Fixed point and multidimensional fixed point theorems with applications to nonlinear matrix equations in terms of weak altering distance functions

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Abstract: The aim of this work is to introduce the notion of weak altering distance functions and prove new fixed point theorems in metric spaces endowed with a transitive binary relation by using weak altering distance functions. We give some examples which support our main results where previous results in literature are not applicable. Then the main results of the paper are applied to the multidimensional fixed point results. As an application, we apply our main results to study a nonlinear matrix equation. Finally, as numerical experiments, we approximate the definite solution of a nonlinear matrix equation using MATLAB.

Keywords: Altering distance functions, Transitive relations, Weak altering distance functions

MSC: 47H10, 54H25

1 Introduction

The classical Banach contraction principle is one of the essential results of analysis. In the recent years, many authors extended fixed point results for weak contractions and generalized contractions, which are generalizations of Banach contraction mapping principle to partially ordered metric spaces (see [1–15]). Some of the above results involve altering distance functions presented by Khan et al. in [16].

Now, we recall the definition of an altering distance function.

Definition 1.1 ([16]). A function $\psi: [0, \infty) \to [0, \infty)$ is said to be an altering distance function if it satisfies the following conditions:

(a) $\psi$ is continuous and nondecreasing;

(b) $\psi(t) = 0$ if and only if $t = 0$.

Example 1.2. Define $\psi_1, \psi_2, \psi_3, \psi_4: [0, \infty) \to [0, \infty)$ by $\psi_1(t) = t$, $\psi_2(t) = t^2$, $\psi_3(t) = te^t$, $\psi_4(t) = \ln(1+t)$ for all $t \geq 0$. We see that $\psi_1, \psi_2, \psi_3$ and $\psi_4$ are altering distance functions because $\psi_1, \psi_2, \psi_3$ and $\psi_4$ are continuous and nondecreasing.

Moreover, $\psi_i(t) = 0$ if and only if $t = 0$ for all $i = 1, 2, 3, 4$. (The graphs of functions $\psi_1, \psi_2, \psi_3$ and $\psi_4$ show in Figure 1).

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In 2012, Yan et al. [17] discussed some results on existence and uniqueness of a fixed point in partially ordered metric spaces by using the concept of an altering distance function as follows.

**Theorem 1.3** ([17]). Let \((X, \preceq)\) be a partially ordered set and suppose that there exists a metric \(d\) in \(X\) such that \((X, d)\) is a complete metric space. Suppose that \(T : X \to X\) is a continuous and nondecreasing mapping such that

\[
\psi(d(Tx, Ty)) \leq \phi(d(x, y)), \quad \forall x, y \in X \text{ with } x \preceq y,
\]

where \(\psi\) is an altering distance function and \(\phi : [0, \infty) \to [0, \infty)\) is a continuous function with the condition: \(\psi(t) > \phi(t)\) for all \(t > 0\). If there exists \(x_0 \in X\) such that \(x_0 \preceq Tx_0\), then \(T\) has a fixed point.

On the other hand, the notion of a coupled fixed point was firstly investigated by Guo and Lakshmikantham in [18]. In 2006, Bhaskar and Lakshmikantham [19] were the first to introduce the notion of mixed monotone property in partially ordered metric spaces. They also established the classical coupled fixed point theorems for mappings by using this property under contractive type conditions. Due to the important role of such results for the investigation of solutions of nonlinear differential and integral equations, several authors have studied various generalizations of these results. In this continuation, several authors introduced concepts of a tripled fixed point, quadruple fixed point and multidimensional fixed point. They also proved new results on the existence and uniqueness of multidimensional fixed points that were presented in partially ordered metric spaces and another spaces. For instance, Rus [20] obtained the existence and uniqueness results for a multidimensional fixed point of nonlinear mappings satisfying the contractive condition with a control function in the following class:

\[
\Phi := \{\varphi : [0, \infty) \to [0, \infty) : \varphi \text{ is nonincreasing such that } \varphi^n(t) \to \infty \text{ as } n \to \infty \text{ for all } t \in [0, \infty)\},
\]

together with an approximating iterative scheme, in the setting of partially ordered metric spaces.

In 2015, Soleimani Rad et al. [21] compared relation between multidimensional fixed point results and fixed point theorems concerning various contractive conditions which are depended on contractive constants in abstract metric spaces and metric-like spaces. Also, they claimed that these results are true for another spaces with several contractive conditions. Recently, Su et al. [22] have discussed the existence and uniqueness results for multidimensional fixed point of contraction mappings involving some control functions in complete metric spaces.
In this paper, we present a new concept of weak altering distance function and establish fixed point theorems for generalized contraction mappings in complete metric spaces endowed with a transitive relation by using the idea of a weak altering distance function. Also, we provide an example that supports our main result where previous results in literature are not applicable. Our results generalize and improve the main result of Yan et al. [17] and several well-known results given by some authors in partially metric spaces. Moreover, we present a new extension of multidimensional fixed point theorems in metric spaces endowed with a transitive relation. Furthermore, we apply our results to prove the existence and uniqueness of a solution of nonlinear matrix equations. Finally, we use some numerical examples to show the iterative method is feasible to confirm the existence and uniqueness of positive definite solution of a nonlinear matrix equation.

