The characterizations of upper approximation operators based on special coverings

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Research Article

1 Introduction

The concept of rough set which was first proposed by Pawlak [11] is an extension of set theory for the study of the intelligent systems characterized by insufficient and incomplete information in 1982. It is a useful and powerful tool in many fields, such as data analysis, granularity or vagueness. It has also been applied successfully in process control, economics, medical diagnosis, biochemistry, environmental science, biology, chemistry, psychology, conflict analysis, and so on. Many researchers have made some significant contributions to developing the rough theory [3,4,7-8,12,14-15,17-26]. However, a problem with Pawlak’s rough set theory is that partition or equivalence relation is explicitly used in the definition of the lower and upper approximations. Such a partition or equivalence relation is too restrictive for many applications because it can only deal with complete information systems. To address this issue, generalizations of rough set theory were considered by scholars. One approach was to extend equivalence relation to tolerance [5-6,32] and others [20-21,33-34]. Another important approach was to relax the partition to a covering of the universe. In 1983, W.Zakowski generalized the classical rough set theory by using coverings of a universe instead of partitions [27]. The covering-based rough sets is one of the most important generalization of the classical Pawlak rough sets. Such generalization leads to various covering approximation operators that are both of theoretical and practical importance [28-30]. The relationships between properties of covering-based approximation and their corresponding coverings have attracted intensive research. Based on the mutual correspondence of the concepts of extension and intension, E.Bryniarski [36] and Z.Bonikowski et al. [37] gave the second type of covering-based rough sets. The third and the fourth type of covering-based rough sets were introduced in [38]. Subsequently, W.Zhu utilized the topological method to characterize covering rough sets [18]. W.Zhu and F. Wang discussed the relationship between properties of four types of covering-based upper approximation operators and their corresponding coverings [21-23]. T. Yang et al. researched attribute reduction...
of covering information systems [39]. G. Liu studied the two types of rough sets induced by coverings and obtained some interesting results. G. Cattaneo, D. Ciucci, and G. Liu obtained the algebraic structures of generalized rough set theory [1, 2, 40, 44]. G. Liu also used axiomatic method to characterize covering-based rough sets [41-43]. X. Bian et al. gave characterizations of covering-based approximation spaces being closure operators [31]. Ge et al. proposed not only general, but also topological characterizations of coverings for these operators being closure operators [25, 29-30]. T. Lin et al. defined some approximation operators which are based on neighborhood systems and did some research on them [52]. In addition, Y. Zhang et al. discussed the relationships between generalized rough sets based on covering and reflexive neighborhood system [49] and proposed the operator \( \overline{\text{appr}}_S \). Based on their works, we will investigate the properties of \( NS(U) \) and \( S \). We also give characterizations for \( \overline{\text{appr}}_{NS}, \overline{\text{appr}}_S \) being closure operators.

This paper is organized as follows. Section 2 recalls the main ideas of generalized rough set and covering approximations. Section 3 gives the properties of \( NS(U) \) and some examples. Section 4 studies the characterization of \( NS(U) \) for \( \overline{\text{appr}}_{NS} \) being a closure operator, while Section 5 considers the properties of \( S \) and \( \overline{\text{appr}}_S \) and obtains the general and topological characterizations of special covering \( S \) for covering-based upper approximation operator \( \overline{\text{appr}}_S \) to be a closure operator. Finally, Section 6 concludes the paper.

## 2 Background

In this section, we introduce the fundamental concepts that are used in this paper. \( U \) is the universe of discourse and \( \mathcal{P}(U) \) denotes the family of all subsets of \( U \). If \( C \) is a family of subsets of \( U \), none of sets in \( C \) is empty, and \( \cup C = U \), then \( C \) is called a covering of \( U \).

**Definition 2.1** ([46]). A mapping \( n : U \to \mathcal{P}(U) \) is called a neighborhood operator. If \( n(x) \neq \emptyset \) for all \( x \in U \), \( n \) is called a serial neighborhood operator. If \( x \in n(x) \) for all \( x \in U \), \( n \) is called a reflexive neighborhood operator.

