Integrals of Frullani type and the method of brackets

1 Introduction

The integral
\[ \int_{0}^{\infty} \frac{e^{-ax} - e^{-bx}}{x} \, dx = \log \left( \frac{b}{a} \right) \]  
(1)
appears as entry 3.434.2 in [12]. It is one of the simplest examples of the so-called Frullani integrals. These are examples of the form
\[ S(a, b) = \int_{0}^{\infty} \frac{f(ax) - f(bx)}{x} \, dx, \]  
(2)
and Frullani's theorem states that
\[ S(a, b) = [f(0) - f(\infty)] \log \left( \frac{b}{a} \right). \]  
(3)
The identity (3) holds if, for example, \( f' \) is a continuous function and the integral in (3) exists. Other conditions for the validity of this formula are presented in [3, 13, 16]. The reader will find in [1] a systematic study of the Frullani integrals appearing in [12].

The goal of the present work is to use the method of brackets, a new procedure for the evaluation of definite integrals, to compute a variety of integrals similar to those in (1). The method itself is described in Section 2. This is based on a small number of heuristic rules, some of which have been rigorously established [2, 8]. The point to be stressed here is that the application of the method of brackets is direct and it reduces the evaluation of a definite integral to the solution of a linear system of equations.
2 The method of brackets

A method to evaluate integrals over the half-line \([0, \infty)\), based on a small number of rules has been developed in [6, 9–11]. This method of brackets is described next. The heuristic rules are currently being placed on solid ground [2]. The reader will find in [5, 7, 8] a large collection of evaluations of definite integrals that illustrate the power and flexibility of this method.

For \(a \in \mathbb{R}\), the symbol

\[
\langle a \rangle = \int_{0}^{\infty} x^{a-1} \, dx,
\]

is the \textit{bracket} associated to the (divergent) integral on the right. The symbol

\[
\phi_n = \frac{(-1)^n}{\Gamma(n + 1)},
\]

is called the \textit{indicator} associated to the index \(n\). The notation

\[
\underbrace{n_1 n_2 \cdots n_r}_r,
\]

denotes the product \(n_1 n_2 \cdots n_r\).

Rules for the production of bracket series

\textbf{Rule P}_1. If the function \(f\) is given by the power series

\[
f(x) = \sum_{n=0}^{\infty} a_n x^{\alpha n + \beta - 1},
\]

with \(\alpha, \beta \in \mathbb{C}\), then the integral of \(f\) over \([0, \infty)\) is converted into a \textit{bracket series} by the procedure

\[
\int_{0}^{\infty} f(x) \, dx = \sum_{n=0}^{\infty} a_n \langle \alpha n + \beta \rangle.
\]

\textbf{Rule P}_2. For \(\alpha \in \mathbb{C}\), the multinomial power \((a_1 + a_2 + \cdots + a_r)^\alpha\) is assigned the \(r\)-dimension bracket series

\[
\sum_{n_1} \sum_{n_2} \cdots \sum_{n_r} \phi_{n_1, n_2, \ldots, n_r} a_1^{n_1} \cdots a_r^{n_r} \frac{(-\alpha + n_1 + \cdots + n_r)}{\Gamma(-\alpha)}.
\]

Rules for the evaluation of a bracket series

\textbf{Rule E}_1. The one-dimensional bracket series is assigned the value

\[
\sum_n \phi_n f(n) \langle an + b \rangle = \frac{1}{|a|} f(n^*) \Gamma(-n^*),
\]

where \(n^*\) is obtained from the vanishing of the bracket; that is, \(an + b = 0\). This is precisely the Ramanujan’s Master Theorem.

The next rule provides a value for multi-dimensional bracket series of index 0, that is, the number of sums is equal to the number of brackets.

\textbf{Rule E}_2. Assume the matrix \(A = (a_{ij})\) is non-singular, then the assignment is

\[
\sum_{n_1} \cdots \sum_{n_r} \phi_{n_1, \ldots, n_r} f(n_1, \cdots, n_r) (a_{11} n_1 + \cdots + a_{1r} n_r + c_1) \cdots (a_{r1} n_1 + \cdots + a_{rr} n_r + c_r)
\]

\[
= \frac{1}{|\det(A)|} f(n_1^*, \cdots, n_r^*) \Gamma(-n_1^*) \cdots \Gamma(-n_r^*)
\]

where \(\{n_i^*\}\) is the (unique) solution of the linear system obtained from the vanishing of the brackets.

