Abstract: We prove that homogeneous problem for PDE of second order in time variable, and generally infinite order in spatial variables with local two-point conditions with respect to time variable, has only trivial solution in the case when the characteristic determinant of the problem is nonzero. In another, opposite case, we prove the existence of nontrivial solutions of the problem, and we propose a differential-symbol method of constructing them.

Keywords: Null space, PDE, Two-point problem, Differential-symbol method

MSC: 35G15

1 Introduction

The problem of finding the solution $T(t)$ of ordinary differential equation (ODE) of order $n \in \mathbb{N} \setminus \{1\}$, which satisfies conditions $T(t_1) = c_1, \ldots, T(t_n) = c_n$, where $t_1 < \ldots < t_n$, $t_i, c_i \in \mathbb{R}$, in the literature is found as the Vallee-Poussin problem or multipoint ($n$-point) problem.

For the first time such problems were studied in the articles [1–4], in which the importance to study the problems is indicated from the point of view of generalization of Cauchy problem. In contrast to Cauchy problem, the multipoint problem is ill-posed, because corresponding homogeneous problem can have nontrivial solutions.

The first results on solving the problems with multipoint time conditions for linear partial differential equations (PDE) have been obtained in [5] based on the metric approach. In particular, this paper points out the problem of small denominators, which is typical for multipoint problems. Also, ill-posedness of these problems was proved, moreover it was shown that the classes of uniqueness of the multipoint in time problem solutions for PDE were significantly different from the classes of uniqueness of the solutions of the corresponding Cauchy problem for the same equations.

The investigation of the $n$-point problem for equations and systems of PDE’s in the bounded domains that is based on the metric approach has significantly developed in recent years (see works [6, 7] and bibliography).

Papers [8–10] are devoted to establish the classes of unique solvability of problems with multipoint conditions in time for PDE’s in unbounded domains (strip, layer). The technique of investigation multipoint problems in spaces of functions in which there is no problem of small denominators is proposed in these works.

The solvability of problems with multipoint conditions for differential-operator equations was studied in [11].
This article is devoted to study of the null space of problem for the PDE of the second order with respect to
time variable, and arbitrary order with respect to spatial variables with local two-point conditions in time, and it is a
continuation of researches [12] for the case of equations with several spatial variables.

2 Problem statement

In the domain $\mathbb{R}^{1+s}$, $s \in \mathbb{N} \setminus \{1\}$, we investigate the set of solutions $U = U(t, x)$, $x = (x_1, \ldots, x_s)$, of the PDE

$$L\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)U \equiv \frac{\partial^2 U}{\partial t^2} + 2a\left(\frac{\partial}{\partial x}\right)\frac{\partial U}{\partial t} + b\left(\frac{\partial}{\partial x}\right)U = 0,$$

in which the operator coefficients $a\left(\frac{\partial}{\partial x}\right)$, $b\left(\frac{\partial}{\partial x}\right)$, are considered as arbitrary differential
expressions with complex coefficients of the finite or infinite order, and the symbols $a\left(v\right)$, $b\left(v\right)$ of those coefficients
are entire functions of complex vector-variable $v \in \mathbb{C}^s$.

Under condition that $a\left(v\right)$ and $b\left(v\right)$ are polynomials, denote their degrees by the set of variables as
$p_a \in \mathbb{Z} \mathbb{C}$ and $p_b \in \mathbb{Z} \mathbb{C}$. Also we assume that $p_a = p_b = 1$ if $a\left(v\right)$ and $b\left(v\right)$ respectively are not polynomials.

Denote $p = \max \{p_a, p_b\}$. Besides, assume $p = \infty$ if $p_a = \infty$ or $p_b = \infty$.

In the solutions set of equation (1), we will find the solutions that satisfy the homogeneous local two-point
conditions:

$$A_1\left(\frac{\partial}{\partial x}\right)U(0, x) + A_2\left(\frac{\partial}{\partial x}\right)\frac{\partial U}{\partial t}(0, x) = 0,$$

$$B_1\left(\frac{\partial}{\partial x}\right)U(h, x) + B_2\left(\frac{\partial}{\partial x}\right)\frac{\partial U}{\partial t}(h, x) = 0,$$

where $A_1\left(\frac{\partial}{\partial x}\right)$, $A_2\left(\frac{\partial}{\partial x}\right)$, $B_1\left(\frac{\partial}{\partial x}\right)$, $B_2\left(\frac{\partial}{\partial x}\right)$ are differential polynomials with complex coefficients. Moreover
their symbols $A_1\left(v\right)$, $A_2\left(v\right)$, $B_1\left(v\right)$, $B_2\left(v\right)$ for $v \in \mathbb{C}^s$ satisfy the conditions:

$$\left|A_1\left(v\right)\right|^2 + \left|A_2\left(v\right)\right|^2 \neq 0,$$

$$\left|B_1\left(v\right)\right|^2 + \left|B_2\left(v\right)\right|^2 \neq 0.$$

In this article, we establish the conditions of existence of nontrivial solutions (nontrivial null space) of this problem.
In the case of existence of such solutions, we propose the way of their construction by using the differential-symbol
method [13].