**Definition 2.2** ([50-51]). A neighborhood system of an object \( x \in U \), denoted by \( NS(x) \), is a non-empty family of neighborhoods of \( x \). The set \( \{ NS(x) : x \in U \} \) is called as a neighborhood system of \( U \), and it is denoted by \( NS(U) \). Let \( NS(U) \) be a neighborhood system of \( U \).

- \( NS(U) \) is said to be serial, if for any \( x \in U \) and \( n(x) \subseteq NS(x) \), \( n(x) \neq \emptyset \) (called Freé(V) Space in [32]).

- \( NS(U) \) is said to be reflexive, if for any \( x \in U \) and \( n(x) \subseteq NS(x), x \in n(x) \).

- \( NS(U) \) is said to be symmetric, if for any \( x, y \in U \), \( n(x) \subseteq NS(y) \) and \( n(y) \subseteq NS(y) \), \( x \in n(y) \Rightarrow y \in n(x) \).

- \( NS(U) \) is said to be transitive, if for any \( x, y, z \in U \), \( n(y) \subseteq NS(y) \) and \( n(z) \subseteq NS(z) \), \( x \in n(y) \) and \( y \in n(z) \Rightarrow x \in n(z) \).

**Definition 2.3** (Covering approximation space [23]). If \( U \) is an universe and \( \mathcal{C} \) is a covering of \( U \), then we call \( U \) together with covering \( \mathcal{C} \) a covering approximation space, denoted by \( (U, \mathcal{C}) \).

**Definition 2.4** ([52]). Let \( NS(U) \) be a neighborhood system of \( U \). The lower and upper operators of \( X \) are defined as follows:

\[
\text{appr}_S(X) = \{ x \in U : \exists n(x) \subseteq NS(x), n(x) \subseteq X \};
\]

\[
\overline{\text{appr}}_{NS}(X) = \{ x \in U : \forall n(x) \subseteq NS(x), n(x) \cap X \neq \emptyset \}.
\]

**Definition 2.5** ([49]). Let \( NS(U) \) be a neighborhood system of \( U \).

- \( NS(U) \) is referred to as weak-unary, if for any \( x \in U \) and \( n_1(x), n_2(x) \subseteq NS(x) \), there exists an \( n_3(x) \subseteq NS(x) \) such that \( n_3(x) \subseteq n_1(x) \cap n_2(x) \);

- \( NS(U) \) is referred to as weak-transitive, if for any \( x \in U \) and \( n(x) \subseteq NS(x) \), there exists an \( n_1(x) \subseteq NS(x) \) satisfying that for any \( y \subseteq n_1(x) \), there exists an \( n(y) \subseteq NS(y) \) such that \( n(y) \subseteq n(x) \);

- \( NS(U) \) is referred to as a weak-\( S_4 \) neighborhood system, if \( NS(x) \) is reflexive and weak-transitive.
The following topological concepts and facts are elementary and can be found in [47, 51]. We list them below for the purpose of this paper being self-contained.

1. A topological space is a pair \((U, \tau)\) consisting of a set \(U\) and a family \(\tau\) of subsets of \(U\) satisfying the following conditions: (a) \(\emptyset \in \tau\) and \(U \in \tau\); (b) if \(U_1, U_2 \in \tau\), then \(U_1 \cap U_2 \in \tau\); (c) if \(A \subseteq \tau\), then \(\cup A \in \tau\). \(\tau\) is called a topology on \(U\) and the members of \(\tau\) are called open sets of \((U, \tau)\). The complementary set of an open set is called a closed set.

2. A set \(F\) is called a clopen set, if \(F\) in \((U, \tau)\) is both an open set and a closed set.

3. A family \(B \subseteq \tau\) is called a base for \((U, \tau)\) if for every non-empty open subset \(O\) of \(U\) and each \(x \in O\), there exists a set \(B \in \tau\) such that \(x \in B \subseteq O\). Equivalently, a family \(B \subseteq \tau\) if every non-empty open subset \(O\) of \(U\) can be represented as the union of a subfamily of \(B\).

4. For any \(x \in U\), a family \(B \subseteq \tau\) is called a local base at \(x\) for \((U, \tau)\) if \(x \in B\) for each \(B \in \tau\), and for every open subset \(O\) of \(U\) with \(x \in O\), there exists a set \(B \in \tau\) such that \(B \subseteq O\).