\textbf{Rule E}_3. The value of a multi-dimensional bracket series of positive index is obtained by computing all the contributions of maximal rank by Rule \(E_2\). These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded.
The formula in one dimension

The goal of this section is to establish Frullani’s evaluation (3) by the method of brackets. The notation $\phi_k = (-1)^k / \Gamma(k + 1)$ is used in the statement of the next theorem.

**Theorem 3.1.** Assume $f(x)$ admits an expansion of the form

$$f(x) = \sum_{k=0}^{\infty} \phi_k C(k)x^{\alpha k},$$

for some $\alpha > 0$ with $C(0) \neq 0$ and $C(0) < \infty$. \hspace{1cm} (1)

Then,

$$S(a, b) := \int_0^\infty \frac{f(ax) - f(bx)}{x} \, dx$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\alpha} \Gamma\left( \frac{\varepsilon}{\alpha} \right) C\left( -\frac{\varepsilon}{\alpha} \right) \left( a^{-\varepsilon} - b^{-\varepsilon} \right)$$

$$= C(0) \log \left( \frac{b}{a} \right),$$

independently of $\alpha$.

**Proof.** Introduce an extra parameter and write

$$S(a, b) = \lim_{\varepsilon \to 0} \int_0^\infty \frac{f(ax) - f(bx)}{x^{1-\varepsilon}} \, dx.$$ \hspace{1cm} (3)

Then,

$$S(a, b) = \lim_{\varepsilon \to 0} \int_0^\infty \sum_{k=0}^{\infty} \phi_k C(k) \left( a^{\alpha k} - b^{\alpha k} \right) \int_0^\infty x^{\alpha k + 1-\varepsilon} \, dx$$

$$= \lim_{\varepsilon \to 0} \sum_{k=0}^{\infty} \phi_k C(k) \left( a^{\alpha k} - b^{\alpha k} \right) (\alpha k + \varepsilon).$$

The method of brackets gives

$$S(a, b) = \lim_{\varepsilon \to 0} \frac{1}{\alpha} \Gamma\left( \frac{\varepsilon}{\alpha} \right) C\left( -\frac{\varepsilon}{\alpha} \right) \left( a^{-\varepsilon} - b^{-\varepsilon} \right).$$ \hspace{1cm} (4)

The result follows from the expansions $\Gamma(\varepsilon/\alpha) = \alpha/\varepsilon - \gamma + O(\varepsilon)$, $C(-\varepsilon/\alpha) = C(0) + O(\varepsilon)$ and $a^{-\varepsilon} - b^{-\varepsilon} = (\log b - \log a) \varepsilon + O(\varepsilon^2)$. \hfill \square

In the examples given below, observe that $C(0) = f(0)$ and that $f(\infty) = 0$ is imposed as a condition on the integrand.

**Example 3.2.** Entry 3.434.2 of [12] states the value

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \, dx = \log \frac{b}{a}.$$ \hspace{1cm} (5)

This follows directly from (2).

**Note 3.3.** The method of brackets gives a direct approach to Frullani style problems if the expansion (1) is replaced by the more general one

$$f(x) = \sum_{k=0}^{\infty} \phi_k C(k)x^{\alpha k + \beta},$$

with $\beta \neq 0$ and if the function $f$ does not necessarily have a limit at infinity.
Example 3.4. Consider the evaluation of

$$I = \int_0^\infty \frac{\sin ax - \sin bx}{x} \, dx,$$

for $a, b > 0$. The integral is evaluated directly as

$$I = \int_0^\infty \frac{\sin ax}{x} \, dx - \int_0^\infty \frac{\sin bx}{x} \, dx,$$

and since $a, b > 0$, both integrals are $\pi/2$, giving $I = 0$. The classical version of Frullani theorem does not apply, since $f(x)$ does not have a limit as $x \to \infty$. Ostrowski [15] shows that in the case $f(x)$ is periodic of period $p$, the value $f(\infty)$ might be replaced by

$$\frac{1}{p} \int_0^p f(x) \, dx.$$

In the present case, $f(x) = \sin x$ has period $2\pi$ and mean 0. This yields the vanishing of the integral.