3 Main results

In equation (1), we replace $\frac{d}{dx}$ with vector-parameter $v$ and the symbol $\frac{d}{dt}$ with $\frac{d}{dt}$. We obtain the ODE

$$L\left(\frac{d}{dt}, v\right)T(t, v) \equiv \left(\frac{d^2}{dt^2} + 2a(v)\frac{d}{dt} + b(v)\right)T(t, v) = 0,$$

in which, from now on, we consider $v \in \mathbb{C}^s$.

The normal fundamental system of solutions of equation (3) at the point $t = 0$ has the form:

$$T_0(t, v) = e^{-a(v)t} \left\{a(v) \frac{\sinh \left[t \sqrt{D(v)}\right]}{\sqrt{D(v)}} + \cosh \left[t \sqrt{D(v)}\right]\right\},$$

$$T_1(t, v) = e^{-a(v)t} \frac{\sinh \left[t \sqrt{D(v)}\right]}{\sqrt{D(v)}}.$$

where $D(v) = a^2(v) - b(v)$. 
Since the coefficients \( a(v), b(v) \) of equation (3) are entire functions by assumption, so by the Poincare Theorem [14, p. 59] the functions \( T_0(t, v), T_1(t, v) \) are also entire functions of vector-parameter \( v \) with order \( p \).

We write down the determinant of the form:

\[
\Delta(v) = \begin{vmatrix}
A_1(v) & A_2(v) \\
B_1(v) T_0(h, v) + B_2(v) \frac{dT_0}{dt}(h, v) & B_1(v) T_1(h, v) + B_2(v) \frac{dT_1}{dt}(h, v)
\end{vmatrix}.
\]

Such determinant will be called the characteristic determinant of problem (1), (2). It can also be represented in a matrix form:

\[
\Delta(v) = \left( B_1(v) B_2(v) \right) \begin{pmatrix}
T_0(h, v) & T_1(h, v) \\
\frac{dT_0}{dt}(h, v) & \frac{dT_1}{dt}(h, v)
\end{pmatrix} \begin{pmatrix}
-A_2(v) \\
A_1(v)
\end{pmatrix}.
\]

Note that function \( \Delta(v) \) as a superposition of entire functions is an entire function.

### 3.1 The case when the set of zeroes of the characteristic determinant is empty

**Theorem 3.1.** If \( \Delta(v) \neq 0 \) \( \forall v \in \mathbb{C}^s \), then problem (1), (2) in the class of entire functions has only trivial solution.

**Proof.** Let there exists a nontrivial integer solution \( U(t, x) \) of equation (1), i.e. entire function of the form

\[
U(t, x) = \sum_{k \in \mathbb{Z}^{s+1}_+} u_k t^{k_0} x^{k_1}, \quad k = (k_0, k_1, \ldots, k_s), \quad u_k \in \mathbb{C},
\]

of variables \( t \) and \( x = (x_1, \ldots, x_s) \), where \( x^k = x_1^{k_1} \ldots x_s^{k_s} \), that satisfies conditions (2).

Let’s denote \( U(0, x) = \psi(x), \frac{\partial U}{\partial t}(0, x) = \psi(x) \). Then \( \psi(x) \) and \( \psi(x) \) are also entire functions. Write down the solution of problem (1), (2) according to differential-symbol method [13] as the solution of Cauchy problem for equation (1) with initial data \( \psi \) and \( \psi \) in the form:

\[
U(t, x) = \left. \psi \left( \left. \frac{\partial}{\partial v} \right|_{v=\mathcal{O}} \| A_1(v) e^{v \cdot x} \right|_{v=\mathcal{O}} + \left. \psi \left( \left. \frac{\partial}{\partial v} \right|_{v=\mathcal{O}} \| T_1(t, v) e^{v \cdot x} \right|_{v=\mathcal{O}} \right) \right|_{v=\mathcal{O}},
\]

where \( v \cdot x = v_1 x_1 + \ldots + v_s x_s, \mathcal{O} = (0, \ldots, 0) \).

Since \( T_0(t, v) e^{v \cdot x} \) and \( T_1(t, v) e^{v \cdot x} \) are entire in \( v \) functions of order \( \pi = \max \{ p, 1 \} \geq 1 \), so both entire functions \( \psi(x) \) and \( \psi(x) \) must have the adjoint [15, p. 316] with \( \pi \) order \( q \), i.e. \( q = \pi / (\pi - 1) \) for \( 1 < \pi < \infty \), and \( q = \infty \) for \( \pi = 1 \) and \( q = 1 \) for \( \pi = \infty \).

Since two-point conditions (2) for function \( \psi \) and \( \psi \) are satisfied, we obtain in \( \mathbb{R}^s \) the system of identities

\[
\left\{ \begin{array}{l}
\psi \left( \left. \frac{\partial}{\partial v} \right|_{v=\mathcal{O}} \| A_1(v) e^{v \cdot x} \right|_{v=\mathcal{O}} \right) + \psi \left( \left. \frac{\partial}{\partial v} \right|_{v=\mathcal{O}} \| A_2(v) e^{v \cdot x} \right|_{v=\mathcal{O}} \right) = 0, \\
\psi \left( \left. \frac{\partial}{\partial v} \right|_{v=\mathcal{O}} \| B_1(v) T_0(h, v) + B_2(v) T_0'(h, v) \right|_{v=\mathcal{O}} \right) + \\
\psi \left( \left. \frac{\partial}{\partial v} \right|_{v=\mathcal{O}} \| B_1(v) T_1(h, v) + B_2(v) T_1'(h, v) \right|_{v=\mathcal{O}} \right) = 0,
\end{array} \right.
\]

where \( T_0'(h, v) = \frac{dT_0}{dt}(h, v), T_1'(h, v) = \frac{dT_1}{dt}(h, v) \).

