On the number of spanning trees, the Laplacian eigenvalues, and the Laplacian Estrada index of subdivided-line graphs

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Abstract: As a generalization of the Sierpiński-like graphs, the subdivided-line graph $\mathcal{G}$ of a simple connected graph $G$ is defined to be the line graph of the barycentric subdivision of $G$. In this paper we obtain a closed-form formula for the enumeration of spanning trees in $\mathcal{G}$, employing the theory of electrical networks. We present bounds for the largest and second smallest Laplacian eigenvalues of $\mathcal{G}$ in terms of the maximum degree, the number of edges, and the first Zagreb index of $G$. In addition, we establish upper and lower bounds for the Laplacian Estrada index of $\mathcal{G}$ based on the vertex degrees of $G$. These bounds are also connected with the number of spanning trees in $\mathcal{G}$.

Keywords: Laplacian Estrada index, Subdivide-line graph, Line graph, Laplacian spectrum, Spanning tree

MSC: 05C50, 05C76, 05C30, 05C05

1 Introduction

In this paper we are concerned with simple connected (molecular) graphs. Let $G$ be such a graph with the vertex set $V(G) = \{v_1, \ldots, v_n\}$ and the edge set $E(G)$. The adjacency matrix of $G$ is $A(G) = (a_{ij}) \in \mathbb{R}^{n \times n}$, where $a_{ij} = 1$ if two vertices $v_i$ and $v_j$ are adjacent in $G$ and $a_{ij} = 0$ otherwise. The Laplacian matrix of $G$ is the matrix $L(G) = D(G) - A(G)$, where $D(G)$ is a diagonal matrix with $d_G(v_1), d_G(v_2), \ldots, d_G(v_n)$ on the main diagonal, in which $d_G(v_i)$ is the degree of the vertex $v_i$ in $G$. Since $L(G)$ is a positive semi-definite matrix and $G$ is connected, the eigenvalues of $L(G)$ can be ordered as $\lambda_1(G) \geq \lambda_2(G) \geq \ldots \geq \lambda_{n-1}(G) > \lambda_n(G) = 0$ [1]. They are referred to as the Laplacian eigenvalues of $G$, and $\lambda_{n-1}(G)$ is also called the algebraic connectivity of $G$ [2].

The line graph of $G$, written $L(G)$, is the graph whose vertex set is in one-to-one correspondence with the edge set $E(G)$ of $G$, and whose two vertices are adjacent if and only if the corresponding edges in $G$ have a common vertex. The barycentric subdivision $B(G)$ of $G$ is the graph obtained from $G$ by inserting a vertex to each edge of $G$. More precisely, $V(B(G)) = V(G) \cup \{v_e | e = \{u, v\} \in E(G)\}$, where $v_e \notin V(G)$, and $E(B(G)) = \{\{u, v_e\}, \{v_e, v\} | e = \{u, v\} \in E(G)\}$. Inspired by self-similar structures of Sierpiński graphs (see, e.g., [3, 4]), Hasunuma [5] introduced recently the subdivided-line graph operation $\mathcal{G}$.

Definition 1.1. The subdivided-line graph $\mathcal{G}(G)$ of $G$ is the line graph of the barycentric subdivision of $G$, namely, $\mathcal{G}(G) = L(B(G))$.

The subdivided-line graph $\mathcal{G}(G)$, combining both notions of line graph and barycentric subdivision, generalizes the class of Sierpiński-like graphs. Various structural properties, such as edge-disjoint Hamilton cycles, hamiltonian-
connectivity, hub sets, connected dominating sets, independent spanning trees, and book-embeddings, have been systematically investigated in [5].