The computation of (7) by the method of brackets begins with the expansion

$$\sin x = x \cdot {}_0 F_1 \left( \frac{n}{2} \mid -\frac{1}{4} x^2 \right).$$

Here

$$p_F q \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right) = \sum_{n=0}^{\infty} \frac{(a_1) n \cdots (a_p) n}{(b_1) n \cdots (b_q) n} \frac{z^n}{n!},$$

with $(a)_n = a(a+1) \cdots (a+n-1)$, is the classical hypergeometric function. The integrand has the series expansion

$$\sum_{n \geq 0} \phi_n \frac{(a_2 n + 1) - b_2 n + 1}{(\frac{3}{2})_n n!} \frac{1}{4^n} x^{2n},$$

that yields

$$I = \sum_n \phi_n \frac{(a_2 n + 1) - b_2 n + 1}{(\frac{3}{2})_n n!} \frac{1}{4^n} (2n + 1).$$

The vanishing of the bracket gives $n^* = -1/2$ and the bracket series vanishes in view of the factor $a^{2n+1} - b^{2n+1}$.

Example 3.5. The next example is the evaluation of

$$I = \int_0^\infty \frac{\cos ax - \cos bx}{x} \, dx = \log \left( \frac{b}{a} \right),$$

for $a, b > 0$. The expansion

$$\cos x = \sum_{n=0}^{\infty} \phi_n \frac{n!}{(2n)!} x^{2n},$$

and $C(n) = \frac{n!}{(2n)!} = \frac{\Gamma(n+1)}{\Gamma(2n+1)}$ in (1). Then $C(0) = 1$ and the integral is $I = \log \left( \frac{b}{a} \right)$, as claimed.

Example 3.6. The integral

$$I = \int_0^\infty \frac{\tan^{-1}(e^{-ax}) - \tan^{-1}(e^{-bx})}{x} \, dx,$$

is evaluated next. The expansion of the integrand is

$$\tan^{-1}(e^{-t}) = e^{-t} \cdot {}_2 F_1 \left( \frac{1}{2}, 1 \mid -e^{-2t} \right)$$
Integrals of Frullani type and the method of brackets

\[ \int \frac{f(ax) - f(bx)}{x} \, dx \]

\[ = \frac{1}{2} \sum_{n=0}^{\infty} \phi_n \frac{\Gamma(n + \frac{1}{2}) \Gamma(n + 1)}{\Gamma(n + \frac{3}{2})} \sum_{k=0}^{\infty} \phi_k (2n + 1)^k \]

\[ = \sum_{k=0}^{\infty} \phi_k \left[ \frac{1}{2} \sum_{n=0}^{\infty} \phi_n \frac{\Gamma(n + \frac{1}{2}) \Gamma(n + 1)}{\Gamma(n + \frac{3}{2})} (2n + 1)^k \right] i^k. \]

Therefore,

\[ C(k) = \frac{1}{2} \sum_{n=0}^{\infty} \phi_n \frac{\Gamma(n + \frac{1}{2}) \Gamma(n + 1)}{\Gamma(n + \frac{3}{2})} (2n + 1)^k. \]

and from here it follows that

\[ C(0) = \frac{1}{2} \sum_{n=0}^{\infty} \phi_n \frac{\Gamma(n + \frac{1}{2}) \Gamma(n + 1)}{\Gamma(n + \frac{3}{2})} = \tan^{-1}(1) = \frac{\pi}{4}. \]

Thus, the integral is

\[ I = C(0) \log \left( \frac{b}{a} \right) = \frac{\pi}{4} \log \left( \frac{b}{a} \right). \]

4 A first generalization

This section describes examples of Frullani-type integrals that have an expansion of the form

\[ f(x) = \sum_{k \geq 0} \phi_k C(k) x^{\alpha k + \beta}. \]

with \( \beta \neq 0 \).