Let’s act by the expression \( B_1 \left( \frac{\partial}{\partial x} \right|_{x=\mathcal{O}} T_1(h, \frac{\partial}{\partial x}) + B_2 \left( \frac{\partial}{\partial x} \right|_{x=\mathcal{O}} T_1'(h, \frac{\partial}{\partial x}) \right) \) onto the first identity of system (5) and act by \( A_2 \left( \frac{\partial}{\partial x} \right|_{x=\mathcal{O}} \) onto the second identity of the system, after that subtract the second identity from the first one. We obtain

\[
\psi \left( \left. \frac{\partial}{\partial v} \right|_{v=\mathcal{O}} \| e^{v \cdot x} \Delta(v) \right|_{v=\mathcal{O}} \right) = 0, \quad x \in \mathbb{R}^s.
\]

Since the function \( \Delta^{-1}(v) \) is entire, so, acting by expression \( \Delta^{-1} \left( \frac{\partial}{\partial x} \right|_{x=\mathcal{O}} \) onto the last identity, we obtain

\[
\psi \left( \left. \frac{\partial}{\partial v} \right|_{v=\mathcal{O}} \| e^{v \cdot x} \Delta(v) \right|_{v=\mathcal{O}} \right) = 0, \text{ whence we have } \psi \left( \left. \frac{\partial}{\partial v} \right|_{v=\mathcal{O}} \| e^{v \cdot x} \right|_{v=\mathcal{O}} \right) = 0 \text{ or } \psi(x) = 0.
\]
Similarly, if we act by expression $-B_1 \left( \frac{\partial}{\partial x} \right) T_0 \left( h, \frac{\partial}{\partial x} \right) - B_2 \left( \frac{\partial}{\partial x} \right) T_0' \left( h, \frac{\partial}{\partial x} \right)$ onto the first identity of system (5) and by expression $A_1 \left( \frac{\partial}{\partial x} \right)$ onto the second identity, and then add the obtained identities, we get

$$
\psi \left( \frac{\partial}{\partial v} \right) \left\{ \hat{e}^{v \cdot x} \Delta (v) \right\} \bigg|_{v = 0} = 0, \quad x \in \mathbb{R}^k,
$$

that implies $\psi (x) \equiv 0$.

We have the identity $U (t, x) \equiv 0$, which contradicts to the assumption on nontriviality of the solution of problem (1), (2).

\[ \square \]

### 3.2 The case when the set of zeroes of the characteristic determinant coincides with $\mathbb{C}^*$

Let’s investigate the solvability of problem (1), (2) in case $\Delta (v) \equiv 0$.

Consider the function

$$
\Phi (t, x, v) = \left\{ A_2 (v) T_0 (t, v) - A_1 (v) T_1 (t, v) \right\} \hat{e}^{v \cdot x},
$$

which is an entire function with respect to the vector-parameter $v \in \mathbb{C}^k$. Let’s denote the order of this function in the set of parameters $v_1, \ldots, v_k$ by $\overline{p}$. Note that $1 \leq \overline{p} \leq \overline{p}^*$.

**Theorem 3.2.** Let $\Delta (v) \equiv 0$ for problem (1), (2), then entire nontrivial solutions of the problem exist, and they can be represented in form

$$
U (t, x) = \varphi \left( \frac{\partial}{\partial v} \right) \left\{ \Phi (t, x, v) \right\} \bigg|_{v = 0},
$$

where $\varphi (x)$ is an arbitrary entire function, that has order, adjoint with the order $\overline{p}$.

**Proof.** Since $\Delta (v) \equiv 0$ then there holds the equality

$$
A_1 (v) \left\{ B_1 (v) T_1 (h, v) + B_2 (v) T_1' (h, v) \right\} = A_2 (v) \left\{ B_1 (v) T_0 (h, v) + B_2 (v) T_0' (h, v) \right\}.
$$

Let’s show firstly that function (7) satisfies equation (1). Taking into account the commutativity of the operations $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial v}$, we obtain

$$
L \left( \frac{d}{dt}, \frac{\partial}{\partial x} \right) \left\{ \frac{\partial}{\partial v} \left\{ \Phi (t, x, v) \right\} \right\} \bigg|_{v = 0} = \varphi \left( \frac{\partial}{\partial v} \right) L \left( \frac{d}{dt}, \frac{\partial}{\partial x} \right) \left\{ \Phi (t, x, v) \right\} \bigg|_{v = 0} =
$$

$$
= \varphi \left( \frac{\partial}{\partial v} \right) \left\{ \hat{e}^{v \cdot x} L \left( \frac{d}{dt}, v \right) \left\{ A_2 (v) T_0 (t, v) - A_1 (v) T_1 (t, v) \right\} \right\} \bigg|_{v = 0} =
$$

$$
= \varphi \left( \frac{\partial}{\partial v} \right) \left\{ \hat{e}^{v \cdot x} \left[ A_2 (v) L \left( \frac{d}{dt}, v \right) T_0 (t, v) - A_1 (v) L \left( \frac{d}{dt}, v \right) T_1 (t, v) \right] \right\} \bigg|_{v = 0} = 0.
$$