Among numerous graph-theoretic concepts, spanning trees have found a wide range of applications in mathematics, chemistry, physics and computer sciences. Denote by \( \tau(G) \) the number of spanning trees in \( G \). Enumeration of spanning trees in graphs with certain symmetry and fractals has been widely studied via ad hoc techniques capitalizing on the particular structures \([6–11]\). In general, we often have to resort to Kirchhoff’s celebrated matrix-tree theorem \([12]\), which asserts that \( n(G) = \prod_{i=1}^{n-1} \lambda_i(G) \). However, numerical computation for large graphs is notoriously difficult since the calculation of eigenvalues is \( \text{NP} \)-hard with respect to graph size \([13]\). Our first main result in this paper is an exact formula for enumeration of spanning trees in \( \Gamma(G) \). To obtain \( \tau(\Gamma(G)) \), the idea of electrically equivalent transformations \([14]\) will be applied, which enables us to determine the relationship of the numbers of spanning trees in networks before and after the transformation.

The Laplacian Estrada index of a (molecular) graph \( G \) with \( n \) vertices is defined as \([15]\)

\[
\text{LEE}(G) = \sum_{i=1}^{n} e^{\lambda_i(G)}.
\]

It is a close relative of the so-called Estrada index put forward by Estrada \([16]\) in 2000, which has already found extensive applications in chemistry and physics. Many properties of \( \text{LEE} \), including upper/lower bounds and extremal graphs, have been established (see e.g. \([15, 17–20]\)). Here, to deal with \( \text{LEE}(\Gamma(G)) \), we first derive bounds for the largest and second smallest eigenvalues \( \lambda_1(\Gamma(G)) \) and \( \lambda_{|V(\Gamma(G))|-1}(\Gamma(G)) \). Based on these estimates and the obtained exact expression for \( \tau(\Gamma(G)) \), we manage to present upper and lower bounds for \( \text{LEE}(\Gamma(G)) \) in terms of some basic graph parameters of \( G \), including degrees and the number of edges.

2 Preliminaries

To begin with, we briefly review the electrically equivalent transformation technique introduced in \([14]\).

An edge-weighted graph \( G \) (with the weight function \( w : E(G) \rightarrow [0, \infty) \)) can be considered as an electrical network with the weights being the conductances of the corresponding edges. The weighted number of spanning trees in \( G \) is defined as

\[
\sigma(G) = \sum_{T \in T(G)} \prod_{e \in E(T)} w(e),
\]

where \( T(G) \) denotes the set of spanning trees of \( G \). Evidently, \( \tau(G) = \sigma(G) \) if \( G \) is a simple graph, namely, \( w(e) = 1 \) for every \( e \in E(G) \). Two edge-weighted graphs \( G \) and \( H \) are called electrically equivalent with respect to \( \Theta \subseteq V(G) \cap V(H) \), if they cannot be distinguished by applying voltages to \( \Theta \) and measuring the resulting currents on \( \Theta \). In \([14]\), Teuff and Wagner showed that if a subgraph of a graph \( G \) is replaced by an electrically equivalent graph (setting the resulting graph \( G' \)), the weighted number of spanning trees only changes by an explicit factor. The effect of each of the two electrically equivalent transformations that will be used later is described as follows.

- Serial edges transformation: If two serial edges with conductances \( a \) and \( b \) are merged into a single edge with conductance \( \frac{ab}{a+b} \), we have \( \sigma(G') = \frac{1}{a+b} \cdot \sigma(G) \).
- Mesh-star transformation: If a complete graph \( K_t \) \( (t \geq 2) \) with conductance \( a \) on all its edges is changed into a star \( K_{1,t} \) with conductance \( ta \) on all its edges, we have \( \sigma(G') = t^2 a \cdot \sigma(G) \).

Fig. 1 shows an example of the above electrically equivalent transformations.

The following two lemmas on the Laplacian eigenvalues will be used in our proofs.

**Lemma 2.1** \([21]\). Let \( G \) be a simple graph. Then

\[
\lambda_1(G) \leq \max_{\{u,v\} \in E(G)} \{d_G(u) + d_G(v)\}.
\]

If \( G \) is connected then the equality holds if and only if \( G \) is bipartite semiregular. Here, a semiregular graph \( G = (V, E) \) is a graph with bipartition \( (V_1, V_2) \) of \( V \) such that all vertices in \( V_i \) have the same degree \( k_i \) for \( i = 1, 2 \).
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Fig. 1. An example of serial edges and mesh-star transformations

![Diagram showing serial edges transformation and mesh-star transformation](image)

Lemma 2.2 ([22]). Let $G$ be a simple connected graph with $n \geq 2$ vertices and $m$ edges. Then

$$\lambda_{n-1}(G) \geq \frac{2m - \sqrt{m(n-2)(n^2 - 2m - n)}}{n - 1}$$

with equality if and only if $G$ is a complete graph.