Theorem 4.1. Assume \( f(x) \) admits an expansion of the form (20). Then,

\[ S(a, b) = \int_0^\infty \frac{f(ax) - f(bx)}{x} \, dx \]

\[ = \lim_{\varepsilon \to 0} \frac{1}{|a|} \Gamma \left( \frac{\beta + \varepsilon}{\alpha} \right) C \left( \frac{-\beta}{\alpha} - \frac{\varepsilon}{\alpha} \right) \left[ a^{-\varepsilon} - b^{-\varepsilon} \right]. \]

Proof. The method of brackets gives

\[ S(a, b; \varepsilon) = \int_0^\infty \frac{f(ax) - f(bx)}{x^{1-\varepsilon}} \, dx \]

\[ = \sum_{k \geq 0} \phi_k C(k) \left[ a^{\alpha k + \beta} - b^{\alpha k + \beta} \right] \int_0^\infty x^{\alpha k + \beta + \varepsilon - 1} \, dx \]

\[ = \sum_{k} \phi_k C(k) \left[ a^{\alpha k + \beta} - b^{\alpha k + \beta} \right] \Gamma(k + \beta + \varepsilon) \]

\[ = \frac{1}{|a|} \Gamma(-k) C(k) \left[ a^{\alpha k + \beta} - b^{\alpha k + \beta} \right] \]

with \( k = -(\beta + \varepsilon)/\alpha \) in the last line. The result follows by taking \( \varepsilon \to 0 \).

Example 4.2. The integral

\[ \int_0^\infty \frac{\tan^{-1} ax - \tan^{-1} bx}{x} \, dx = \frac{\pi}{2} \log \left( \frac{b}{a} \right) \]
appears as entry 4.536.2 in [12]. It is evaluated directly by the classical Frullani theorem. Its evaluation by the method of brackets comes from the expansion

\[
\tan^{-1} x = x \cdot 2 \, _2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; -x^2 \right)
\]

\[
= \sum_{k \geq 0} \phi_k \frac{(\frac{1}{2})_k \, (1)_k}{(\frac{1}{2})_k} x^{2k+1}.
\]

Therefore, \( \alpha = 2, \beta = 1 \) and

\[
C(k) = \frac{\Gamma \left( \frac{1}{2} + k \right) \Gamma (1 + k)}{2 \Gamma \left( \frac{3}{2} + k \right)} = \frac{\Gamma (1 + k)}{2k + 1}.
\]

Then

\[
\int_0^\infty \frac{\tan^{-1} ax - \tan^{-1} bx}{x} \, dx = \lim_{\varepsilon \to 0} \frac{1}{2} \Gamma \left( \frac{1 + \varepsilon}{2} \right) \Gamma \left( \frac{1 - \varepsilon}{2} \right) \left[ a^{-\varepsilon} - b^{-\varepsilon} \right]
\]

\[
= \lim_{\varepsilon \to 0} \frac{1}{2} \Gamma \left( \frac{1 + \varepsilon}{2} \right) \Gamma \left( \frac{1 - \varepsilon}{2} \right) \left[ a^{-\varepsilon} - b^{-\varepsilon} \right] \frac{a - b}{-\varepsilon}
\]

\[
= -\frac{\pi}{2} \log \left( \frac{b}{a} \right).
\]

5 A second class of Frullani type integrals

Let \( f_1, \ldots, f_N \) be a family of functions. This section uses the method of brackets to evaluate

\[
I = I(f_1, \ldots, f_N) = \int_0^\infty \frac{1}{x} \sum_{k=1}^{N} f_k(x) \, dx,
\]

subject to the condition \( \sum_{k=1}^{N} f_k(0) = 0 \), required for convergence.

The functions \( \{f_k(x)\} \) are assumed to admit a series representation of the form

\[
f_k(x) = \sum_{n=0}^{\infty} \phi_n C_k(n) x^{\alpha n},
\]

where \( \alpha > 0 \) is independent of \( k \) and \( C_k(0) \neq 0 \). The coefficients \( C_k \) are assumed to admit a meromorphic extension from \( n \in \mathbb{N} \) to \( n \in \mathbb{C} \).