Let’s prove that the first condition (2) is satisfied:

$$
A_1 \left( \frac{\partial}{\partial x} \right) U (0, x) + A_2 \left( \frac{\partial}{\partial x} \right) \frac{\partial U}{\partial t} (0, x) =
$$

$$
= A_1 \left( \frac{\partial}{\partial x} \right) \varphi \left( \frac{\partial}{\partial v} \right) \left\{ \Phi (0, x, v) \right\} \bigg|_{v = O} + A_2 \left( \frac{\partial}{\partial x} \right) \varphi \left( \frac{\partial}{\partial v} \right) \left\{ \frac{\partial \Phi}{\partial v} (0, x, v) \right\} \bigg|_{v = 0} =
$$

$$
= \varphi \left( \frac{\partial}{\partial v} \right) \left\{ A_1 \left( \frac{\partial}{\partial x} \right) \left[ A_2 (v) \hat{e}^{v \cdot x} \right] \right\} \bigg|_{v = 0} - \varphi \left( \frac{\partial}{\partial v} \right) \left\{ A_2 \left( \frac{\partial}{\partial x} \right) \left[ A_1 (v) \hat{e}^{v \cdot x} \right] \right\} \bigg|_{v = O} =
$$

$$
= \varphi \left( \frac{\partial}{\partial v} \right) \left\{ \hat{e}^{v \cdot x} \left[ A_1 (v) A_2 (v) - A_1 (v) A_2 (v) \right] \right\} \bigg|_{v = O} = 0.
$$
Due to equality (8), function (7) also satisfies the second condition (2):

\[ B_1 \left( \frac{\partial}{\partial t} \right) U(h, x) + B_2 \left( \frac{\partial}{\partial x} \right) \frac{\partial U}{\partial t}(h, x) = \]

\[ = B_1 \left( \frac{\partial}{\partial v} \right) \left\{ \Phi(h, x, v) \right\} \bigg|_{v=O} + B_2 \left( \frac{\partial}{\partial x} \right) \left\{ \frac{\partial \Phi}{\partial t}(h, x, v) \right\} \bigg|_{v=O} = \]

\[ = \phi \left( \frac{\partial}{\partial v} \right) \left\{ B_1(v) \Phi(h, x, v) \right\} \bigg|_{v=O} + \phi \left( \frac{\partial}{\partial x} \right) \left\{ B_2(v) \frac{\partial \Phi}{\partial t}(h, x, v) \right\} \bigg|_{v=O} = \]

\[ = \phi \left( \frac{\partial}{\partial v} \right) \left\{ e^{\nu x} \Delta(v) \right\} \bigg|_{v=O} = 0. \]

Thus, we have proved that function (7) is a solution of problem (1), (2), in which \( \varphi(x) \) is an arbitrary entire function with the order adjoint to the order \( \overline{p} \).

Let’s show that function (7) is nontrivial. In fact,

\[ \frac{\partial U}{\partial t}(0, x) = \phi \left( \frac{\partial}{\partial v} \right) \left\{ \frac{\partial \Phi}{\partial t}(0, x, v) \right\} \bigg|_{v=O} = -\phi \left( \frac{\partial}{\partial x} \right) \left\{ \frac{\partial \Phi}{\partial t}(0, x, v) \right\} \bigg|_{v=O}. \]

Since \( |A_1(v)|^2 + |A_2(v)|^2 > 0 \) for arbitrary \( v \in \mathbb{C}^s \), thus either \( U(0, x) \) or \( \frac{\partial U}{\partial t}(0, x) \) are nontrivial. Hence, function (7) is also nontrivial as a solution of Cauchy problem with nonzero initial data.

### 3.3 The case when the set of zeroes of the characteristic determinant is not empty and does not coincide with \( \mathbb{C}^s \)

Consider the set

\[ M = \{ v \in \mathbb{C}^s : \Delta(v) = 0 \}, \]

moreover we assume \( M \neq \emptyset \) and \( M \neq \mathbb{C}^s \).

Let \( \alpha \in M \). For complex vector \( \alpha \) consider the set of multi-indexes:

\[ \Omega_1(\alpha) = \left\{ \omega = (\omega_1, \ldots, \omega_s) \in \mathbb{Z}_+^s : \left( \frac{\partial}{\partial v} \right) \omega \Delta(v) \bigg|_{v=\alpha} = \Delta^{(\omega)}(\alpha) \neq 0 \right\}, \]

where \( \left( \frac{\partial}{\partial v} \right) \omega \Delta(v) = \frac{\partial^{(0)}}{\partial v_{\omega_1} \cdots \partial v_{\omega_s}} \Delta(v), \) \( |\omega| = \omega_1 + \ldots + \omega_s \).