To conclude this section, we present an inequality which will be instrumental in bounding $LEE(\Gamma(G))$ later. It is also interesting in its own right.

Lemma 2.3. Given an integer $n \geq 1$ and a sequence $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$, we have

$$\sum_{i=1}^{n} a_i \leq n\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} + (n-1)(a_1 - a_n).$$

The equality holds if $a_1 = \ldots = a_n$.

Proof. Notice that $\sum_{i=2}^{n-1} a_i \leq (n-2)a_1$ and $a_n^2 \leq \prod_{i=1}^{n} a_i$. Therefore, $\sum_{i=2}^{n-1} a_i + n a_n \leq (n-2)a_1 + n\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}$. The result follows immediately. The equality condition is also clear.

3 Number of spanning trees related to degree sequence

The main result in this section is the following exact formula for the number of spanning trees in $\Gamma(G)$ in terms of the degree sequence of $G$.

Theorem 3.1. Let $G$ be a simple connected graph. Then

$$\tau(\Gamma(G)) = \prod_{v \in V(G)} \frac{(d(v) + 2)^{d(v)}}{d^2(v)} \sum_{T \in T(G)} \left(\prod_{e = \{u,v\} \in E(T)} \frac{d(u)d(v)}{(d(u) + 2)(d(v) + 2)} \cdot \prod_{e = \{u,v\} \in E(G) \setminus E(T)} \frac{d(u)d(v) + d(u) + d(v)}{(d(u) + 2)(d(v) + 2)}\right).$$

(3)

where $T(G)$ is the set of spanning trees in $G$, and $d(v) := d_G(v)$ for short.

Proof. First, recall that a vertex of a graph is said to be pendant if its neighborhood contains exactly one vertex. The edge incident to a pendant vertex is called a pendant edge. Let $\overline{B(G)}$ represent the graph obtained from $B(G)$ by subdividing all pendant edges in $B(G)$ (if they exist). Clearly, by Definition 1.1 we have

$$\tau(\Gamma(G)) = \tau(\mathcal{L}(\overline{B(G)})).$$

(4)

See Fig. 2 for an illustration (the purple node is inserted to subdivide the pendant edge).
To proceed, we will need the following edge-weighted graphs \[23\]. Define \( H \) as the edge-weighted version of \( G \) with the weight function \( w : E(G) \rightarrow [0, \infty) \) satisfying \( w(e) = \frac{d(u)v(v)}{d(u)+d(v)} \) for \( e = \{u, v\} \in E(G) \). Define \( H' \) as the edge-weighted version of \( B(G) \) with the weight function \( w' : E(B(G)) \rightarrow [0, \infty) \) satisfying \( w'(e) = \frac{d(u)v(v)}{d(u)+d(v)} \) for \( e = \{u, v\} \in E(B(G)) \). Define \( H'' \) as the edge-weighted version of \( B(B(G)) \) with the weight function \( w'' \) such that \( w''(e) = d(u) \) for \( e = \{u, v\} \in E(B(B(G))) \) with \( u \in V(B(G)) \). Finally, define \( H''' \) as the edge-weighted version of \( L(B(G)) \) with the weight function \( w''' \) such that \( w'''(e) \equiv 1 \) for every \( e \in E(L(B(G))) \).