**Theorem 5.1.** The integral \( I \) is given by

\[
I = -\frac{1}{|\alpha|} \sum_{k=1}^{N} C_k'(0),
\]

where

\[
C_k'(0) = \left. \frac{dC_k(\varepsilon)}{d\varepsilon} \right|_{\varepsilon=0}.
\]

**Proof.** The proof begins with the expansion

\[
\frac{f_k(x)}{x^{1-\varepsilon}} = \sum_{n=0}^{\infty} \phi_n C_k(n) x^{\alpha n - 1 + \varepsilon}
\]
and the bracket series for the integral is

$$I = \lim_{\varepsilon \to 0} \sum_n \phi_n \left( \sum_{k=1}^N C_k(n) \right) (\alpha n + \varepsilon)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\alpha} \Gamma \left( -\frac{\varepsilon}{\alpha} \right) \sum_{k=1}^N C_k \left( -\frac{\varepsilon}{\alpha} \right).$$

The result follows by letting $\varepsilon \to 0$. \hfill \Box

**Example 5.2.** Entry 3.4.29 in [12] states that

$$I = \int_0^\infty \left[ e^{-x} - (1 + x)^{-\mu} \right] \frac{dx}{x} = \psi(\mu).$$

where $\mu > 0$ and $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function. This is one of many integral representation for this basic function. The reader will find a classical proof of this identity in [14]. The method of brackets gives a direct proof.

The functions appearing in this example are

$$f_1(x) = e^{-x} = \sum_{n=0}^\infty \phi_n x^n,$$

and

$$f_2(x) = -(1 + x)^{-\mu} = -\sum_{n=0}^\infty \phi_n (\mu)_n x^n,$$

where $(\mu)_n = \mu(\mu+1)\cdots(\mu+n-1)$ is the Pochhammer symbol (this comes directly from the binomial theorem).

The condition $f_1(0) + f_2(0) = 0$ is satisfied and the coefficients are identified as

$$C_1(n) = 1 \text{ and } C_2(n) = -(\mu)_n = -\frac{\Gamma(\mu + n)}{\Gamma(\mu)}.$$

Then, $C_1'(0) = 0$ and $C_2'(0) = -\frac{\Gamma'(\mu)}{\Gamma(\mu)}$. This gives the evaluation.

**Example 5.3.** The elliptic integrals $K(x)$ and $E(x)$ may be expressed in hypergeometric form as

$$K(x) = \frac{\pi}{2} {}_2F_1 \left( \frac{1}{2}, \frac{1}{2} \mid x^2 \right) \text{ and } E(x) = \frac{\pi}{2} {}_2F_1 \left( \frac{-1}{2}, \frac{1}{2} \mid x^2 \right).$$

The reader will find information about these integrals in [4, 17].

Theorem 5.1 is now used to establish the value

$$\int_0^\infty \frac{e^{-ax^2} - K(bx) - E(cx)}{x} \, dx = \frac{\pi}{2} \left[ \log \left( \frac{bc}{a} \right) - \gamma - 4 \log 2 + 1 \right].$$

Here $\gamma = -\Gamma'(1)$ is Euler’s constant.

The first step is to compute series expansions of each of the terms in the integrand. The exponential term is easy:

$$\pi e^{-ax^2} = \pi \sum_{n_1=0}^\infty \frac{(-a)^{n_1}}{n_1!} = \sum_{n_1} \phi_{n_1} a^{n_1} x^{2n_1},$$

and this gives $C_1(n) = a^n$. For the first elliptic integral,

$$K(bx) = \frac{\pi}{2} {}_2F_1 \left( \frac{1}{2}, \frac{1}{2} \mid b^2 x^2 \right).$$
\[
\begin{align*}
&= \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{\binom{\frac{1}{2}}{n_2} \binom{\frac{1}{2}}{n_2} b^{2n_2} 2^{2n_2}}{(1)_{n_2} n_2!} x^{2n_2} \\
&= \sum_{n_2} \phi_{n_2} \frac{\pi}{2} \left( \frac{(-1)^{n_2} b^{2n_2}}{n_2!} \right) x^{2n_2}.
\end{align*}
\]

Therefore,
\[
C_2(n) = \frac{\pi}{2} \cos(\pi n) \Gamma^2(n + \frac{1}{2}) b^{2n},
\]
where the term \((-1)^n\) has been replaced by \(\cos(\pi n)\). A similar calculation gives
\[
C_3(n) = \frac{\pi}{4} \cos(\pi n) \Gamma(n - \frac{1}{2}) \Gamma(n + \frac{1}{2}) c^{2n}.
\]

