{yn}, g_{y}, g_{y}))
\]
\[
< G(g_{x}, g_{(xn+1)}), g_{(yn+1)}) + \frac{G(g_{yn}, g_{x}, g_{x}) + G(g_{xn}, g_{y}, g_{y})}{2}
\]
Similarly, using (15), we obtain
\[
G(g_{y}, F(y, x), F(x, y)) \leq G(g_{y}, g_{(yn+1)}), g_{(xn+1)}) + G(g_{xn+1}), F(y, x), F(y, x))
\]
\[
\leq G(g_{y}, g_{(yn+1)}), g_{(xn+1)}) + G(F(g_{yn}, g_{xn}), F(y, x), F(y, x))
\]
\[
\leq G(g_{y}, g_{(yn+1)}), g_{(xn+1)}) + \varphi(G(g_{yn}, g_{x}, g_{x}) + G(g_{xn}, g_{y}, g_{y}))
\]
\[
< G(g_{y}, g_{(yn+1)}), g_{(xn+1)}) + \frac{G(g_{yn}, g_{x}, g_{x}) + G(g_{xn}, g_{y}, g_{y})}{2}
\]
Taking the limit as \( n \to \infty \) in the above two inequalities, we get
\[
G(g_{x}, F(x, y), F(x, y)) = 0 \quad \text{and} \quad G(g_{y}, F(y, x), F(y, x)) = 0.
\]
Hence, \( F(x, y) = g_{x} \) and \( F(y, x) = g_{y} \). Thus, \( F \) and \( g \) have a coupled coincidence point. The proof is completed.

\[\square\]

**Remark 2.15.** In Corollaries 2.9–2.13, if we omit the continuity of \( F \), then Corollaries 2.9–2.13 will not hold, since by the proof of Theorem 2.7, we must use the continuity of \( F \) to get limit. But the function \( F \) in Theorem 2.14 is not continuous because the function \( \alpha \) meets the conditions (ii) in Theorem 2.14 instead of the continuity of \( F \). Therefore, Theorem 2.14 is essentially different from Theorem 2.7. They are two parallel conclusions.
Now, we shall prove the uniqueness of the coupled fixed point. Note that, if \((X, \preceq)\) is a partially ordered set, then we endow the product \(X \times X\) with the following partial order relation:
\[
(x, y), (u, v) \in X \times X, (x, y) \preceq (u, v) \iff x \preceq u, y \preceq v.
\]

**Theorem 2.16.** In addition to the hypotheses of Theorem 2.7, suppose that for all \((x, y), (x^*, y^*) \in X \times X\), there exists \((u, v) \in X \times X\) such that
\[
\alpha(gx, gy), (gu, gv), (gu, gv)) \geq 1 \quad \text{and} \quad \alpha((gx^*, gy^*), (gu, gv)), (gu, gv)) \geq 1
\]
and also assume that \((F(u, v), F(v, u))\) is comparable with \((F(x, y)F(y, x))\) and \((F(x^*, y^*), F(y^*, x^*))\). Then \(F\) and \(g\) have a unique coupled common fixed point, that is, there exists a unique \((x, y) \in X \times X\) such that
\[
x = gx = F(x, y) \quad \text{and} \quad y = gy = F(y, x).
\]  \hspace{1cm} (16)