2 Preliminaries

We start our consideration by giving a brief review of the definitions and basic properties in this work. Henceforth, \( X, N \) and \( N_0 \) denote a nonempty set, the set of positive integers and nonnegative integers, respectively, and \( N \) denotes a positive integer. Throughout this paper, unless otherwise specified, \( \mathfrak{R} \) denotes a binary relation on \( X \) and \( \mathfrak{R}^N \) denotes a binary relation on the \( N \)-fold Cartesian product \( X^N \) which is defined by

\[
((x_1, x_2, \ldots, x_N), (y_1, y_2, \ldots, y_N)) \in \mathfrak{R}^N \iff (x_1, y_1) \in \mathfrak{R}, (x_2, y_2) \in \mathfrak{R}, \ldots, (x_N, y_N) \in \mathfrak{R}.
\]

**Definition 2.1** ([23]). Let \( X \) be a nonempty set and \( T : X^N \to X \) be a given mapping. An element \((x_1, x_2, \ldots, x_N) \in X^N\) is said to be a fixed point of \( N \)-order of the mapping \( T \) if

\[
\begin{align*}
T(x_1, x_2, \ldots, x_N) &= x_1, \\
T(x_2, x_3, \ldots, x_N, x_1) &= x_2, \\
& \vdots \\
T(x_N, x_1, \ldots, x_{N-1}) &= x_N.
\end{align*}
\]

**Definition 2.2** ([24]). Let \( X \) be a nonempty set. A subset \( \mathfrak{R} \) of \( X^2 \) is called a binary relation on \( X \). Notice that for each pair \( x, y \in X \), one of the following conditions holds:

(i) \((x, y) \in \mathfrak{R}\), which amounts to saying that “\( x \) is \( \mathfrak{R} \)-related to \( y \)” or “\( x \) relates to \( y \) under \( \mathfrak{R} \).” Sometimes, we write \( x\mathfrak{R}y \) instead of \((x, y) \in \mathfrak{R}\);

(ii) \((x, y) \notin \mathfrak{R}\) which means that “\( x \) is not \( \mathfrak{R} \)-related to \( y \)” or “\( x \) does not relate to \( y \) under \( \mathfrak{R} \).”

**Definition 2.3.** A binary relation \( \mathfrak{R} \) defined on a nonempty set \( X \) is called transitive if \((x, y) \in \mathfrak{R} \) and \((y, z) \in \mathfrak{R} \) implies \((x, z) \in \mathfrak{R} \).

**Definition 2.4** ([25]). Let \( \mathfrak{R} \) be a binary relation defined on a nonempty set \( X \) and \( x, y \in X \). We say that \( x \) and \( y \) are \( \mathfrak{R} \)-comparative if either \((x, y) \in \mathfrak{R} \) or \((y, x) \in \mathfrak{R} \). We denote it by \([x, y] \in \mathfrak{R}\).

**Definition 2.5** ([25]). Let \( X \) be a nonempty set and \( T \) be a self-mapping on \( X \). A binary relation \( \mathfrak{R} \) defined on \( X \) is called \( T \)-closed if for any \( x, y \in X \), \((x, y) \in \mathfrak{R} \implies (Tx, Ty) \in \mathfrak{R} \).

**Definition 2.6** ([25]). Let \( X \) be a nonempty set and \( \mathfrak{R} \) be a binary relation on \( X \). A sequence \( \{x_n\} \subset X \) is called \( \mathfrak{R} \)-preserving if

\[
(x_n, x_{n+1}) \in \mathfrak{R} \quad \forall n \in \mathbb{N}.
\]

**Definition 2.7** ([25]). Let \((X, d)\) be a metric space. A binary relation \( \mathfrak{R} \) defined on \( X \) is called \( d \)-self-closed if whenever \( \{x_n\} \) is an \( \mathfrak{R} \)-preserving sequence and

\[
x_n \xrightarrow{d} x \quad \text{as} \quad n \to \infty,
\]
then there exists a subsequence \{x_{n_k}\} of \{x_n\} with \[x_{n_k}, x\] \in \mathcal{R}\) for all \(k \in \mathbb{N}\).

**Definition 2.8** ([26]). Let \(X\) be a nonempty set and \(\mathcal{R}\) a binary relation on \(X\). A subset \(E\) of \(X\) is called \(\mathcal{R}\)-directed if for each \(x, y \in E\), there exists \(z \in X\) such that \((x, z) \in \mathcal{R}\) and \((y, z) \in \mathcal{R}\).

In this paper, we use the following notations for a binary relation \(\mathcal{R}\) on a nonempty set \(X\):

\[X(T; \mathcal{R}) := \{x \in X : (x, Tx) \in \mathcal{R}\},\]

where \(T : X \to X\) is a given mapping.

### 3 Main results

We first give the definition of a weak altering distance function as follows:

**Definition 3.1.** A function \(\psi : [0, \infty) \to [0, \infty)\) is said to be a weak altering distance function if it satisfies the following conditions:

(a) \(\psi\) is lower semicontinuous and nondecreasing;
(b) \(\psi(t) = 0\) if and only if \(t = 0\).

Every continuous function is lower semicontinuous and so the class of weak altering distance functions is larger than the class of altering distance functions. In general, a weak altering distance function need not necessarily be an altering distance function. Next, we give some examples of the weak altering distance functions which show that the weak altering distance functions are real generalization of altering distance functions.

**Example 3.2.** Define \(\psi_1, \psi_2, \psi_3 : [0, \infty) \to [0, \infty)\) by

\[
\psi_1(t) = \begin{cases} 
\ln(1 + t) & \text{if } t \leq 1 \\
\frac{t}{2} & \text{if } t > 1
\end{cases},
\]

\[
\psi_2(t) = \begin{cases} 
\frac{t^2}{2} & \text{if } t \leq 1 \\
e^t - 1 & \text{if } t > 1
\end{cases},
\]

\[
\psi_3(t) = \begin{cases} 
\frac{t^2}{2} & \text{if } t \leq 1 \\
\frac{t^2}{4} & \text{if } t > 1
\end{cases}.
\]

We see that \(\psi_1, \psi_2\) and \(\psi_3\) are weak altering distance functions because \(\psi_1, \psi_2\) and \(\psi_3\) are lower semicontinuous and nondecreasing. Moreover, \(\psi_i(t) = 0\) if and only if \(t = 0\) for all \(i = 1, 2, 3\). (The graphs of functions \(\psi_1, \psi_2\) and \(\psi_3\) show in Figure 2).

![Graphs of \(\psi_1, \psi_2, \psi_3\) in Example 3.2.](image-url)
Now we give an useful proposition concerning a contractive condition given for comparable elements with respect to a binary relation.