A direct calculation gives
\[
C_1'(0) = \log a, \quad C_2'(0) = -\frac{\gamma}{2} - \log b - \psi \left( \frac{1}{2} \right) \quad \text{and} \quad C_3'(0) = -\frac{\gamma}{2} - \log c - \psi \left( -\frac{1}{2} \right).
\]

The result now comes from the values
\[
\psi \left( \frac{1}{2} \right) = -2 \log 2 - \gamma \quad \text{and} \quad \psi \left( -\frac{1}{2} \right) = -2 \log 2 - \gamma + 2.
\]

**Example 5.4.** Let \(a, b \in \mathbb{R}\) with \(a > 0\). Then
\[
\int_{0}^{\infty} \frac{\exp(-ax^2) - \cos bx}{x} \, dx = \frac{\gamma - \log a + 2 \log b}{2}.
\]

To apply Theorem 5.1 start with the series
\[
f_1(x) = e^{-ax^2} = \sum_{n} \phi_n a^n x^{2n}
\]
and
\[
f_2(x) = \cos bx = \sum_{n} \phi_n \left[ \frac{\Gamma(n + 1)}{\Gamma(2n + 1)} b^{2n} \right] x^{2n}.
\]

In both expansions \(\alpha = 2\) and the coefficients are given by
\[
C_1(n) = a^n \quad \text{and} \quad C_2(n) = \frac{\Gamma(n + 1)}{\Gamma(2n + 1)} b^{2n}.
\]

Then, \(C_1'(0) = \log a\) and \(C_2'(n) = \frac{b^{2n} \Gamma(n + 1)}{\Gamma(2n + 1)} \left[ 2 \log b + \psi(n + 1) - \psi(2n + 1) \right] \) yield \(C_2'(0) = 2 \log b - \psi(1) = 2 \log b + \gamma\). The value (17) follows from here.

**Example 5.5.** The next example in this section involves the Bessel function of order 0
\[
J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \frac{x}{2} \right)^{2n}
\]
and Theorem 5.1 is used to evaluate
\[
\int_{0}^{\infty} \frac{J_0(x) - \cos ax}{x} \, dx = \log 2a.
\]

This appears as entry 6.693.8 in [12]. The expansions
\[
J_0(x) = \sum_{n=0}^{\infty} \phi_n \frac{1}{n!} \frac{1}{2^{2n}} x^{2n} \quad \text{and} \quad \cos ax = \sum_{n=0}^{\infty} \phi_n \frac{n!}{(2n)!} a^{2n} x^{2n}.
\]
show $a = 2$ and
\[ C_1(n) = \frac{1}{\Gamma(n+1)2^{2n}} \text{ and } C_2(n) = \frac{\Gamma(n+1)}{\Gamma(2n+1)}a^{2n}. \]  
(24)

Differentiation gives
\[ C_1'(n) = -\frac{2\ln 2 + \psi(n+1)}{2^{2n}\Gamma(n+1)}, \]
and
\[ C_2'(n) = -\frac{a^{2n}\Gamma(n+1)(2\log a + \psi(n+1) - 2\psi(2n+1))}{\Gamma(2n+1)}. \]  
(26)

Then,
\[ C_1'(0) = \gamma - 2\log 2 \text{ and } C_2'(0) = -(\gamma + 2\log a), \]
and the result now follows from Theorem 5.1. The reader is invited to use the representation
\[ J_2^2(0,x) = {}_1F_2 \left( \frac{1}{2}, 1 \mid -x^2 \right). \]
(28)

to verify the identity
\[ \int_0^\infty \frac{J_2^2(x) - \cos x}{x} \, dx = \log 2. \]  
(29)

Example 5.6. The final example in this section is
\[ I = \int_0^\infty \frac{J_2^2(x) - e^{-x^2}\cos x}{x} \, dx. \]
(30)
The evaluation begins with the expansions
\[ J_0(x) = \sum_{k=0}^\infty \phi_k \frac{x^{2k}}{4^k\Gamma(k+1)} \text{ and } \cos x = \sum_{k=0}^\infty \phi_k \frac{\sqrt{\pi}}{4^k\Gamma\left(k + \frac{1}{2}\right)} \]  
(31)