5. If \(P\) is a partition of \(U\), the topology \(\tau = \{O \subseteq U : O\) is the union of some members of \(P\} \cup \{\emptyset\}\) is called a pseudo-discrete topology in [13] (also called a closed-open topology in [12]).

6. Let \((U, \tau)\) be a topological space. If for each pair of points \(x, y \in U\) with \(x \neq y\), there exist open sets \(O, O'\) such that \(x \in O, y \in O'\) and \(O \cap O' = \emptyset\), then \((U, \tau)\) is called a \(T_2\)-space and \(\tau\) is called a \(T_2\)-topology.

Definition 2.6 (Induced topology and subspace). Let \((U, \tau)\) be a topological space and \(X \subseteq U\). It is easy to check that \(\tau' = \{O \cap X : O \in \tau\}\) is a topology on \(X\). \(\tau'\) is called a topology induced by \(X\), and the topology space \((X, \tau')\) is called a subspace of \((U, \tau)\).

Definition 2.7 (Closure operator). An operator \(H : P(U) \to P(U)\) is called a closure operator on \(U\) if it satisfies the following conditions: for any \(X, Y \subseteq U\),

\[(H_1)\quad H(X \cup Y) = H(X) \cup H(Y);\]
\[(H_2)\quad X \subseteq H(X);\]
\[(H_3)\quad H(\emptyset) = \emptyset;\]
\[(H_4)\quad H(H(X)) = H(X).\]

Definition 2.8 (Interior operator). An operator \(I : P(U) \to P(U)\) is called an interior operator on \(U\) if it satisfies the following conditions: for any \(X, Y \subseteq U\),

\[(I_1)\quad I(X \cap Y) = I(X) \cap I(Y);\]
\[(I_2)\quad I(X) \subseteq X;\]
\[(I_3)\quad I(U) = U;\]
\[(I_4)\quad I(I(X)) = I(X).\]

Definition 2.9 (Dual operator). Assume that \(H, I : P(U) \to P(U)\) are two operators on \(U\). If for any \(X \subseteq U\), \(H(X) = I(I(X))\). We say that \(H, I\) are dual operators or \(H\) is the dual operator of \(I\).

Proposition 2.10 ([49]). Let \(NS(U)\) be a neighborhood system of \(U\). Then the following are equivalent:

1. \(\overline{pr}_{NS}(X \cup Y) = \overline{pr}_{NS}(X) \cup \overline{pr}_{NS}(Y)\) for all \(X, Y \subseteq U\);\n2. \(apr_{NS}(X \cup Y) = apr_{NS}(X) \cup apr_{NS}(Y)\) for all \(X, Y \subseteq U\);\n3. \(NS(U)\) is weak-unary.

Proposition 2.11 ([49]). Let \(NS(U)\) be a neighborhood system of \(U\). Then the following are equivalent:

1. \(\overline{pr}_{NS}(\overline{pr}_{NS}(X)) \subseteq \overline{pr}_{NS}(X)\) for all \(X \subseteq U\);\n2. \(apr_{NS}(X) \subseteq apr_{NS}(apr_{NS}(Y))\) for all \(X \subseteq U\);\n3. \(NS(U)\) is weak-transitive.
3 Some propositions of \( NS(U) \)

In this section, we will discuss some properties of \( NS(U) \).

**Definition 3.1.** Let \( NS(U) \) be a neighborhood system of \( U \). \( NS(U) \) is said to be Euclidean, if for any \( x, y \in U \), and \( n(x) \in NS(x) \), \( y \in n(x) \Rightarrow n(x) \subseteq n(y) \).

**Proposition 3.2.** Let \( NS(U) \) be a neighborhood system of \( U \). If \( NS(U) \) is Euclidean, then \( \overline{\text{apr}}_{NS}(\text{apr}_{NS}(X)) \subseteq \text{apr}_{NS}(X) \) and \( \overline{\text{apr}}_{NS}(X) \subseteq \text{apr}_{NS}(\overline{\text{apr}}_{NS}(X)) \) for any \( X \subseteq U \).