For two vectors \( \omega = (\omega_1, \ldots, \omega_s) \) and \( \sigma = (\sigma_1, \ldots, \sigma_s) \in \mathbb{Z}_+^s \), we assume \( \omega \geq \sigma \), when \( \omega_1 \geq \sigma_1, \ldots, \omega_s \geq \sigma_s \). Also let’s denote \( \mathbb{C}_\omega^\sigma = \frac{\omega!}{\sigma!(\omega-\sigma)!} \), where \( \omega! = \prod_{k=1}^{s} \omega_k! \), \( \omega_k! = 1 \cdot 2 \cdot \ldots \cdot \omega_k \), \( 0! = 1 \).

Besides the set (10) let us consider such sets:

\[ \Omega(\alpha) = \{ \bar{\omega} \in \mathbb{Z}_+^s : \bar{\omega} \geq \omega, \ \omega \in \Omega_1(\alpha) \}, \quad \Omega(\alpha) = \mathbb{Z}_+^s \setminus \Omega(\alpha). \]

Note the following fact: if \( \omega \in \overline{\Omega}(\alpha) \), then \( \Delta^{(\sigma)}(\alpha) = 0 \) for all \( \sigma \in \mathbb{Z}_+^s \) for which \( \sigma \leq \omega \).

**Theorem 3.3.** If \( \omega \in \overline{\Omega}(\alpha) \), then

\[ U(t, x) = \left( \frac{\partial}{\partial v} \right) \omega \left\{ \Phi(t, x, v) \right\} \bigg|_{v=\alpha} \]

is a nontrivial solution of problem (1), (2), where \( \Phi(t, x, v) \) is the function (6).
Proof. The function \( U(t, x) \) is the solution of equation (1). It follows from equality (7) for entire function \( \varphi(x) = x^\omega \). The first condition from two-point conditions (2) is fulfilled:

\[
A_1 \left( \frac{\partial}{\partial x} \right) U(0, x) + A_2 \left( \frac{\partial}{\partial t} \right) U(0, x) = \left( \frac{\partial}{\partial x} \right) ^\omega \left[ \{ A_1(v) A_2(v) - A_1(v) A_2(v) \} e^{\nu x} \right] \bigg| _{x=\alpha} = 0.
\]

Let’s show that the second condition in (2) is satisfied:

\[
B_1 \left( \frac{\partial}{\partial x} \right) \left\{ \} \right. + B_2 \left( \frac{\partial}{\partial t} \right) \left\{ \} \right. = \left( \frac{\partial}{\partial x} \right) ^\omega \left[ \left\{ A_2(v) T_0(h, v) - A_1(v) T_1(h, v) \right\} e^{\nu x} \right] \bigg| _{x=\alpha} = \left( \frac{\partial}{\partial x} \right) ^\omega \left[ B_1(v) \left\{ A_2(v) T_0(h, v) - A_1(v) T_1(h, v) \right\} e^{\nu x} \right] \bigg| _{x=\alpha} + \left( \frac{\partial}{\partial x} \right) ^\omega \left[ B_2(v) \left\{ A_2(v) T_0(h, v) - A_1(v) T_1(h, v) \right\} e^{\nu x} \right] \bigg| _{x=\alpha} = \left( \frac{\partial}{\partial x} \right) ^\omega \Delta(v) e^{\nu x} \bigg| _{x=\alpha} = e^{\alpha x} \left( \frac{\partial}{\partial x} + x \right) ^\omega \Delta(\alpha) = e^{\alpha x} \sum_{O \leq q \leq \alpha} C^q \alpha^{\omega-q} \Delta^{(q)}(\alpha).
\]

Since \( \omega \in \Omega(\alpha) \), then according to the remark for arbitrary \( q \in Z^+_\alpha \) such that \( q \leq \omega \), equalities \( \Delta^{(q)}(\alpha) = 0 \) are fulfilled. Hence, the function (11) satisfies the second condition (2).

Let’s prove that the formula (11) defines nontrivial solution of problem (1), (2). Let’s calculate the value of the function (11) and its derivative at the point \( t = 0 \):

\[
U(0, x) = \left( \frac{\partial}{\partial x} \right)^\omega \{ \Phi(0, x, v) \} \bigg| _{x=\alpha} = \left( \frac{\partial}{\partial v} \right)^\omega \left\{ A_2(v) e^{\nu x} \right\} \bigg| _{x=\alpha} = e^{\alpha x} \left( \frac{\partial}{\partial x} + x \right)^\omega A_2(\alpha) = e^{\alpha x} \sum_{O \leq q \leq \alpha} C^q \alpha^{\omega-q} A_2^{(q)}(\alpha);
\]

\[
\frac{\partial U}{\partial t}(0, x) = \left( \frac{\partial}{\partial v} \right)^\omega \left\{ \left[ A_2(v) T_0(0, v) - A_1(v) T_1(0, v) \right] e^{\nu x} \right\} \bigg| _{x=\alpha} = \left( \frac{\partial}{\partial v} \right)^\omega \left\{ A_2(v) e^{\nu x} \right\} \bigg| _{x=\alpha} = -e^{\alpha x} \left( \frac{\partial}{\partial x} + x \right)^\omega A_1(\alpha) = -e^{\alpha x} \sum_{O \leq q \leq \alpha} C^q \alpha^{\omega-q} A_1^{(q)}(\alpha).
\]