**Proposition 3.3.** If \((X, d)\) is a metric space, \(\mathcal{R}\) is a binary relation on \(X\), \(T\) is a self-mapping on \(X\), \(\psi\) is a weak altering distance function and \(\phi : [0, \infty) \rightarrow [0, \infty)\) is a right upper semicontinuous function, then the following contractivity conditions are equivalent:

(i) \(\psi(d(Tx, Ty)) \leq \phi(d(x, y)), \quad \forall x, y \in X \text{ with } (x, y) \in \mathcal{R},\)

(ii) \(\psi(d(Tx, Ty)) \leq \phi(d(x, y)), \quad \forall x, y \in X \text{ with } [x, y] \in \mathcal{R}.\)

**Proof.** First, we will show that the implication (i) \(\Rightarrow\) (ii) holds.

Assume that (i) holds. Take \(x, y \in X\) with \([x, y] \in \mathcal{R}\). If \((x, y) \in \mathcal{R}\), then (ii) directly follows from (i). Now, suppose that \((y, x) \in \mathcal{R}\), then using the symmetry of \(d\) and (i), we get

\[
\psi(d(Tx, Ty)) = \psi(d(Ty, Tx)) \leq \phi(d(y, x)) = \phi(d(x, y)).
\]

This shows that (i) \(\Rightarrow\) (ii).

Conversely, the implication (ii) \(\Rightarrow\) (i) is trivial. This completes the proof. \(\square\)

**Theorem 3.4.** Let \((X, d)\) be a complete metric space and \(\mathcal{R}\) be a transitive relation on \(X\). Suppose that \(T : X \rightarrow X\) is a continuous mappings and \(\mathcal{R}\) is \(T\)-closed such that

\[
\psi(d(Tx, Ty)) \leq \phi(d(x, y)), \quad \forall x, y \in X \text{ with } (x, y) \in \mathcal{R},
\]

where \(\psi\) is a weak altering distance function and \(\phi : [0, \infty) \rightarrow [0, \infty)\) is a right upper semicontinuous function such that \(\psi(t) > \phi(t)\) for all \(t > 0\). If \(X(T; \mathcal{R})\) is a nonempty set, then \(T\) has a fixed point.

**Proof.** Let \(x_0\) be an arbitrary point in \(X(T; \mathcal{R})\). Put \(x_n = T^n x_0\) for all \(n \in \mathbb{N}\). If \(x_{n^*} = x_{n^*+1}\) for some \(n^* \in \mathbb{N}\), then \(x_{n^*}\) is a fixed point of \(T\). Thus we will assume that \(x_n \neq x_{n+1}\) for all \(n \in \mathbb{N}\). Since \((x_0, Tx_0) \in \mathcal{R}\), using the \(T\)-closedness of \(\mathcal{R}\), we get

\[
(Tx_0, T^2 x_0, T^3 x_0, \ldots, T^n x_0, T^{n+1} x_0, \ldots) \in \mathcal{R}
\]

and so \((x_n, x_{n+1}) \in \mathcal{R}\) for all \(n \in \mathbb{N}\). Thus the sequence \(\{x_n\}\) is \(\mathcal{R}\)-preserving. From contractive condition (1), we have

\[
\psi(d(x_n, x_{n+1})) = \psi(d(Tx_{n-1}, Tx_n)) \leq \phi(d(x_{n-1}, x_n)) < \psi(d(x_{n-1}, x_n))
\]

for all \(n \in \mathbb{N}\). Since \(\psi\) is a nondecreasing function, we have

\[
d(x_n, x_{n+1}) < d(x_{n-1}, x_n)
\]

for all \(n \in \mathbb{N}\). Thus, the sequence \(\{d(x_n, x_{n+1})\}\) is decreasing and bounded below. Consequently, there exists \(s \geq 0\) such that

\[
d(x_n, x_{n+1}) \rightarrow s \quad \text{as} \quad n \rightarrow \infty.
\]

From (3), letting \(n \rightarrow \infty\) and using the property of \(\psi\) and \(\phi\) we get

\[
\psi(s) \leq \liminf_{n \rightarrow \infty} \psi(d(x_n, x_{n+1})) \leq \limsup_{n \rightarrow \infty} \psi(d(x_n, x_{n+1})) \leq \limsup_{n \rightarrow \infty} \phi(d(x_{n-1}, x_n)) \leq \phi(s).
\]

Since \(\psi(t) > \phi(t)\) for all \(t > 0\), we have \(s = 0\) and so \(\{d(x_n, x_{n+1})\}\) converges to 0. Now, we will show that \(\{x_n\}\) is a Cauchy sequence.

Assume on the contrary, there is an \(\epsilon > 0\) and subsequences \(\{x_{m_k}\}\) and \(\{x_{n_k}\}\) of \(\{x_n\}\) with \(m_k > m_k \geq k\) such that

\[
d(x_{m_k}, x_{n_k}) \geq \epsilon \quad \text{for all} \quad k \in \mathbb{N}.
\]
Choosing $n_k$ to be the smallest integer exceeding $m_k$ for which (5) holds, we obtain that

$$d(x_{m_k}, x_{n_k - 1}) < \epsilon. \quad (6)$$

Using (5) and (6), we get

$$\epsilon \leq d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{n_k - 1}) + d(x_{n_k - 1}, x_{n_k}) < \epsilon + d(x_{n_k - 1}, x_{n_k}).$$

Hence, $d(x_{m_k}, x_{n_k}) \to \epsilon$ as $k \to \infty$. Furthermore, we have

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k - 1}) + d(x_{m_k - 1}, x_{n_k - 1}) + d(x_{n_k - 1}, x_{n_k}) \quad (7)$$

and

$$d(x_{m_k - 1}, x_{n_k - 1}) \leq d(x_{m_k - 1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k - 1}) \quad (8)$$

Letting $k \to \infty$ in (7) and (8) and using the fact that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ and $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon$, we have

$$\lim_{k\to\infty} d(x_{m_k - 1}, x_{n_k - 1}) = \epsilon.$$ 

Since $n_k > m$ and $\mathcal{R}$ is a transitive relation, we get $(x_{m_k - 1}, x_{n_k - 1}) \in \mathcal{R}$. This implies that

$$\psi(d(x_{m_k}, x_{n_k})) \leq \phi(d(x_{m_k - 1}, x_{n_k - 1})). \quad (9)$$

From (9), letting $k \to \infty$ and using the property of $\psi$ and $\phi$ we get

$$\psi(\epsilon) \leq \liminf_{k\to\infty} \psi(d(x_{m_k}, x_{n_k})) \leq \limsup_{k\to\infty} \psi(d(x_{m_k}, x_{n_k})) \leq \limsup_{k\to\infty} \phi(d(x_{m_k - 1}, x_{n_k - 1})) \leq \phi(\epsilon).$$

It yields that $\epsilon = 0$, which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence.