**Proof.** For any \( x \in U \), if \( x \in \overline{\text{apr}}_{NS}(\text{apr}_{NS}(X)) \) and \( n(x) \in NS(U) \), we have \( n(x) \cap \text{apr}_{NS}(X) \neq \emptyset \). Pick \( y \in n(x) \cap \text{apr}_{NS}(X) \), then there exists an \( n(y) \in NS(y) \) such that \( n(y) \subseteq X \). Since \( NS(U) \) is Euclidean, we have \( n(x) \subseteq n(y) \). It follows that \( n(x) \subseteq X \). From the Definition 2.4, we have \( x \in \text{apr}_{NS}(X) \), therefore \( \overline{\text{apr}}_{NS}(\text{apr}_{NS}(X)) \subseteq \text{apr}_{NS}(X) \).

We can get \( \overline{\text{apr}}_{NS}(X) \subseteq \text{apr}_{NS}(\overline{\text{apr}}_{NS}(X)) \) by the duality between \( \overline{\text{apr}}_{NS} \) and \( \text{apr}_{NS} \).

The inverse of Proposition 2.10 does not hold. An example is given as follows:

**Example 3.3.** Let \( U = \{a, b, c\} \), \( NS(a) = \{\{a\}\} \), \( NS(b) = \{\{b\}, \{a, b\}\} \) and \( NS(c) = \{\{c\}\} \). Then:

- \( \text{apr}_{NS}(\emptyset) = \emptyset = \overline{\text{apr}}_{NS}(\text{apr}_{NS}(\emptyset)) \);
- \( \text{apr}_{NS}(\{a\}) = \{a\} = \overline{\text{apr}}_{NS}(\text{apr}_{NS}(\{a\})) \);
- \( \text{apr}_{NS}(\{b\}) = \{b\} = \overline{\text{apr}}_{NS}(\text{apr}_{NS}(\{b\})) \);
- \( \text{apr}_{NS}(\{c\}) = \{c\} = \overline{\text{apr}}_{NS}(\text{apr}_{NS}(\{c\})) \);
- \( \text{apr}_{NS}(\{a, b\}) = \{a, b\} = \overline{\text{apr}}_{NS}(\text{apr}_{NS}(\{a, b\})) \);
- \( \text{apr}_{NS}(\{a, c\}) = \{a, c\} = \overline{\text{apr}}_{NS}(\text{apr}_{NS}(\{a, c\})) \);
- \( \text{apr}_{NS}(\{b, c\}) = \{b, c\} = \overline{\text{apr}}_{NS}(\text{apr}_{NS}(\{b, c\})) \);
- \( \text{apr}_{NS}(U) = U = \overline{\text{apr}}_{NS}(\text{apr}_{NS}(U)) \).

Hence \( \overline{\text{apr}}_{NS}(\text{apr}_{NS}(X)) \subseteq \text{apr}_{NS}(X) \) for any \( X \subseteq U \). Since \( a \in n(b) = \{a, b\} \in NS(b) \) and \( n(b) \nsubseteq n(a) \). We obtain that \( NS(U) \) is not Euclidean.

**Proposition 3.4.** Let \( NS(U) \) be a neighborhood system of \( U \). Then the following are equivalent:

1. \( NS(U) \) is transitive;
2. For any \( x, y \in U \), if \( x \in n(y) \), then \( n(x) \subseteq n(y) \).

**Proof.** (1) \( \Rightarrow \) (2) For any \( x, y \in U \) and \( x \in n(y) \), if \( n(x) \nsubseteq n(y) \), there exists \( p \in n(x) \) and \( p \notin n(y) \). Since \( x \in n(y) \) and \( NS(U) \) is transitive, we have \( p \in n(y) \). It is contradictory to \( p \notin n(y) \).

(2) \( \Rightarrow \) (1) for any \( x, y, z \in U, n(y) \in NS(y) \) and \( n(z) \in NS(z) \), \( x \in n(y) \) and \( y \in n(z) \). It is easy to prove \( NS(U) \) is transitive.

**Proposition 3.5.** Let \( NS(U) \) be a neighborhood system of \( U \) and \( n : U \rightarrow P(U) \) is a reflexive mapping. Then the following are equivalent:

1. \( \{n(x) : x \in U\} \) forms a partition of \( U \);
2. \( NS(U) \) is reflexive, transitive and Euclidean.