In the last sums, the coefficients of \( x^\omega \) are \( A_2(\alpha) \) and \( A_1(\alpha) \). From the condition \( |A_1(\alpha)|^2 + |A_2(\alpha)|^2 \neq 0 \), it follows that at least one of the expressions \( U(0, x) \) or \( \frac{\partial U}{\partial t}(0, x) \) is a nonzero quasipolynomial. This implies that function (11) is a nontrivial solution of problem (1), (2).

\[\square\]

4 Examples

Example 4.1. Find in domain \( (t, x_1, x_2) \in \mathbb{R}^3 \) the solutions of equation

\[
\left[ \frac{\partial^2}{\partial t^2} + 2 \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) \frac{\partial}{\partial t} + \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right)^2 + 1 \right] U(t, x_1, x_2) = 0.
\]

that satisfy local two-point conditions

\[
\left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) U(0, x) + \frac{\partial U}{\partial t}(0, x) = 0.
\]

\[
\left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) U(h, x) + \frac{\partial U}{\partial t}(h, x) = 0.
\]
For this problem, we obtain $a(v) = v_1 + v_2$, $b(v) = (v_1 + v_2)^2 + 1$, $D(v) = -1$, $A_1(v) = B_1(v) = v_1 + v_2$, $A_2(v) = B_2(v) = 1$, $v = (v_1, v_2)$, $x = (x_1, x_2)$, $s = 2$.

The normal at the point $t = 0$ fundamental system of solutions of the corresponding to (12) ODE

$$\left[ \frac{d^2}{dt^2} + 2(v_1 + v_2) \frac{d}{dt} + ((v_1 + v_2)^2 + 1) \right] T(t, v) = 0$$

has the form

$$T_0(t, v) = e^{-(v_1 + v_2)t} \left\{ (v_1 + v_2) \sin t + \cos t \right\}, \quad T_1(t, v) = e^{-(v_1 + v_2)t} \sin t.$$ 

Let’s write down the characteristic determinant of problem (12), (13):

$$\Delta(v) = \begin{vmatrix} v_1 + v_2 & 1 \\ (v_1 + v_2) T_0(h, v) + T_0'(h, v) (v_1 + v_2) T_1(h, v) + T_1'(h, v) \end{vmatrix} = e^{-(v_1 + v_2)h} \sin h.$$ 

In the case $h \neq \pi k$, $k \in \mathbb{N}$, the condition $\Delta(v) \neq 0$ is satisfied for arbitrary $v \in C^2$, so by Theorem 3.1, problem (12), (13) has only trivial solution.

If $h = \pi k$, where $k \in \mathbb{N}$, then there holds the equality $\Delta(v) \equiv 0$. So, by Theorem 3.2, there exist nontrivial solutions of problem (12), (13). Function (6) gets the form:

$$\Phi(t, x, v) = \left\{ T_0(t, v) - (v_1 + v_2) T_1(t, v) \right\} e^{v \cdot x} = e^{-(v_1 + v_2)t + v \cdot x} \cos t.$$ 

We search the nontrivial solutions of problem (12), (13) in the case $h = \pi k$, where $k \in \mathbb{N}$, by formula (7):

$$U(t, x) = \Phi \left( \frac{\partial}{\partial v} \right) \left\{ e^{-(v_1 + v_2)t + v \cdot x} \cos t \right\} \big|_{v = 0} = \\
\cos t \Phi \left( \frac{\partial}{\partial v} \right) \left\{ e^{-(v_1 + v_2)t + v_1 x_1 + v_2 x_2} \right\} \big|_{v = 0} = \varphi(x_1 - t, x_2 - t) \cos t.$$ 

The null space of the problem is infinite-dimensional. Note that the condition of entirety of the function $\varphi$ can be weakened, considering classical solutions of problem (12), (13): $\varphi$ can be arbitrary twice continuously differentiable function on $\mathbb{R}^2$.

\[ \square \]

**Example 4.2.** In the domain $(t, x_1, x_2) \in \mathbb{R}^3$, find the solutions of the two-point problem

$$\frac{\partial^2}{\partial t^2} U(t, x_1, x_2) + 2 \frac{\partial^2}{\partial t \partial x_1} U(t, x_1, x_2) + \frac{\partial^2}{\partial x_1^2} U(t, x_1, x_2 + 1) = 0,$$

$$\frac{\partial U}{\partial x_1}(0, x_1, x_2) + \frac{\partial U}{\partial t}(0, x_1, x_2) = 0, \quad U(1, x_1, x_2) = 0.$$ 

(14)

For this problem, we have $a(v) = v_1$, $b(v) = v_1^2 e^{v_2}$, $D(v) = v_1^2 (1 - e^{v_2})$, $A_1(v) = v_1$, $B_2(v) = 0$, $A_2(v) = B_1(v) = 1$, $s = 2$, $h = 1$. The functions

$$T_0(t, v) = e^{-v_1 t} \cosh \left[ t v_1 \sqrt{1 - e^{v_2}} \right] + \sinh \left[ t v_1 \sqrt{1 - e^{v_2}} \right]$$

$$T_1(t, v) = e^{-v_1 t} \frac{\sinh \left[ t v_1 \sqrt{1 - e^{v_2}} \right]}{v_1 \sqrt{1 - e^{v_2}}}$$ 

form the normal at the point $t = 0$ fundamental system of solutions of ODE

$$\left[ \frac{d^2}{dt^2} + 2 v_1 \frac{d}{dt} + v_1^2 e^{v_2} \right] T(t, v) = 0.$$ 