Since $(X, d)$ is a complete metric space, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Thus, by the continuity of $T$, we get $Tx^* = x^*$. This completes the proof. \(\square\)

Remark 3.5. It is fascinating to point out that we use the result in Theorem 3.4 to derive a criterion for the existence of fixed points in some cases wherein several results contained in [17, 20–22] cannot guarantee the existence of fixed points. Indeed, the main results of Yan et al. in [17] (Theorem 1.3) are not applicable in the following cases:

- $(X, \preceq)$ is not a partially ordered set;
- $\psi$ is not an altering distance function;
- $\phi$ is not continuous,

the main results of Rus in [20] are not applicable in the following cases:

- $(X, \preceq)$ is not a partially ordered set (or quasi-ordered set);
- $T$ does not satisfy the contractive condition with the control function $\varphi \in \Phi$,

the main results of Soleimani Rad et al. in [21] are not applicable in the following cases:

- $T$ does not satisfy the contractive condition with several contractive constant,
- $T$ is not a complete metric space without the transitive relation;
- $T$ does not satisfy the contractive condition with two control functions $\psi$ and $\phi$.

Now, we give an example to illustrate utility of Theorem 3.4.

Example 3.6. Let $X = [0, \infty)$ with usual metric $d$. Thus $(X, d)$ is a complete metric space. Define a binary relation $\mathcal{R}$ on $X$ by

$$\mathcal{R} := \{(x, y) \in X \times X : x^2 + 2x = y^2 + 2y\}.$$ 

It is easy to prove that $\mathcal{R}$ is a transitive relation on $X$. Also, we define two functions $\phi, \psi : [0, \infty) \to [0, \infty)$ by

$$\psi(t) = \begin{cases} \ln(t + 1) & \text{if } t \leq 1, \\ \frac{3t}{t + 1} & \text{if } t > 1 \end{cases}.$$
and \( \phi(t) = \frac{2t}{T} \). Then \( \psi \) is a weak altering distance function and \( \phi \) is a right upper semicontinuous function such that \( \psi(t) > \phi(t) \) for all \( t > 0 \).

Let \( T : X \rightarrow X \) be defined by

\[
Tx = \ln(x^2 + 2x + 1)
\]

for all \( x \in X \). Then \( T \) is continuous.

Next, we will show that \( X \) is \( T \)-closed. Assume that \( x, y \in X \) such that \( (x, y) \in X \) and then \( \ln(x^2 + 2x + 1) = \ln(y^2 + 2y + 1) \). This means that \( Tx = Ty \) and so \( (Tx)^2 + 2Tx = (Ty)^2 + 2Ty \). This implies that \( (Tx, Ty) \in X \).

Therefore, \( X \) is \( T \)-closed. Moreover, there exists \( 0 \in X \) such that \( (0, T0) \in X \). This shows that \( X(T; X) \) is a nonempty set.

Finally, it is clear that for each \( x, y \in X \) with \( (x, y) \in X \), we get

\[
\psi(d(Tx, Ty)) \leq \phi(d(x, y)).
\]

So \( T \) satisfies the contractive condition (1).

Now all of the conditions in Theorem 3.4 hold and hence \( T \) has at least one fixed point. For instance, the point \( x = 0 \) is one of the fixed point of \( T \).

**Remark 3.7.** We note that Yan et al.’s result in [17] (Theorem 1.3) is not applicable in the above example since \( \psi \) is not an altering distance function. This implies that the Banach contraction principle is not also applicable in the above example.

The following theorem guarantees the uniqueness of fixed point in Theorem 3.4.

**Theorem 3.8.** In addition to the hypothesis of Theorem 3.4, suppose that \( \phi(0) = 0 \) and \( X \) is \( \mathfrak{A} \)-directed. Then \( T \) has a unique fixed point.

**Proof.** Suppose that there exist \( x^*, y^* \in X \) which are fixed points. We distinguish two cases.

**Case 1.** If \( (x^*, y^*) \in X \), then \( (T^n x^*, T^n y^*) \in X \) for all \( n \in \mathbb{N}_0 \). It yields that

\[
\psi(d(x^*, y^*)) = \psi(d(T^n x^*, T^n y^*)) \leq \phi(d(T^{n-1} x^*, T^{n-1} y^*)) = \phi(d(x^*, y^*))
\]

for all \( n \in \mathbb{N} \). From the fact that \( \psi(t) > \phi(t) \) for all \( t > 0 \), we get \( d(x^*, y^*) = 0 \). Therefore, \( x^* = y^* \).

**Case 2.** If \( (x^*, y^*) \notin X \), then there exists \( z^* \in X \) such that \( (x^*, z^*) \in X \) and \( (y^*, z^*) \in X \). Since \( X \) is \( T \)-closed, we get \( (T^n x^*, T^n z^*) \in X \) and \( (T^n y^*, T^n z^*) \in X \) for all \( n \in \mathbb{N}_0 \).