**Proof.** (1) \( \Rightarrow \) (2) For any \( x, y \in U \) and \( y \in n(x) \). Since \( \{n(x) : x \in U\} \) forms a partition of \( U \) and \( n : U \rightarrow P(U) \) is a reflexive mapping, then \( n(x) = n(y) \). By the Definition 3.1 and Proposition 2.11, it is easy to prove \( NS(U) \) is reflexive, transitive and Euclidean.

(2) \( \Rightarrow \) (1) for any \( x, y \in U \), if \( n(x) \neq n(y) \), then \( n(x) \cap n(y) = \emptyset \). Otherwise, we take \( z \in n(x) \cap n(y) \), since \( NS(U) \) is reflexive, transitive and Euclidean, then \( n(z) = n(x) = n(y) \). It is contradictory to \( n(x) \neq n(y) \).

**Proposition 3.6.** Let \( NS(U) \) be a reflexive neighborhood system of \( U \). \( NS(U) \) is weak-uniary if and only if there is a topology on \( U \) such that \( NS(U) \) is a local base for any \( x \in U \).
Proof. (1) ⇒ (2) For any \( x \in U, n_1(x), n_2(x) \in NS(x) \) and \( x \in n_1(x) \cap n_2(x) \), since \( NS(U) \) is weak-uniary, there exists \( n_3(x) \) such that \( n_3(x) \subseteq n_1(x) \cap n_2(x) \). We have \( x \in n_3(x) \) because \( NS(U) \) is a reflexive neighborhood system of \( U \). So \( NS(x) \) is a local base for any \( x \in U \).

(2) ⇒ (1) By the definition of local base, it is easy to prove \( NS(U) \) is weak-uniary.

\[
\begin{align*}
\text{4 Characterization of } NS(U) \text{ for } \overline{ap}_NS \text{ being a closure operator} \\
\text{In Section 3, we discuss the properties of } NS(U). \text{ One natural question thus arise: When is } \overline{ap}_NS \text{ a closure operator?}
\end{align*}
\]

**Theorem 4.1** (General characterization of of \( NS(U) \) for \( \overline{ap}_NS \) being a closure operator). Let \( NS(U) \) be a neighborhood system of \( U \). \( \overline{ap}_NS \) is a closure operator if and only if \( NS(U) \) is reflexive, weak-uniary and weak-transitive.

**Proof.** (⇒) Assume that \( \overline{ap}_NS \) is a closure operator. By the Definition 2.7, we get \( X \subseteq \overline{ap}_NS(X), \overline{ap}_NS(\overline{ap}_NS(X)) = \overline{ap}_NS(X) \) and \( \overline{ap}_NS(X \cup Y) = \overline{ap}_NS(X) \cup \overline{ap}_NS(Y) \) for all \( X, Y \subseteq U \). According to Proposition 6 \([49]\) and Proposition 7 \([49]\), we obtain \( NS(U) \) is weak-uniary and weak-transitive.

(⇐) Assume that \( NS(U) \) is weak-uniary and weak-transitive. We prove that \( \overline{ap}_NS \) satisfies the conditions (H) \( i = 1, 2, 3, 4 \). By the Definition of \( \overline{ap}_NS \), it is easy to check that \( \overline{ap}_NS(\emptyset) = \emptyset \). Hence (H3) is satisfied. We prove (H1) and (H4). Since \( NS(U) \) is weak-uniary and weak-transitive, by the Proposition 6 \([49]\) and Proposition 7 \([49]\), we obtain \( \overline{ap}_NS(\overline{ap}_NS(X)) \subseteq \overline{ap}_NS(X) \) and \( \overline{ap}_NS(X \cup Y) = \overline{ap}_NS(X) \cup \overline{ap}_NS(Y) \) for all \( X, Y \subseteq U \). Since \( NS(U) \) is a reflexive neighborhood system of \( U \), so \( \overline{ap}_NS(X) \subseteq \overline{ap}_NS(\overline{ap}_NS(X)) \) for any \( X \subseteq U \). Therefore \( \overline{ap}_NS(X) = \overline{ap}_NS(\overline{ap}_NS(X)) \) for any \( X \subseteq U \). It is obvious that (H2) holds. Thus \( \overline{ap}_NS \) is a closure operator.