The characteristic determinant of problem (14) and the corresponding set $M$ will have the form:

$$
\Delta(v) = e^{-v_1} \cosh \left[ v_1 \sqrt{1 - e^{v_2}} \right], \quad M = \left\{ v \in \mathbb{C}^2 : \quad v_1 \sqrt{1 - e^{v_2}} = \left( \frac{\pi}{2} + \pi k \right) i, \quad k \in \mathbb{Z}, \quad i^2 = -1 \right\}.
$$

The set $M$ consists of the following vectors:

$$
\alpha_k(\mu) = \left( \frac{\pi + \pi k}{\sqrt{1 - e^{\mu}}}, \mu \right), \quad \mu \in \mathbb{C} \setminus \{ 2\pi mi, \ m \in \mathbb{Z} \}, \ k \in \mathbb{Z}.
$$

Let’s find the first order partial derivatives of the function $\Delta(v)$ at the point $v = \alpha_k(\mu)$:

$$
\Delta^{(1,0)}(\alpha_k(\mu)) = e^{-v_1} \left\{ \sqrt{1 - e^{v_2}} \sinh \left[ v_1 \sqrt{1 - e^{v_2}} \right] - \cosh \left[ v_1 \sqrt{1 - e^{v_2}} \right] \right\}_{v=\alpha_k(\mu)} =
$$

$$
= (-1)^k \sqrt{1 - e^{\mu}} \left( \frac{\pi + \pi k}{\sqrt{1 - e^{\mu}}} \right) \neq 0,
$$

$$
\Delta^{(0,1)}(\alpha_k(\mu)) = -\frac{v_1}{2} \sqrt{1 - e^{v_2}} e^{v_2 - v_1} \sinh \left[ v_1 \sqrt{1 - e^{v_2}} \right]_{v=\alpha_k(\mu)} =
$$

$$
= \left( \frac{\pi}{2} + \pi k \right)i \left( 1 - \frac{1}{(1 - e^{\mu})^2} \right) e^{\mu} \left( \frac{\pi + \pi k}{\sqrt{1 - e^{\mu}}} \right) \neq 0.
$$

Hence, $\mathfrak{M}(\alpha_k(\mu)) = \{ (0,0) \}$. Function (6) gets the form

$$
\Phi(t,x,v) = \cosh \left[ t v_1 \sqrt{1 - e^{v_2}} \right] e^{v_1(x_1 - t) + v_2 x_2}.
$$

By Theorem 3.3, for arbitrary $\mu \in \mathbb{C} \setminus \{ 2\pi mi, \ m \in \mathbb{Z} \}$ and $k \in \mathbb{Z}$, we find such nontrivial solutions of problem (14):

$$
U_k(t,x_1,x_2,\mu) = \Phi(t,x,v_k(\mu)) = \cos \left[ \left( \frac{\pi}{2} + \pi k \right) t \right] e^{\mu t} e^{\frac{(\pi + \pi k)i}{\sqrt{1 - e^{\mu}}} (x_1 - t) + \mu x_2}.
$$

Note that the obtained solutions of problem (14) are linearly independent.

\[\triangle\]

**Example 4.3.** In the domain $(t, x) \in \mathbb{R}^4$ find the solutions of two-point problem

$$
\begin{align*}
\left[ \frac{\partial}{\partial t} + a \left( \frac{\partial}{\partial x} \right) \right]^2 U(t, x) &= 0, \\
\left[ a \left( \frac{\partial}{\partial t} \right) + c \left( \frac{\partial}{\partial x} \right) \right] U(0, x) + \frac{\partial U}{\partial t}(0, x) &= 0, \quad U(1, x) = 0,
\end{align*}
$$

in which $a \left( \frac{\partial}{\partial x} \right)$, $c \left( \frac{\partial}{\partial x} \right)$ are differential polynomials with complex coefficients.

Problem (15) is problem (1), (2), in which $b(v) = a^2(v)$, $A_1(v) = a(v) + c(v)$, $A_2(v) = B_1(v) = 1$, $B_2(v) = 0$, $h = 1$, $s = 3$.

The functions

$$
T_0(t,v) = e^{-a(v)t} \{ a(v)t + 1 \}, \quad T_1(t,v) = t \ e^{-a(v)t}
$$

form the normal at the point $t = 0$ fundamental system of solutions of ODE

$$
\left[ \frac{d}{dt} + a(v) \right]^2 T(t,v) = 0.
$$

We calculate the characteristic determinant of problem (15):

$$
\Delta(v) = \begin{vmatrix}
  a(v) + c(v) & 1 \\
  T_0(1,v) & T_1(1,v)
\end{vmatrix} = e^{-a(v)} \{ c(v) - 1 \}.
$$

Let’s consider the cases.
Case 1. Let \( c(v) = c \in \mathbb{C}\setminus \{1\} \). Then for arbitrary \( v \in \mathbb{C}^3 \) condition
\[
\Delta(v) = e^{-a(v)} \{c - 1\} \neq 0
\]
is satisfied. Therefore by Theorem 3.1 problem (15) has only trivial solution.