Moreover, we have

\[
\psi(d(x^*, T^n z^*)) = \psi(d(T^n x^*, T^n z^*)) \leq \phi(d(T^{n-1} x^*, T^{n-1} z^*))
\]

(10)

for all \( n \in \mathbb{N} \). Since \( \psi \) is a nondecreasing function, we have

\[
d(x^*, T^n z^*) \leq d(x^*, T^{n-1} z^*)
\]

(11)

for all \( n \in \mathbb{N} \). Thus, the sequence \( \{d(x^*, T^n z^*)\} \) is non-increasing. Thus, there exists \( \xi \) such that

\[
\lim_{n \rightarrow \infty} d(x^*, T^n z^*) = \xi.
\]

From (10), letting \( n \rightarrow \infty \) and using the property of \( \psi \) and \( \phi \) we get

\[
\psi(\xi) \leq \lim sup_{n \rightarrow \infty} \psi(d(x^*, T^n z^*)) \leq \lim sup_{n \rightarrow \infty} \phi(d(x^*, T^{n-1} z^*)) \leq \phi(\xi).
\]

(12)

From (12) and the condition: \( \psi(t) > \phi(t) \) for all \( t > 0 \), it implies that \( \xi = 0 \). Similarly, we can show that

\[
\lim_{n \rightarrow \infty} d(y^*, T^n z^*) = 0.
\]

Therefore, \( T^n z^* \rightarrow x^* \) and \( T^n z^* \rightarrow y^* \) as \( n \rightarrow \infty \). This implies that \( x^* = y^* \). This completes the proof. \( \square \)
Now we use the following notation for a binary relation \( \mathcal{R} \) on a nonempty set \( X \) for all \( N \in \mathbb{N} \),

\[
X^N(T; \mathcal{R}^N) := \{(x_1, x_2, \ldots, x_N) \in X^N : (T(x_1, x_2, \ldots, x_N), T(x_2, x_3, \ldots, x_N, x_1), \ldots, T(x_N, x_1, \ldots, x_{N-1})) \in \mathcal{R}^N \},
\]

where \( T : X^N \to X \) is a given mapping.

**Definition 3.9** ([27]). Let \( X \) be a nonempty set. Given \( N \in \mathbb{N} \) and \( T : X^N \to X \) is a mapping. A binary relation \( \mathcal{R} \) defined on \( X \) is called \( T_N \)-closed if for any \((x_1, x_2, \ldots, x_N), (y_1, y_2, \ldots, y_N) \in X^N \),

\[
\begin{align*}
(x_1, y_1) \in \mathcal{R} & \quad \Rightarrow \quad (T(x_1, x_2, \ldots, x_N), T(y_1, y_2, \ldots, y_N)) \in \mathcal{R} \\
(x_2, y_2) \in \mathcal{R} & \quad \Rightarrow \quad (T(x_2, x_3, \ldots, x_N, x_1), T(y_2, y_3, \ldots, y_N, y_1)) \in \mathcal{R} \\
& \cdots \\
(x_N, y_N) \in \mathcal{R} & \quad \Rightarrow \quad (T(x_N, x_1, \ldots, x_{N-1}), T(y_N, y_1, \ldots, y_{N-1})) \in \mathcal{R}
\end{align*}
\]

**Definition 3.10.** Let \( X \) be a nonempty set and \( \mathcal{R} \) a binary relation on \( X \). Given \( N \in \mathbb{N} \). A subset \( E^N \) of \( X^N \) is called \( \mathcal{R}^N \)-directed if for each \((x_1, x_2, \ldots, x_N), (y_1, y_2, \ldots, y_N) \in E^N \), there exists \((z_1, z_2, \ldots, z_N) \in X \) such that

\[
((x_1, x_2, \ldots, x_N), (z_1, z_2, \ldots, z_N)) \in \mathcal{R}^N
\]

and

\[
((y_1, y_2, \ldots, y_N), (z_1, z_2, \ldots, z_N)) \in \mathcal{R}^N.
\]

Next, we illustrate how to prove multidimensional results from the unidimensional result by involving simple tools.

Given \( N \in \mathbb{N} \) and a mapping \( T : X^N \to X \), let us denote by \( G_T^N : X^N \to X^N \) the mapping

\[
G_T^N(x_1, x_2, \ldots, x_N) = (T(x_1, x_2, \ldots, x_N), T(x_2, x_3, \ldots, x_N, x_1), \ldots, T(x_N, x_1, \ldots, x_{N-1})).
\]

The following lemmas will be useful later.

**Lemma 3.11** ([27]). Given \( N \in \mathbb{N} \), \( T : X^N \to X \), a point \((x_1, x_2, \ldots, x_N) \in X^N \) is a fixed point of \( N \)-order of mapping \( T \) if and only if it is a fixed point of \( G_T^N \).

**Lemma 3.12** ([27]). Given \( N \in \mathbb{N} \), \( T : X^N \to X \) and \( G_T^N : X^N \to X^N \) are two mappings. A binary relation \( \mathcal{R} \) defined on \( X \) is \( T_N \)-closed if and only if a binary relation \( \mathcal{R}^N \) defined on \( X^N \) is \( G_T^N \)-closed.

**Lemma 3.13** ([27]). Given \( N \in \mathbb{N} \), \( T : X^N \to X \) and \( G_T^N : X^N \to X^N \) are two mappings. A point \((x_1, x_2, \ldots, x_N) \in X^N(T; \mathcal{R}^N) \) if and only if a point \((x_1, x_2, \ldots, x_N) \in X^N(G_T^N; \mathcal{R}^N) \).

**Lemma 3.14** ([27]). Given \( N \in \mathbb{N} \). Let \((X, d)\) be a metric space and a mapping \( D^N : X^N \times X^N \to \mathbb{R} \) defined by

\[
D^N(A, B) = \sum_{i=1}^{N} d(a_i, b_i)
\]

for all \( A = (a_1, a_2, \ldots, a_N), B = (b_1, b_2, \ldots, b_N) \in X^N \). Then the following properties hold.

1. \((X^N, D^N)\) is also a metric space.
2. Let \( \{a_n \} = (a_1^n, a_2^n, \ldots, a_N^n) \) be a sequence on \( X^N \) and let \( A = (a_1, a_2, \ldots, a_N) \in X^N \). Then \( \{a_n \} \xrightarrow{D^N} A \) if and only if \( a_i^n \xrightarrow{d} a_i \) for all \( i \in \{1, 2, \ldots, N\} \).
3. If \( \{a_n \} = (a_1^n, a_2^n, \ldots, a_N^n) \) is a sequence on \( X^N \), then \( \{a_n \} \) is \( D^N \)-Cauchy if and only if \( a_i^n \) is Cauchy for all \( i \in \{1, 2, \ldots, N\} \).
4. \((X, d)\) is complete if and only if \((X^N, D^N)\) is complete.