**Theorem 4.2** (Topological characterization of of \( NS(U) \) for \( \overline{ap}_NS \) being a closure operator). Let \( NS(U) \) be a reflexive neighborhood system of \( U \). Then \( \overline{ap}_NS \) is a closure operator if and only if \( N = \{ n(x) : n(x) \in NS(x), x \in U \} \) is a base for some topology \( \tau \) on \( U \).

**Proof.** (⇒) Assume that \( \overline{ap}_NS \) is a closure operator. We prove \( N = \{ n(x) : n(x) \in NS(x), x \in U \} \) is a base for some topology \( \tau \) on \( U \). Since \( NS(U) \) is a reflexive neighborhood system of \( U \). It is easy to prove \( x \in n(x) \) for each \( n(x) \in NS(x) \) and \( x \in U \). Thus \( N \) is a cover of \( U \). For any \( x, y, z \in U \) and \( n(y) \in NS(y), n(z) \in NS(z) \) and \( x \in n(y) \cap n(z) \), there exists \( n_0(x) \in NS(x) \) such that \( n_0(x) \subseteq n(y) \cap n(z) \). By the definition of base, we have \( N = \{ n(x) : n(x) \in NS(x), x \in U \} \) is a base for some topology \( \tau \) on \( U \).

(⇐) Assume that \( N = \{ n(x) : n(x) \in NS(x), x \in U \} \) is a base for some topology \( \tau \) on \( U \). We prove that \( \overline{ap}_NS \) satisfies the conditions (H) \( i = 1, 2, 3, 4 \). By Definition of \( \overline{ap}_NS \), it is easy to check that \( \overline{ap}_NS(\emptyset) = \emptyset \). Hence (H3) is satisfied. So to prove (H1), we prove (H1) holds. Since \( N = \{ n(x) : n(x) \in NS(x), x \in U \} \) is a base for some topology \( \tau \) on \( U \), by Definition 2.5, for any \( x \in U \) and \( n_1(x), n_2(x) \in NS(x) \), there exists an \( n_3(x) \in NS(x) \) such that \( n_3(x) \subseteq n_1(x) \cap n_2(x) \); so \( NS(U) \) is weak-uniary. By Proposition 2.10, we have \( \overline{ap}_NS(X \cup Y) = \overline{ap}_NS(X) \cup \overline{ap}_NS(Y) \) for all \( X, Y \subseteq U \). Since \( NS(U) \) is a reflexive neighborhood system of \( U \), \( \overline{ap}_NS(X) \subseteq \overline{ap}_NS(\overline{ap}_NS(X)) \) for any \( X \subseteq U \). So to prove (H1), we only need to prove \( \overline{ap}_NS(\overline{ap}_NS(A)) \subseteq \overline{ap}_NS(A) \cup \overline{ap}_NS(A) \) for any \( A \subseteq U \). Let \( x \in \overline{ap}_NS(\overline{ap}_NS(A)) \), by Definition of \( \overline{ap}_NS \), \( n(x) \cap \overline{ap}_NS(A) \neq \emptyset \) for any \( n(x) \in NS(x) \). Pick \( p \in n(x) \cap \overline{ap}_NS(A) \). Then \( p \in n(x) \) and \( n(p) \subseteq A \) for any \( n(p) \in NS(p) \). From \( p \in n(x) \), we obtain \( n(p) \subseteq n(x) \), therefore \( n(x) \subseteq A \neq \emptyset \). By the arbitrariness of \( n \), we have \( x \in \overline{ap}_NS(A) \). Thus \( \overline{ap}_NS(A) = \overline{ap}_NS(\overline{ap}_NS(A)) \) for any \( A \subseteq U \). Since \( NS(U) \) is a reflexive neighborhood system of \( U \), (H2) is obvious satisfied. Therefore \( \overline{ap}_NS \) is a closure operator.

**Corollary 4.3**. Let \( NS(U) \) be a reflexive neighborhood system of \( U \). Then \( \overline{ap}_NS \) is a closure operator if and only if \( N = \{ n(x) : n(x) \in NS(x), x \in U \} \) is a base for some topology \( \tau \) on \( U \) and \( NS(U) \) is transitive.
Proof. It is easy to prove by Proposition 3.4 and Theorem 4.2.

