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A remark on local fractional calculus and ordinary derivatives

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Abstract: In this short note we present a new general definition of local fractional derivative, that depends on an unknown kernel. For some appropriate choices of the kernel we obtain some known cases. We establish a relation between this new concept and ordinary differentiation. Using such formula, most of the fundamental properties of the fractional derivative can be derived directly.

Keywords: Local fractional derivative, Conformable derivative

MSC: 26A33, 26A24

1 Introduction

Fractional calculus is a generalization of ordinary calculus, where derivatives and integrals of arbitrary real or complex order are defined. These fractional operators may model more efficiently certain real world phenomena, especially when the dynamics is affected by constraints inherent to the system. There exist several definitions for fractional derivatives and fractional integrals, like the Riemann–Liouville, Caputo, Hadamard, Riesz, Grünwald–Letnikov, Marchaud, etc. (see e.g., [14, 17] and references therein). Although most of them are already well-studied, some of the usual features concerning the differentiation of functions fail, like the Leibniz rule, the chain rule, the semigroup property, to name a few. As it was mentioned in [6], “These definitions, however, are non-local in nature, which makes them unsuitable for investigating properties related to local scaling or fractional differentiability”. Recently, the concept of local fractional derivatives have gained relevance, namely because they kept some of the properties of ordinary derivatives, although they loss the memory condition inherent to the usual fractional derivatives. For example, in [9], a concept similar to the Caputo fractional derivative is presented, but the first-order derivative $f'(t)$ is replaced by another operator: $\lim_{\epsilon \rightarrow 0} (f(x + \epsilon t) - f(x))/\epsilon^\alpha$; in [12], the local fractional derivative is given by $f^{(\alpha)}(t) = \lim_{\epsilon \rightarrow 0} (f(t \exp(\epsilon t^{-\alpha}) - f(t)))/\epsilon$, and some fundamental properties like the algebraic rules or the mean value theorem are obtained; later, in [3], the same concept of local fractional derivative is considered and an anti-derivative operator is defined, as well some applications to quantum mechanics; in [15, 16], the local fractional derivative is defined by the expression $f^{(\alpha)}(t) = \lim_{x \rightarrow t} (D^\alpha(f(x) - f(t)))$, where D^α denotes the Riemann–Liouville fractional derivative, and it is proven that the Weierstrass Function is α -differentiable. One question is what is the best local fractional derivative definition that we should consider, and the answer is not unique. Similarly to what happens to the classical definitions of fractional operators, the best choice depends on the experimental data that fits better in the theoretical model, and because of this we find already a vast number of definitions for local fractional derivatives.

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2 Local fractional derivative

We present a definition of local fractional derivative using kernels.

Definition 2.1. Let $k : [a, b] \rightarrow \mathbb{R}$ be a continuous nonnegative map such that $k(t) \neq 0$, whenever $t > a$. Given a function $f : [a, b] \rightarrow \mathbb{R}$ and $\alpha \in (0, 1)$ a real, we say that f is α -differentiable at $t > a$, with respect to kernel k , if the limit

$$f^{(\alpha)}(t) := \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon k(t)^{1-\alpha}) - f(t)}{\epsilon} \quad (1)$$

exists. The α -derivative at $t = a$ is defined by

$$f^{(\alpha)}(a) := \lim_{t \rightarrow a^+} f^{(\alpha)}(t),$$

if the limit exists.

Consider the limit $\alpha \rightarrow 1^-$. In this case, for $t > a$, we obtain the classical definition for derivative of a function, $f^{(\alpha)}(t) = f'(t)$. Our definition is a more general concept, compared to others that we find in the literature. For example, taking $k(t) = t$ and $a = 0$, we get the definition from [7, 8, 10, 11, 13] (also called conformable fractional derivative); when $k(t) = t - a$, the one from [1, 2, 18]; for $k(t) = t + 1/\Gamma(\alpha)$, the definition in [4, 5].

The following result is trivial, and we omit the proof.

Theorem 2.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function and $t > a$. Then, f is α -differentiable at t and

$$f^{(\alpha)}(t) = k(t)^{1-\alpha} f'(t), \quad t > a.$$

Also, if f' is continuous at $t = a$, then

$$f^{(\alpha)}(a) = k(a)^{1-\alpha} f'(a).$$

However, there exist α -differentiable functions which are not differentiable in the usual sense. For example, consider the function $f(t) = \sqrt{t}$, with $t \geq 0$. If we take the kernel $k(t) = t$, then $f^{(\alpha)}(t) = 1/2 t^{1/2-\alpha}$. Thus, for $\alpha \in (0, 1/2)$, $f^{(\alpha)}(0) = 0$ and for $\alpha = 1/2$, $f^{(\alpha)}(0) = 1/2$. In general, if we consider the function $f(t) = \sqrt[n]{t}$, with $t \geq 0$ and $n \in \mathbb{N} \setminus \{1\}$, we have $f^{(\alpha)}(t) = 1/n t^{1/n-\alpha}$ and so $f^{(\alpha)}(0) = 0$ if $\alpha \in (0, 1/n)$ and for $\alpha = 1/n$, $f^{(\alpha)}(0) = 1/n$.

Theorem 2.3. If $f^{(\alpha)}(t)$ exists for $t > a$, then f is differentiable at t and

$$f'(t) = k(t)^{\alpha-1} f^{(\alpha)}(t).$$

Proof. It follows from

$$f'(t) = \lim_{\delta \rightarrow 0} \frac{f(t + \delta) - f(t)}{\delta} = k(t)^{\alpha-1} \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon k(t)^{1-\alpha}) - f(t)}{\epsilon} = k(t)^{\alpha-1} f^{(\alpha)}(t). \quad \square$$

Of course we can not conclude anything at the initial point $t = a$, as was discussed before.

Combining the two previous results, we have the main result of our paper.

Theorem 2.4. A function $f : [a, b] \rightarrow \mathbb{R}$ is α -differentiable at $t > a$ if and only if it is differentiable at t . In that case, we have the relation

$$f^{(\alpha)}(t) = k(t)^{1-\alpha} f'(t), \quad t > a. \quad (2)$$

3 Conclusion

In this short note we show that some of the existent notions about local fractional derivative are very close related to the usual derivative function. In fact, the α -derivative of a function is equal to the first-order derivative, multiplied

by a continuous function. Also, using formula (2), most of the results concerning α -differentiation can be deduced trivially from the ordinary ones. In the authors opinion, local fractional calculus is an interesting idea and deserves further research, but definitions like (1) are not the best ones and a different path should be followed.

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