Here, we show how to use Theorem 3.4 in order to deduce multidimensional fixed point results.
Theorem 3.15. Let $(X, d)$ be a complete metric space and $\mathcal{R}$ be a transitive relation on $X$. Given $N \in \mathbb{N}$. Suppose that $T : X^N \to X$ is a continuous and $\mathcal{R}$ is $T_N$-closed such that for each $(x_1, x_2, \ldots, x_N), (y_1, y_2, \ldots, y_N) \in X^N$ with $(x_1, x_2, \ldots, x_N), (y_1, y_2, \ldots, y_N) \in \mathcal{R}^N$,
\[
\psi \left( d(T(x_1, x_2, \ldots, x_N), T(y_1, y_2, \ldots, y_N)) \right) + d(T(x_2, x_3, \ldots, x_N, x_1), T(y_2, y_3, \ldots, y_N, y_1)) + \cdots + d(T(x_N, x_1, \ldots, x_{N-1}), T(y_N, y_1, \ldots, y_{N-1})) \right) \leq \phi \left( \sum_{i=1}^{N} d(x_i, y_i) \right),
\]
where $\psi$ is a weak altering distance function and $\phi : [0, \infty) \to [0, \infty)$ is a right upper semicontinuous function such that $\psi(t) > \phi(t)$ for all $t > 0$. If $X^N(T; \mathcal{R})$ is a nonempty set, then $T$ has a fixed point of $N$-order.

Proof. Using items 1 and 4 of Lemma 3.14, we obtain that $(X^N, D^N)$ is a complete metric space. By Lemma 3.12, a binary relation $\mathcal{R}^N$ defined on $X^N$ is $G_T^N$-closed. Assume that $(x_0^1, x_0^2, \ldots, x_0^N) \in X^N(T; \mathcal{R}^N)$, by Lemma 3.13 we get $(x_1^1, x_1^2, \ldots, x_1^N) \in X^N(G_T^N; \mathcal{R}^N)$.

Now, let $A = (a_1, a_2, \ldots, a_N), B = (b_1, b_2, \ldots, b_N) \in X^N$ such that $(A, B) \in \mathcal{R}^N$. Then
\[
\psi(D^N(G_T^N A, G_T^N B)) = \psi(D^N(G_T^N(a_1, a_2, \ldots, a_N), G_T^N(b_1, b_2, \ldots, b_N)))
\]
\[
= \psi \left( \left( T(a_1, a_2, \ldots, a_N), T(a_2, a_3, \ldots, a_N, a_1), \ldots, T(a_N, a_1, \ldots, a_{N-1}) \right) \right)
\]
\[
= \psi \left( \left( T(b_1, b_2, \ldots, b_N), T(b_2, b_3, \ldots, b_N, b_1), \ldots, T(b_N, b_1, \ldots, b_{N-1}) \right) \right)
\]
\[
\leq \phi \left( \sum_{i=1}^{N} d(a_i, b_i) \right)
\]
\[
= \phi(D^N(A, B)).
\]
Applying Theorem 3.4, there exists $\hat{X} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_N) \in X^N$ such that $G_T^N \hat{X} = \hat{X}$. That is, $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_N)$ is a fixed point of $G_T^N$. Using Lemma 3.11, $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_N)$ is a fixed point of $N$-order of mapping $T$. This completes the proof.

By using Theorem 3.8, we get the following uniqueness result of fixed point of $N$-order.

Theorem 3.16. In addition to the hypothesis of Theorem 3.15, suppose that $\phi(0) = 0$ and $X^N$ is $\mathcal{R}^N$-directed. Then $T$ has a unique fixed point of $N$-order.

4 Applications

In this section, we will use the following notations:
- $M(n)$ denotes the set of all $n \times n$ complex matrices;
- $H(n) \subset M(n)$ is the set of all $n \times n$ Hermitian matrices;
- $P(n) \subset H(n)$ is the set of all $n \times n$ positive definite matrices;
- $H^+(n) \subset H(n)$ is the set of all $n \times n$ positive semidefinite matrices.

Furthermore, we will use the following notations for $X, Y \in M(n)$:
- $X > 0 \iff X \in P(n)$;
- $X \geq 0 \iff X \in H^+(n)$;
- $X - Y > 0 \iff X > Y$;
- $X - Y \geq 0 \iff X \geq Y$.

It is well-known that for each $X, Y \in H(n)$, there is the greatest lower bound and the least upper bound.
We use the symbol $\| \cdot \|$ which stands for the spectral norm of a matrix $A$ unless and until it is stated, i.e.,

$$\| A \| = \sqrt{\lambda^+ (A^* A)},$$

where $\lambda^+(A^* A)$ is the largest eigenvalue of $A^* A$ and $A^*$ is the conjugate transpose of $A$.

In the sequel, we use the metric induced by the trace norm $\| \cdot \|_{tr}$ defined by $\| A \|_{tr} = \sum_{j=1}^{n} s_j(A)$, where $s_j(A)$, $j = 1, 2, \ldots, n$, are the singular values of $A \in M(n)$. The set $H(n)$ endowed with this norm is a complete metric space. See [28–30] for more details. Moreover, we see that $H(n)$ is a partially ordered set with partial order $\preceq$ which is defined by

$$X \preceq Y \iff Y \succeq X.$$

In 2003, Ran and Reurings [28] discussed an analogue of Banach contraction mapping principle in partially ordered sets and applied this result to linear and nonlinear matrix equations. Later, Petrušel and Rus [31] presented fixed point results in ordered $L$-spaces and applied it to nonlinear matrix equations which are generalization and extension of Ran and Reurings [28].

Inspired by the work mentioned above and the basis of the fixed point results in previous section, we investigate the nonlinear matrix equation

$$X = Q + \sum_{i=1}^{m} A_i^* G(X) A_i,$$  

(13)

where $A_i$ is an arbitrary $n \times n$ matrices, $Q$ is a Hermitian positive definite matrix and $G$ is continuous order preserving\(^1\) maps from $H(n)$ into $P(n)$ such that $G(0) = 0$.