It is critical to observe that, as electrical networks, \( H'' \) can be obtained from \( H''' \) by performing a series of mesh-star transformations taking each vertex \( v \in V(B(G)) \) as the center of the star (see e.g. the blue and red nodes in Fig. 2). Hence, it follows from (2) and the effect of mesh-star transformation that

\[
\tau(L(B(G))) = \sigma(H''') = \sigma(H'') \cdot \prod_{v \in V(B(G))} \frac{1}{d^2(v)} = \frac{1}{4^m} \cdot \sigma(H') \cdot \prod_{v \in V(G)} \frac{1}{d^2(v)}. \tag{5}
\]

where \( m := |E(G)| \) since each vertex in \( V(B(G)) \setminus V(G) \) has degree two and \( |V(B(G)) \setminus V(G)| = m \) (see e.g. the red nodes in Fig. 2).

Since \( H' \) can be obtained from \( H'' \) by applying a series of serial edges transformations, we have

\[
\sigma(H'') = \sigma(H') \cdot \prod_{e = \{u, v\} \in E(B(G))} (d(u) + d(v)) = \sigma(H') \cdot \prod_{v \in V(G)} (d(v) + 2)^{d(v)}, \tag{6}
\]

where the second equality holds since each edge in \( B(G) \) must have a degree-two vertex. Likewise, we obtain

\[
\sigma(H') = \sigma(H) \cdot \prod_{e = \{u, v\} \in E(G)} \left( \frac{2d(u)}{d(u)+2} + \frac{2d(v)}{d(v)+2} \right) = 4^m \cdot \sigma(H) \cdot \prod_{e = \{u, v\} \in E(G)} \frac{d(u)d(v) + d(u) + d(v)}{(d(u)+2)(d(v)+2)}. \tag{7}
\]

again by noting that each edge in \( B(G) \) contains a degree-two vertex.

Now, combining (5), (6), and (7) with (4), we have

\[
\tau(B(G)) = \tau(L(B(G))) = \frac{1}{4^m} \cdot \sigma(H'') \cdot \prod_{v \in V(G)} \frac{1}{d^2(v)} = \frac{1}{4^m} \cdot \sigma(H') \cdot \prod_{v \in V(G)} \frac{(d(v) + 2)^{d(v)}}{d^2(v)}.
\]
In view of (2), we obtain
\[
\sigma(H) = \prod_{v \in V(G)} \left( \frac{(d(v) + 2)^{d(v)}}{d^2(v)} \right) \prod_{\{u, v\} \in E(G)} \frac{d(u)d(v) + d(u) + d(v)}{(d(u) + 2)(d(v) + 2)}. \tag{8}
\]

In view of (2), we obtain
\[
\sigma(H) = \sum_{T \in T(G)} \prod_{e \in E(T)} \frac{d(u)d(v)}{d(u)d(v) + d(u) + d(v)}. \tag{9}
\]

Hence, we readily obtain the expression (3) for \( \tau(\Gamma(G)) \) by plugging (9) into (8). The proof is complete. \( \square \)

4 Bounds for Laplacian eigenvalues

We begin with the following upper bound for the largest Laplacian eigenvalue of a subdivided-line graph.

**Theorem 4.1.** Let \( G \) be a simple connected graph. Then
\[
\lambda_1(\Gamma(G)) \leq 2\Delta(G),
\]
where \( \Delta(G) \) is the maximum degree of \( G \). The equality holds if and only if \( G \) is a regular bipartite graph.

**Proof.** For each edge \( \{u, v\} \in E(\Gamma(G)) \), the vertices \( u \) and \( v \) correspond to two incident edges, say, \( \{u_1, u_2\}, \{u_2, u_3\} \), in \( B(G) \). If \( u \) and \( v \) have \( k \) common neighbors, then we have
\[
d_{\Gamma(G)}(u) + d_{\Gamma(G)}(v) = 2k = d_{B(G)}(u_1) + d_{B(G)}(u_3)
\]
and \( d_{B(G)}(u_2) = k + 2 \). Consequently,
\[
d_{\Gamma(G)}(u) + d_{\Gamma(G)}(v) = d_{B(G)}(u_1) + d_{B(G)}(u_3) + 2d_{B(G)}(u_2) - 4
\]
holds.