Then,
\[ J_2^2(x) = \sum_{k,n} \phi_{k,n} \frac{1}{4^{k+n}\Gamma(k+1)\Gamma(n+1)} x^{2k+2n}, \]
(32)

and
\[ e^{-x^2}\cos x = \sum_{k,n} \phi_{k,n} \frac{\sqrt{\pi}}{4^k\Gamma\left(k + \frac{1}{2}\right)} x^{2k+2n}. \]  
(33)

Integration yields
\[ I = \int_0^\infty \frac{J_2^2(x) - e^{-x^2}\cos x}{x^{1-\varepsilon}} \, dx \]

\[ = \sum_{k,n} \phi_{k,n} \left[ \frac{1}{4^{k+n}\Gamma(k+1)\Gamma(n+1)} - \frac{\sqrt{\pi}}{4^k\Gamma\left(k + \frac{1}{2}\right)} \right] \int_0^\infty x^{2k+2n+\varepsilon-1} \, dx \]

\[ = \sum_{k,n} \phi_{k,n} \left[ \frac{1}{4^{k+n}\Gamma(k+1)\Gamma(n+1)} - \frac{\sqrt{\pi}}{4^k\Gamma\left(k + \frac{1}{2}\right)} \right] (2k + 2n + \varepsilon). \]

The method of brackets now gives
\[ I = \lim_{\varepsilon \to 0} \frac{1}{2} \sum_{k=0}^\infty \frac{(-1)^k \Gamma\left(k + \frac{1}{2}\right)}{k!} \left[ \frac{1}{2^{-\varepsilon}\Gamma(k+1)\Gamma(1-k-\varepsilon/2)} - \frac{\sqrt{\pi}}{2^{2k}\Gamma\left(k + \frac{1}{2}\right)} \right]. \]
The term corresponding to \( k = 0 \) gives
\[
\lim_{\varepsilon \to 0} \frac{1}{2} \Gamma \left( \frac{\varepsilon}{2} \right) \left[ \frac{1}{2^{-\varepsilon} \Gamma \left( 1 - \frac{\varepsilon}{2} \right)} - 1 \right] = \log 2 - \frac{\gamma}{2}
\] (34)

and the terms with \( k \geq 1 \) as \( \varepsilon \to 0 \) give
\[
-\frac{\sqrt{\pi}}{2} \sum_{k=1}^{\infty} \phi_k \frac{\Gamma(k)}{2^{2k} \Gamma \left( k + \frac{1}{2} \right)} = \frac{1}{4} \left( F_2 \left( \frac{1}{2}, 1 \middle| \frac{1}{2}, 1 \middle| \frac{1}{4} \right) \right)
\] (35)

Therefore,
\[
\int_{0}^{\infty} J_0^2(x) - \frac{e^{-x^2}}{x} \cos x \, dx = \frac{1}{4} \left( 4 \log 2 - 2 \gamma + 2 F_2 \left( \frac{1}{2}, 1 \middle| \frac{1}{2}, 1 \middle| \frac{1}{4} \right) \right).
\] (36)

No further simplification seems to be possible.

### 6 A multi-dimensional extension

The method of brackets provides a direct proof of the following multi-dimensional extension of Frullani’s theorem.

**Theorem 6.1.** Let \( a_j, b_j \in \mathbb{R}^+ \). Assume the function \( f \) has an expansion of the form
\[
f(x_1, \ldots, x_n) = \sum_{\ell_1, \ldots, \ell_n=0} (-1)^{\ell_1} \ell_1! \cdots (-1)^{\ell_n} \ell_n! C(\ell_1, \ldots, \ell_n) x_1^{\gamma_1} \cdots x_n^{\gamma_n},
\] (1)

where the \( \gamma_j \) are linear functions of the indices given by
\[
\gamma_1 = a_{11} \ell_1 + \cdots + a_{1n} \ell_n + \beta_1 \\
\vdots \\
\gamma_n = a_{n1} \ell_1 + \cdots + a_{nn} \ell_n + \beta_n.
\] (2)