The following lemmas will be useful later.

**Lemma 4.1** ([28]). If $A, B \in H^+(n)$, then

$$0 \leq tr(AB) \leq \| A \| tr(B).$$

**Lemma 4.2** ([32]). If $A \in H(n)$ and $A \prec I$, then $\| A \| < 1$.

**Theorem 4.3.** Consider the matrix equation (13). Assume that there is a positive number $M$ such that:

(i) for every $X, Y \in H(n)$ such that $(X, Y) \in \preceq$, we have

$$|tr(G(Y) - G(X))| \leq \frac{1}{M} |h(tr(Y - X))|$$

where $h : [0, \infty) \rightarrow [0, \infty)$ is a right upper semicontinuous function with $h(t) < t^2$ for all $t > 0$;

(ii) $\sum_{i=1}^{m} A_i A_i^* < M I_n$.

If $\sum_{i=1}^{m} A_i G(Q) A_i \succ 0$, then the matrix equation (13) has a solution. Moreover, the iteration

$$X_n = Q + \sum_{i=1}^{n} A_i^* G(X_{n-1}) A_i$$

(14)

where $X_0 \in H(n)$ such that $X_0 \preceq Q + \sum_{i=1}^{n} A_i^* G(X_0) A_i$, converges in the sense of trace norm $\| \cdot \|_{tr}$ to the solution of matrix equation (13).

**Proof.** Throughout this proof, we define the mapping $F : H(n) \rightarrow H(n)$ by

$$F(X) = Q + \sum_{i=1}^{m} A_i^* G(X) A_i$$

for all $X \in H(n)$.  

(15)

Then $F$ is well defined and $\preceq$ on $H(n)$ is $F$-closed and a fixed point of $F$ is a solution of equation (13).

---

\(^1\) $G$ is order preserving if $A, B \in H(n)$ with $A \preceq B$ implies that $G(A) \preceq G(B)$.
Next, we will show that the contractive condition (1) holds with \( F \).
Let \( X, Y \in H(n) \) such that \((X, Y) \in \leq \). This means that \( X \preceq Y \) and then \( G(X) \preceq G(Y) \). Therefore,

\[
\| F(Y) - F(X) \|_{tr} = tr(F(Y) - F(X))
\]

\[
= tr\left( \sum_{i=1}^{m} A_i^* (G(Y) - G(X)) A_i \right)
\]

\[
= \sum_{i=1}^{m} tr(A_i^* (G(Y) - G(X)) A_i)
\]

\[
= \sum_{i=1}^{m} tr(A_i A_i^* (G(Y) - G(X)))
\]

\[
= tr\left( \sum_{i=1}^{m} A_i A_i^* \right) (G(Y) - G(X))
\]

\[
\leq \left\| \sum_{i=1}^{m} A_i A_i^* \right\| \| G(Y) - G(X) \|_{tr}
\]

\[
\leq \frac{\| \sum_{i=1}^{m} A_i A_i^* \|}{M} \sqrt{h(\| Y - X \|_{tr})}
\]

This yields that,

\[
(\| F(Y) - F(X) \|_{tr})^2 \leq h(\| Y - X \|_{tr}).
\]  

Putting \( \psi(t) = t^2 \) and \( \phi(t) = h(t) \), obviously \( \psi \) is a weak altering distance function and \( \phi \) is a right upper semicontinuous function such that \( \psi(t) > \phi(t) \) for all \( t > 0 \).

From the inequality (16), we have

\[
\psi(\| F(Y) - F(X) \|_{tr}) \leq \phi(\| Y - X \|_{tr}).
\]

Thus, the contractive condition (1) in Theorem 3.4 is satisfied for all \( X, Y \in H(n) \) such that \((X, Y) \in \leq \). From \( \sum_{i=1}^{m} A_i^* G(Q)A_i > 0 \), we have \( Q \preceq F(Q) \). This means that \( Q \in H(n)(F; \preceq) \). Now from Theorem 3.4, there exists \( \tilde{X} \in H(n) \) such that \( F(\tilde{X}) = \tilde{X} \), that is, the matrix equation (13) has a solution. \( \square \)

**Theorem 4.4.** Consider the matrix equation (13). Assume that there is a positive number \( M \) such that:

(i) for every \( X, Y \in H(n) \) such that \((X, Y) \in \leq \), we have

\[
| tr(G(Y) - G(X)) | \leq \frac{1}{M} | \ln(1 + h(tr(Y - X))) |
\]

where \( h : [0, \infty) \rightarrow [0, \infty) \) is a right upper semicontinuous function with \( h(t) < e^t - 1 \), for all \( t > 0 \);

(ii) \( \sum_{i=1}^{m} A_i A_i^* < M I_n \).

If \( \sum_{i=1}^{m} A_i^* G(Q)A_i > 0 \), then the matrix equation (13) has a solution. Moreover, the iteration

\[
X_n = Q + \sum_{i=1}^{n} A_i^* G(X_{n-1}) A_i
\]  

where \( X_0 \in H(n) \) such that \( X_0 \preceq Q + \sum_{i=1}^{n} A_i^* G(X_0) A_i \), converges in the sense of trace norm \( \| \cdot \|_{tr} \) to the solution of matrix equation (13).

**Proof.** Throughout this proof, we define the mapping \( F : H(n) \rightarrow H(n) \) by

\[
F(X) = Q + \sum_{i=1}^{m} A_i^* G(X) A_i \quad \text{for all } X \in H(n).
\]  

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Then $\mathcal{F}$ is well defined and $\leq$ on $H(n)$ is $\mathcal{F}$-closed and a fixed point of $\mathcal{F}$ is a solution of equation (13).

Next, we will show that the contractive condition (1) holds with $\mathcal{F}$.