We consider two situations. (1) If \( u_2 \in V(B(G)) \setminus V(G) \), then \( d_{B(G)}(u_2) = 2 \). Hence, \( d_{\Gamma(G)}(u) + d_{\Gamma(G)}(v) = d_{G}(u_1) + d_{G}(u_3) \), where \( u_1 \) and \( u_3 \) are adjacent in \( G \). (2) If \( u_2 \in V(G) \), then \( d_{B(G)}(u_1) = d_{B(G)}(u_3) = 2 \). Hence, \( d_{\Gamma(G)}(u) + d_{\Gamma(G)}(v) = 2d_{G}(u_2) \).

Thanks to Lemma 2.1, we obtain
\[
\lambda_1(\Gamma(G)) \leq \max \left\{ 2 \max_{v \in V(G)} d_G(v), \max_{\{u, v\} \in E(G)} \{d_G(u) + d_G(v)\} \right\} \leq 2\Delta(G),
\]
with equality if and only if \( G \) is regular bipartite. \( \square \)

The first Zagreb index [24] of a graph \( G \) is defined as \( Z_G(G) = \sum_{v \in V(G)} d_G^2(v) \). The next result gives us a lower bound for the second smallest eigenvalue of \( \Gamma(G) \).

**Theorem 4.2.** Let \( G \) be a simple connected graph with \( |V(G)| \geq 2 \). Then
\[
\lambda_{|V(\Gamma(G))|-1}(\Gamma(G)) \geq \frac{Z_G(G) - \sqrt{Z_G(G)(m-1)(4m^2 - Z_G(G) - 2m)}}{2m - 1}, \tag{11}
\]
where \( 2m = 2|E(G)| = |V(\Gamma(G))| \). The equality holds if and only if \( G \) is a single edge.
Proof. We know that \(|V(\Gamma(G))| = |E(B(G))| = 2|E(G)|\) by Definition 1.1. Moreover, based on the property of line graphs (see e.g. [25, Theorem 8.1]), we have

\[
|E(\Gamma(G))| = \frac{1}{2} \sum_{v \in V(B(G))} d_{B(G)}^2(v) - |E(B(G))| = \frac{1}{2} \left( \sum_{v \in V(G)} d_G^2(v) + 4|E(G)| \right) - 2|E(G)| = \frac{1}{2} Z_g(G).
\]

Therefore, we readily arrive at (11) by employing Lemma 2.2.

We now discuss the sharpness of (11). If \(G\) is a single edge, then \(\ell(G) = 0\). Lemma 2.2 implies that the equality holds in (11). Conversely, if the equality holds in (11), it follows from Lemma 2.2 that \(\ell(G)\) must be a complete graph. But this is true only if \(G\) is a single edge. (Indeed, if \(G\) is not a single edge, \(G\) must contain a 2-path \(P_2\). Clearly, there are two vertices in \(\ell(P_2)\) that are not adjacent, and hence \(\ell(G)\) cannot be complete.)

This completes the proof. \(\square\)

In [26], Mohar showed that \(\lambda_{n-1}(G) \geq \frac{4}{n \text{diam}(G)}\), where \(G\) is a simple connected graph with \(n\) vertices and diameter \(\text{diam}(G)\). Since the line graph can change the diameter only by at most one, up or down [27, 28], we obtain, in particular,

\[
\text{diam}(\ell(G)) \leq \text{diam}(B(G)) + 1 \leq 2\text{diam}(G) + 1.
\]

Hence,

\[
\lambda_{2m-1}(\ell(G)) \geq \frac{2}{m(2\text{diam}(G) + 1)},
\]

Obviously, the bounds of (11) and (12) are incomparable.

5 Bounds for Laplacian Estrada index

In the light of the matrix-tree theorem which relates the Laplacian eigenvalues to the number of spanning trees, we in this section convert the above obtained results into bounds of the Laplacian Estrada index \(LEE(\ell(G))\).