**Proof.** From Theorem 2.7, the set of coupled coincidences is non-empty. We shall show that if \((x, y)\) and \((x^*, y^*)\) are coupled coincidence points, that is, if \(g(x) = F(x, y), g(y) = F(y, x), g(x^*) = F(x^*, y^*)\) and \(g(y^*) = F(y^*, x^*)\), then \(gx = gx^*\) and \(gy = gy^*\). By assumption, there exists \((u, v) \in X \times X\) such that \((F(u, v), F(v, u))\) is comparable with \((F(x, y), F(y, x))\) and \((F(x^*, y^*), F(y^*, x^*))\). Without restriction to the generality, we can assume that
\[
(F(x, y), F(y, x)) \preceq (F(u, v), F(v, u))
\]
and
\[
(F(x^*, y^*), F(y^*, x^*)) \preceq (F(u^*, v^*), F(v^*, u^*)�)
\]
Put \(u_0 = u, v_0 = v\), and choose \(u_1, v_1 \in X\) such that \(gu_1 = F(u_0, v_0), gv_1 = F(v_0, u_0)\). Then, similarly as in the proof of Theorem 2.7, we can inductively define sequences \((gu_n)\) and \((gv_n)\) in \(X\) by
\[
gu_{n+1} = F(u_n, v_n) \quad \text{and} \quad gv_{n+1} = F(v_n, u_n).
\]
Further, let \(x_0 = x, y_0 = y, x_0^* = x^*, y_0^* = y^*\). and, in the same way, define the sequences \((gx_n), (gy_n), (gx_n^*)\) and \((gy_n^*)\). Since
\[
(F(x, y), F(y, x)) = (gx_1, gy_1) = (gx, gy) \preceq (F(u, v), F(v, u)) = (gu_1, gv_1),
\]
then \(gx \preceq gu_1\) and \(gv_1 \preceq gy\). Since for all \((x, y), (x^*, y^*) \in X \times X\), there exists \((u, v) \in X \times X\) such that
\[
\alpha((gx, gy), (gu, gv)), (gu, gv)) \geq 1 \quad \text{and} \quad \alpha((gx^*, gy^*), (gu, gv)), (gu, gv)) \geq 1.
\]  \hspace{1cm} (17)
Since \(F\) is \(g\)-\(\alpha\)-admissible, so from (17), we have
\[
\alpha((gx, gy), (gu, gv)), (gu, gv)) \geq 1 \Rightarrow \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u)), (F(u, v), F(u, v))) \geq 1.
\]
Since \(u = u_0\) and \(v = v_0\), we get
\[
\alpha((gx, gy), (gu, gv)), (gu, gv)) \geq 1 \\
\Rightarrow \alpha((F(x, y), F(y, x)), (F(u_0, v_0), F(v_0, u_0)), (F(u_0, v_0), F(v_0, u_0))) \geq 1.
\]
Thus,
\[
\alpha((gx, gy), (gu, gv)), (gu, gv)) \geq 1 \Rightarrow \alpha((gx, gy), (gu_1, gv_1), (gu_1, gv_1)) \geq 1.
\]
Therefore, by mathematical induction, we obtain
\[
\alpha((gx, gy), (gu_n, gv_n), (gu_n, gv_n)) \geq 1
\]  \hspace{1cm} (18)
for all \(n \in \mathbb{N}\) and similarly, \(\alpha((gy, gx), (gv_n, gu_n), (gv_n, gu_n)) \geq 1\). From (17) and (18), we get
\[
G(gx, gu_{n+1}, gu_{n+1}) = G(F(x, y), F(u_n, v_n), F(u_n, v_n)) \\
\leq \alpha(gx, gy), (gu_n, gu_n), (gu_n, gu_n))G(F(x, y), F(u_n, v_n), F(u_n, v_n)) \\
\leq \phi \left( \frac{1}{2} G(gx, gu_n, gu_n) + G(gy, gu_n, gu_n) \right).
\]  \hspace{1cm} (19)
Consider a mapping \( F \). Let \( \alpha : X \times X \to [0,1] \) be a mapping
\[
\alpha((x,y),(u,v)) = \begin{cases} 
1 & \text{if } gx \geq gy, gu \geq gv, gz \geq gw \\
0 & \text{otherwise}
\end{cases}
\]
\[
\theta(n) = \frac{G(gx,gu_{n+1}) + G(gy,gv_{n+1})}{2}.
\]
Adding (19) and (20), we get
\[
\frac{G(gx,gu_{n+1}) + G(gy,gv_{n+1})}{2} \leq \varphi\left(\frac{G(gx,gu_{n}) + G(gy,gv_{n})}{2}\right).
\]
Thus,
\[
\frac{G(gx,gu_{n+1}) + G(gy,gv_{n+1})}{2} \leq \varphi^n\left(\frac{G(gx,gu_{1}) + G(gy,gv_{1})}{2}\right)
\]
for each \( n \geq 1 \). Letting \( n \to \infty \) in (21) and using the properties of \( \varphi \), we get
\[
\lim_{n \to \infty} G(gx,gu_{n+1}) + G(gy,gv_{n+1}) = 0.
\]
This implies
\[
\lim_{n \to \infty} G(gx,gu_{n+1}) = 0 \text{ and } \lim_{n \to \infty} G(gy,gv_{n+1}) = 0.
\]
Similarly, one can show that
\[
\lim_{n \to \infty} G(gx^*,gu_{n+1}) = 0 \text{ and } \lim_{n \to \infty} G(gy^*,gv_{n+1}) = 0.
\]
Therefore, from (22), (23) and the uniqueness of the limit, we can get
\[
gx = gx^* \text{ and } gy = gy^*.
\]
Since \( gx = F(x,y) \) and \( gy = F(y,x) \), by commuting \( F \) with \( g \), we have
\[
g(gx) = g(F(x,y)) = F(gx,gy) \text{ and } g(gy) = g(F(y,x)) = F(gy,gx).
\]
Put \( gx = z \) and \( gy = w \), then by (25), we get
\[
gz = F(z,w) \text{ and } gw = F(w,z).
\]
Thus, \((z,w)\) is a coincidence point. Then by (24) with \( x^* = z \) and \( y^* = w \), we have \( gx = gz \) and \( gy = gw \), that is,
\[
gz = gz = z \text{ and } gy = gw = w.
\]
From (26) and (27), we get \( z = gz = F(z,w) \) and \( w = gw = F(w,z) \). Then, \((z,w)\) is a coupled fixed point of \( F \) and \( g \). To prove the uniqueness, assume that \((p,q)\) is another coupled fixed point. Then by (27), we have \( p = gp = gz = z \) and \( q = gq = gw = w \). The proof is completed.