Let $X, Y \in H(n)$ such that $(X, Y) \in \leq$. This means that $X \leq Y$ and then $G(X) \leq G(Y)$. Using the same technique to Theorem 4.3, we get

$$\|\mathcal{F}(Y) - \mathcal{F}(X)\|_{tr} \leq \ln(1 + h(\|Y - X\|_{tr})).$$

(19)

Putting $\psi(t) = t$, $\phi(t) = \ln(1 + h(t))$, obviously $\psi$ is a weak altering distance function and $\phi$ is a right upper semicontinuous function such that $\psi(t) > \phi(t)$ for all $t > 0$. From the inequality (19), we have

$$\psi(\|\mathcal{F}(Y) - \mathcal{F}(X)\|_{tr}) \leq \phi(\|Y - X\|_{tr}).$$

Thus, the contractive condition (1) in Theorem 3.4 is satisfied for all $X, Y \in H(n)$ such that $(X, Y) \in \leq$. From $\sum_{i=1}^{m} A_i^t G(Q) A_i > 0$, we have $Q \leq \mathcal{F}(Q)$. This means that $Q \in H(n)(\mathcal{F}; \leq)$. Now from Theorem 3.4, there exists $\hat{X} \in H(n)$ such that $\mathcal{F}(\hat{X}) = \hat{X}$, that is, the matrix equation (13) has a solution.$\square$

**Theorem 4.5.** In addition to the hypothesis of Theorem 4.3 (resp. Theorem 4.4), suppose that $h(0) = 0$. Then the equation (13) has a unique solution $\hat{X} \in H(n)$.

**Proof.** It follows from $h(0) = 0$ that $\phi(0) = 0$. Since for every $X, Y \in H(n)$ there is the greatest lower bound and the least upper bound, we obtain that $H(n)$ is $\leq$-directed. Thus, we deduce from Theorem 3.8 that $\mathcal{F}$ has a unique fixed point in $H(n)$. This implies that Equation (13) has a unique solution in $H(n)$.$\square$

### 5 Numerical experiments

Next, we use some numerical examples to confirm the correctness of Theorem 4.5.

**Example 5.1.** Let

$$Q = \begin{pmatrix} 5 & 2 & 0 & 0 \\ 2 & 5 & 2 & 0 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 2 & 5 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0.002 & 0.195 & 0 & 0.367 \\ 0 & 0.045 & 0 & 0.006 \\ 0.129 & 0 & 0.245 & 0.028 \\ 0.023 & 0.054 & 0 & 0.147 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.301 & 0.02 & 0.021 & 0.074 \\ 0.11 & 0.002 & 0.1 & 0 \\ 0 & 0 & 0.201 & 0.045 \\ 0.135 & 0.01 & 0.005 & 0.06 \end{pmatrix}.$$

Define $h : [0, \infty) \to [0, \infty)$ by $h(t) = \frac{t^2}{4}$. We consider Equation (13) with $G(X) = X$ that is

$$X = Q + A_1^t X A_1 + A_2^t X A_2.$$

(20)

All the hypotheses of Theorem 4.5 are satisfied with $M = \frac{1}{2}$. We will consider the iteration

$$X_n = Q + A_1^t X_{n-1} A_1 + A_2^t X_{n-1} A_2,$$

(21)

where $X_0 = Q$, and the error $E_n := \|X_n - X_{n-1}\|_{tr}$. After 8 iterations, we can approximate a solution $\hat{X}$ of Equation (20) by

$$\hat{X} \approx X_8 = \begin{pmatrix} 6.0198 & 2.1079 & 0.5126 & 0.3661 \\ 2.1079 & 5.3083 & 2.0944 & 0.5640 \\ 0.5126 & 2.0944 & 5.7344 & 2.2998 \\ 0.3661 & 0.5640 & 2.2998 & 6.1132 \end{pmatrix}$$

with $E_8 = 2.3565e - 05$. 

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Example 5.2. Let

\[ Q = \begin{pmatrix} 9 & 2 & 0 & 0 \\ 2 & 9 & 2 & 0 \\ 0 & 2 & 9 & 2 \\ 0 & 0 & 2 & 9 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0.0325 & 0.0057 & 0.0746 & 0.0069 \\ 0.0125 & 0.0215 & 0 & 0.215 \\ 0.0201 & 0.257 & 0 & 0.201 \\ 0 & 0 & 0.1874 & 0.0424 \end{pmatrix}, \]

\[ A_2 = \begin{pmatrix} 0.0058 & 0.0871 & 0.0526 & 0 \\ 0.0514 & 0.0215 & 0.0321 & 0 \\ 0 & 0 & 0.0808 & 0 \\ 0.0098 & 0 & 0.0165 & 0.0587 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0.0171 & 0 & 0.0751 & 0.0358 \\ 0 & 0 & 0.0221 & 0.012 \\ 0.0325 & 0 & 0.0316 & 0.0955 \\ 0 & 0 & 0 & 0.0147 \end{pmatrix}. \]

Define \( h : [0, \infty) \to [0, \infty) \) by \( h(t) = \frac{t^2}{2} \). We consider Equation (13) with \( g(X) = 2X \) that is

\[ X = Q + A_1^* (2X) A_1 + A_2^* (2X) A_2 + A_3^* (2X) A_3. \]  \hfill (22)

All the hypotheses of Theorem 4.5 are satisfied with \( M = \frac{1}{6} \). We will consider the iteration

\[ X_n = Q + A_1^* X_{n-1} A_1 + A_2^* X_{n-1} A_2 + A_3^* X_{n-1} A_3, \]  \hfill (23)

where \( X_0 = Q \), and the error \( E_n := \| X_n - X_{n-1} \|_{l_1} \). After 7 iterations, we can approximate a solution \( \hat{X} \) of Equation (22) by

\[ \hat{X} \approx X_7 = \begin{pmatrix} 9.1296 & 2.1938 & 0.1818 & 0.3602 \\ 2.1938 & 10.5937 & 2.4096 & 1.5582 \\ 0.1818 & 2.4096 & 10.3194 & 2.8325 \\ 0.3602 & 1.5582 & 2.8325 & 11.8107 \end{pmatrix} \]

with \( E_7 = 2.3990e - 004 \).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.
Fig. 4. The error of iteration process 23 for the Equation (22) given in Example 5.2.

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