Then,
\[
I = \int_{0}^{\infty} \cdots \int_{0}^{\infty} \frac{f(b_1 x_1, \ldots, b_n x_n) - f(a_1 x_1, \ldots, a_n x_n)}{x_1^{1+\rho_1} \cdots x_n^{1+\rho_n}} \, dx_1 \cdots dx_n
\]
\[
= \frac{1}{\det A} \lim_{\varepsilon \to 0} \left[ \left( \Gamma(-\ell_1, \ldots, \ell_n) \Gamma(-\ell_1, \ldots, \ell_n) \right) \frac{\Gamma(-\ell_1, \ldots, \ell_n)}{\Gamma(-\ell_1, \ldots, \ell_n)} \right],
\]
where \( A = (a_{ij}) \) is the matrix of coefficients in (2) and \( \ell_j^* \), \( 1 \leq j \leq n \) is the solution to the linear system
\[
\begin{align*}
\alpha_{11} \ell_1 + \cdots + \alpha_{1n} \ell_n + \beta_1 - \rho_1 + \varepsilon &= 0 \\
\vdots & \\
\alpha_{n1} \ell_1 + \cdots + \alpha_{nn} \ell_n + \beta_n - \rho_n + \varepsilon &= 0.
\end{align*}
\] (3)

**Proof.** The proof is a direct extension of the one-dimensional case, so it is omitted. \( \square \)

**Example 6.2.** The evaluation of the integral
\[
I = \int_{0}^{\infty} \int_{0}^{\infty} \frac{e^{-\mu s x^2} \cos(ax t) - e^{-\mu s t^2} \cos(b x s)}{\sqrt{s}} \, ds \, dt
\] (4)
Integral of Frullani type and the method of brackets

The method of brackets consists of a small number of heuristic rules that reduce the evaluation of a definite integral to the solution of a linear system of equations. The method has been used to establish a classical theorem of Frullani and to evaluate, in an algorithmic manner, a variety of integrals of Frullani type. The flexibility of the method yields a direct and simple solution to these evaluations.

Example 6.3. The method is now used to evaluate

\[ \int_0^\infty \int_0^\infty \frac{\sin(\mu x y^2) \cos(ax y) - \sin(\mu x y^2) \cos(bx y)}{xy} \, dx \, dy = \frac{\pi}{2} \log \frac{b}{a}. \]  

The evaluation begins with the expansion

\[ f(x, y) = \sin(xy^2) \cos(xy) = \sum_{n_1 \geq 0} \sum_{n_2 \geq 0} \frac{\phi_{n_1} \Gamma \left( \frac{3}{2} \right) (x y^2)^{2n_1}}{\Gamma(n_1 + \frac{3}{2}) 4^{n_1}} \frac{\phi_{n_2} \Gamma \left( \frac{1}{2} \right) (xy)^{2n_2}}{\Gamma(n_2 + \frac{1}{2}) 4^{n_2}}. \]

The parameters are \( b_1 = a^2/\mu, \ b_2 = \mu/a, \ a_1 = b^2/\mu, \ a_2 = \mu/b \) and \( \rho_1 = \rho_2 = 0 \). The solution to the linear system is \( n_1^* = -\frac{1}{2} \) and \( n_2^* = -\frac{\epsilon}{4\pi} \) and \( |\det A| = 4 \). Then,

\[ I = \lim_{\epsilon \to 0} \frac{a^{-\epsilon} - b^{-\epsilon}}{4} \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{\epsilon}{2} \right) \sqrt{\pi \Gamma(1 - \epsilon)}}{2 \Gamma(1 - \frac{\epsilon}{2}) 4^{-\epsilon - 1}/2} \]

\[ = \lim_{\epsilon \to 0} \frac{\pi^3/2 e^{\epsilon/2}}{4} \frac{b^2 - a^2}{ab} \frac{2^{1-2\epsilon} \sqrt{\pi \Gamma(\epsilon)}}{\pi \csc \left( \frac{\pi}{4} + \frac{\epsilon}{2} \right)} \]

\[ = \frac{\pi}{2} \log \left( \frac{b}{a} \right). \]

as claimed.

7 Conclusions

The method of brackets consists of a small number of heuristic rules that reduce the evaluation of a definite integral to the solution of a linear system of equations. The method has been used to establish a classical theorem of Frullani and to evaluate, in an algorithmic manner, a variety of integrals of Frullani type. The flexibility of the method yields a direct and simple solution to these evaluations.

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