In what follows, firstly, we give a linear example (i.e. trivial example) of \( F \) and \( g \) to illustrate the usefulness of Theorem 2.7.

**Example 2.17.** Let \( X = [0,1] \) and \((X,\leq)\) be a partially ordered set with the natural ordering of real numbers. Let \( G(x,y,z) = |x-y| + |y-z| + |z-x| \) for all \( x, y, z \in X \). Then \((X,G)\) is a complete G-metric space. Define a mapping \( F : X \times X \to X \) by \( F(x,y) = \frac{x+y}{2} \) for all \( x, y \in X \) and \( g : X \to X \) by \( g(x) = x \) for all \( x \in X \). Consider a mapping \( \alpha : X^2 \times X^2 \to [0,\infty) \)
\[
\alpha((gx,gy),(gu,gv),(gz,gw)) = \begin{cases} 
1 & \text{if } gx \geq gy, gu \geq gv, gz \geq gw \\
0 & \text{otherwise}
\end{cases}
\]
for all $x, y \in X$. Obviously, $F$ is commuting with $g$. Then, we get

$$
G((F(x, y)), F(u, v), F(z, w)) = G\left(\frac{x + y}{4}, \frac{u + v}{4}, \frac{z + w}{4}\right)
$$

$$
= \frac{1}{4}\left((x + y) - (u + v) + (u + v) - (z + w) + |(z + w) - (x + y)|\right)
$$

$$
\leq \frac{1}{4}\left(|x - u| + |u - z| + |z - x| + |y - v| + |v - w| + |w - y|\right)
$$

and

$$
\frac{G(gx, gu, gz) + G(gy, gv, gw)}{2} = G((x, u, z)) + G(y, v, w))
$$

Therefore, (3) holds for $\varphi(t) = \frac{1}{2}t$ for all $t > 0$, and also the hypothesis of Theorem 2.7 is fulfilled. Then there exists a coincide coupled point of $F$ and $g$. In this case, $(0,0)$ is a coincide coupled point of $F$ and $g$.

Secondly, we give a nonlinear example (i.e. nontrivial example) of $F$ and $g$ to illustrate the usefulness of Theorem 2.7.

**Example 2.18.** Let $X = [0, 1]$ and $(X, \leq)$ be a partially ordered set with the natural ordering of real numbers. Let $G(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in X$. Then $(X, G)$ is a complete $G$-metric space. Define a mapping $F : X \times X \rightarrow X$ by $F(x, y) = \left(\frac{(x + y)^2}{24}\right)$ for all $x, y \in X$ and $g : X \rightarrow X$ by $g(x) = x$ for all $x \in X$. Consider a mapping $\alpha : X^2 \times X^2 \times X^2 \rightarrow [0, +\infty)$

$$
\alpha((gx, gy), (gu, gv), (gz, gw)) = \begin{cases} e^{x-y} & \text{if } 0 \leq x - y \leq \frac{2}{3} \\ 0 & \text{otherwise} \end{cases}
$$

for all $x, y \in X$. Obviously, $F$ is commuting with $g$. Then, we get

$$
G((F(x, y)), F(u, v), F(z, w)) = G\left(\frac{(x + y)^2}{6}, \frac{(u + v)^2}{6}, \frac{(z + w)^2}{24}\right)
$$

$$
= \frac{1}{24}\left(\frac{(x + y)^2}{6} - (u + v)^2\right) + \left(\frac{(u + v)^2}{6} - (z + w)^2\right) + \left((z + w)^2 - (x + y)^2\right)
$$

$$
\leq \frac{1}{24}\left(\frac{(x + y)^2}{6} - (u + v)^2\right) + \left|\frac{(u + v)^2}{6} - (z + w)^2\right| + \left|((z + w)^2 - (x + y)^2)\right|
$$

$$
\leq \frac{1}{6}\left(|x - u| + |u - z| + |z - x| + |y - v| + |v - w| + |w - y|\right).
$$

and

$$
\frac{G(gx, gu, gz) + G(gy, gv, gw)}{2} = G((x, u, z)) + G(y, v, w))
$$

Therefore, (3) holds for $\varphi(t) = \frac{e^3}{3}t$ for all $t > 0$, and also the hypothesis of Theorem 2.7 is fulfilled. Then there exists a coincide coupled point of $F$ and $g$. In this case, $(0,0)$ is a coincide coupled point of $F$ and $g$.

**Competing interests**

The authors declare that they have no competing interests.

**Authors’ contributions**

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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