Theorem 5.1. Let \(G\) be a simple connected graph with \(|V(G)| \geq 2\). Then

\[
2m + (2m - 1) \left( e^{(2m\tau(\ell(G)))^{\frac{1}{2m-1}}} - 1 \right) \leq LEE(\ell(G))
\]

\[
\leq (2m - 1) \left( e^{(2m\tau(\ell(G)))^{\frac{1}{2m-1}}} - 1 \right) + (2m - 2) \left( \frac{m}{m - 1} + e^{\lambda_1(\ell(G))} - e^{\lambda_{2m-1}(\ell(G))} \right),
\]

where \(2m = 2|E(G)| = |V(\ell(G))|\). In the first inequality, equality holds if and only if \(G\) is a single edge, while the second equality holds if \(G\) is a single edge.

Furthermore,

\[
LEE(\ell(G)) \leq (2m - 1) \left( e^{(2m\tau(\ell(G)))^{\frac{1}{2m-1}}} - 1 \right) + (2m - 2) \left( \frac{m}{m - 1} + e^{2\Delta(G)} - e^{Z_{\ell(G)} - \sqrt{Z_{\ell(G)}(2m(2m-1)4m^2-4m^2-Z_{\ell(G)}-2m)}} \right)
\]

with equality if \(G\) is a single edge.

Proof. By (1) and \(2m = 2|E(G)| = |V(\ell(G))|\),

\[
LEE(\ell(G)) = \sum_{i=1}^{2m} e^{\lambda_i(\ell(G))} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i=1}^{2m} \lambda_i^k(\ell(G)) = 2m + \sum_{k=1}^{\infty} \sum_{i=1}^{2m-1} \frac{\lambda_i^k(\ell(G))}{k!}
\]

where we have used the fact that \(\ell(G)\) is connected (and hence \(\lambda_{2m}(\ell(G)) = 0\).
Recall that the matrix-tree theorem tells us that $\tau(\Gamma(G)) = \frac{1}{2m} \prod_{i=1}^{2m-1} \lambda_i(\Gamma(G))$, where $\tau(\Gamma(G))$ is given by (3). The first inequality in (13) follows from (15) and the arithmetic-geometric mean inequality:

$$\text{LEE}(\Gamma(G)) \geq 2m + \sum_{k=1}^{\infty} \frac{2m - 1}{k!} \left( \prod_{i=1}^{2m-1} \lambda_i(\Gamma(G)) \right)^{\frac{k}{2m-1}} = 2m + (2m - 1) \sum_{k=1}^{\infty} \frac{1}{k!} \left( 2m \tau(\Gamma(G)) \right)^{\frac{k}{2m-1}}$$

$$= 2m + (2m - 1) \left( e^{2m \tau(\Gamma(G))} - 1 \right),$$

where the equality holds if and only if $\Gamma(G)$ is a complete graph, which is again equivalent to the condition that $G$ is a single edge (see the proof of Theorem 4.2).

For the second inequality in (13), we need to resort to Lemma 2.3. Similarly, we have

$$\text{LEE}(\Gamma(G)) \leq 2m + \sum_{k=1}^{\infty} \frac{2m - 1}{k!} \left( \prod_{i=1}^{2m-1} \lambda_i(\Gamma(G)) \right)^{\frac{k}{2m-1}} + \sum_{k=1}^{\infty} \frac{2m - 2}{k!} \left( \lambda_k(\Gamma(G)) - \lambda_{2m-1}(\Gamma(G)) \right)$$

$$= (2m - 1) \left( e^{2m \tau(\Gamma(G))} - 1 \right) + (2m - 2) \left( e^{\lambda_1(\Gamma(G))} - e^{\lambda_{2m-1}(\Gamma(G))} \right),$$

where the equality holds if $G$ is a single edge.

The last statement concerning the inequality (14) follows by applying Theorem 4.1 and Theorem 4.2 to (13). This completes the proof.

To show the availability of Theorem 5.1, we still use the graph $G$ depicted in Fig. 2 as an example. Direct calculation shows $\text{LEE}(\Gamma(G)) = \sum_{i=1}^{10} e^{\lambda_i(\Gamma(G))} = 259.7$. The respective lower bound and upper bound are 57.1 and 772.1 by Theorem 5.1.

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