Case 2. If \( c(v) = 1 \), then \( \Delta(v) \equiv 0 \). Function (6) has the form
\[
\Phi(t, x, v) = \left\{ T_0(t, v) - [a(v) + 1] T_1(t, v) \right\} e^{v \cdot x} = (1-t) e^{-a(v) x + v \cdot x}.
\]
By Theorem 3.2 we find such nontrivial solutions of problem (15):
\[
U(t, x) = (1-t) \varphi \left( \frac{\partial}{\partial v} \right) \left. \left\{ e^{-a(v) x + v \cdot x} \right\} \right|_{v=0},
\]
where \( \varphi \) is an arbitrary entire function of three variables. The order of this function is adjoint with number \( p = \max \{1, \deg a(v)\} \) (\( \deg a(v) \) defines degree of polynomial \( a(v) \) by the set of variables \( v_1, v_2, v_3 \)).

Case 3. Let \( c(v) = v_3, a(v) = v_3 - v_1 v_2 \). Then problem (15) gets the form
\[
\left[ \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_1 \partial x_2} + 1 \right] U(t, 0, x) + \frac{\partial U}{\partial t}(0, x) = 0, \quad U(1, x) = 0.
\]
For problem (16), we obtain
\[
\Phi(t, x, v) = \left\{ 1 - (v_3 + 1)t \right\} e^{(v_3 + v_1 v_2) t + v \cdot x},
\]
\[
\Delta(v) = v_3 e^{-v_3 + v_1 v_2}, \quad M = \left\{ v \in \mathbb{C}^3 : v_3 = 0 \right\}.
\]
The set \( M \) consists of the following vectors: \( \alpha \equiv \alpha(\mu_1, \mu_2) = (\mu_1, \mu_2, 0) \), where \( \mu_1, \mu_2 \in \mathbb{C} \). We calculate
\[
\Delta^{(f, 0, 0)}(\alpha) = v_2^f v_3 e^{-v_3 + v_1 v_2} \bigg|_{v = \alpha} = 0,
\]
\[
\Delta^{(0, f, 0)}(\alpha) = v_1^f v_3 e^{-v_3 + v_1 v_2} \bigg|_{v = \alpha} = 0,
\]
\[
\Delta^{(0, 0, 1)}(\alpha) = (1 - v_3) e^{-v_3 + v_1 v_2} \bigg|_{v = \alpha} = \mu_1^j \mu_2 \neq 0, \quad j \in \mathbb{N}.
\]
The set \( \Omega(\alpha) \) contains all multi-indexes \( \omega \in \mathbb{Z}^3 \) such that \( \omega \geq (0, 0, 1) \). Since for arbitrary \( m, n \in \mathbb{Z}_+ \) equalities hold \( \Delta^{(m, n, 0)}(\alpha) = 0 \), then
\[
\overline{\Omega}(\alpha) = \left\{ (m, n, 0), m, n \in \mathbb{Z}_+ \right\}.
\]
Using the Theorem 3.3 for arbitrary \( m, n \in \mathbb{Z}_+ \) and \( \mu_1, \mu_2 \in \mathbb{C} \) we find such nontrivial solutions of problem (16):
\[
U_{mn}(t, x, \mu_1, \mu_2) = \frac{\partial^{m+n}}{\partial t^m \partial x^2} \left( \Phi(t, x, v) \right) \bigg|_{v = \alpha} =
\]
\[
= (1-t) \frac{\partial^{m+n}}{\partial t^m \partial \mu_1^n} \frac{\partial^{m+n}}{\partial t^m \partial \mu_2^n} e^{\mu_1 t x + \mu_2 x_2 + v x_3} \bigg|_{v = \alpha} = (1-t) \frac{\partial^{m+n}}{\partial \mu_1^m \partial \mu_2^n} e^{\mu_1 t x + \mu_2 x_2 + v x_3} \bigg|_{v = \alpha} =
\]
\[
= m! n! (1-t) e^{\mu_1 t x + \mu_2 x_2 + v x_3} \sum_{j=0}^{\min(m,n)} \frac{t^j}{j!} \frac{\mu_1^{j+1} x_2^{m-j} (\mu_2 t + x_2)^{n-j}}{(m-j)! (n-j)!}.
\]
It is easy to see that nontrivial solutions of problem (16) are also functions of the form
\[
U(t, x, \mu_1, \mu_2) = (1-t) \varphi \left( \frac{\partial}{\partial \mu_1}, \frac{\partial}{\partial \mu_2} \right) e^{\mu_1 t x + \mu_2 x_2},
\]
where \( \varphi \) is an arbitrary nontrivial entire function of two variables, whose order in the set of variables is not greater than second, and \( \mu_1, \mu_2 \) are arbitrary complex parameters.
5 Conclusions

In this work we proved that the homogeneous problem for PDE of the second order in time variable, in which local two-point conditions are imposed, and generally infinite order in spatial variables, has only trivial solution in the case when the characteristic determinant of the problem is not equal to zero. In the other case, when the characteristic determinant possesses a nonzero value, we proved the existence of nontrivial solutions of the problem and proposed the differential-symbol method for their construction. We also gave some examples of using this method.

The investigation of the null space, of the problem given herein, provide an opportunity for future to specify the classes of unique solvability of the corresponding nonhomogeneous problem.

References