5 Characterization of covering $S$ for $\overline{apr}_S$ being a closure operator

Zhang, Li and Lin defined a special covering $S$ and investigated twenty-three types of covering-based rough sets proposed in [49] which can be treated as the generalized rough sets based on neighborhood systems. In this section, we will discuss the properties of $\overline{apr}_S$ and give the Characterization of covering $S$ for $\overline{apr}_S$ being a closure operator.

Definition 5.1 ([49]). A family of subsets of universe $U$ is called a closure system over $U$ if it contains $U$ and is closed under set intersection. Given a closure system $\overline{S}$, one can define its dual system $\underline{S}$ as follows:

$$\overline{S} = \{ X : X \in \overline{S} \}$$

Definition 5.2 (Subsystem based definition [49]). Suppose $S = (\overline{S}, \underline{S})$ is a pair of subsystems of $\mathcal{P}(U)$, $\overline{S}$ is a closure system and $\underline{S}$ is the dual system of $\overline{S}$. A pair of lower and upper approximation operators $\overline{apr}_S, \underline{apr}_S$ with respect to $S$ is defined as:

$$\overline{apr}_S(X) = \bigcap \{ K \subseteq X : K \in \overline{S} \}$$

$$\underline{apr}_S(X) = \bigcup \{ K \supseteq X : K \in \underline{S} \}$$

for any $X \subseteq U$.

Remark 5.3.
(1) $(\overline{S}, \subseteq)$ is a complete lattice.
(2) $\overline{S}$ may not have element $\emptyset$. Fig. ?? gives an intuitive illustration (2) of the Remark 5.3:

Fig. 1

(3) If $\overline{S}$ has no less than two single sets, then $\emptyset \in \overline{S}$. The converse may not hold:

Fig. 2

By Definition 5.2, it is easy to obtain properties of lower and upper approximation operators as follows:

Proposition 5.4. Let $S = (\overline{S}, \underline{S})$ be a pair of subsystems of $\mathcal{P}(U)$, for any $X, Y \subseteq U$, we have:

1. $\overline{apr}_S(\emptyset) = \emptyset$;
2. $\overline{apr}_S(U) = U$;
(3) \( X \subseteq Y \Rightarrow \overline{apr}_S(X) \subseteq \overline{apr}_S(Y), \overline{apr}_S(X) \subseteq \overline{apr}_S(Y) \); 
(4) \( \overline{apr}_S(\overline{apr}_S(X)) = \overline{apr}_S(X) \) for any \( X \subseteq U \); 
(5) \( \overline{apr}_S(X) = -\overline{apr}_S(-X), \overline{apr}_S(X) = -\overline{apr}_S(-X) \).

However, the following properties may not hold:

(1) \( \overline{apr}_S(\emptyset) = \emptyset \);

**Example 5.5.** Let \( U = \{a, b, c\}, \overline{S} = \{\{a\}, \{a, b\}, U\} \). It is easy to see \( \overline{S} \) contains \( U \) and is closed under set intersection. But \( \overline{apr}_S(\emptyset) = \{a\} \).

(2) \( \overline{apr}_S(U) = U \);

**Example 5.6.** Let \( U = \{a, b, c\}, \overline{S} = \{\{a\}, \{a, b\}, U\}, \overline{S} = \{\{b, c\}, \{c\}, \emptyset\} \). Then \( \overline{apr}_S(U) = \{b, c\} \neq U \).

(3) \( \overline{apr}_S(X \cup Y) \subseteq \overline{apr}_S(X) \cup \overline{apr}_S(Y) \) for any \( X, Y \subseteq U \).

**Example 5.7.** Let \( U = \{a, b, c\}, \overline{S} = \{\emptyset, \{a\}, \{b\}, U\} \). Let \( X = \{a\}, Y = \{b\}, \overline{apr}_S(X) = \{a\}, \overline{apr}_S(Y) = \{b\}, \overline{apr}_S(X \cup Y) = U \). Hence \( \overline{apr}_S(X \cup Y) \not\subseteq \overline{apr}_S(X) \cup \overline{apr}_S(Y) \).

**Definition 5.8 ([53]).** Suppose \( R \) is an arbitrary relation on \( U \). With respect to \( R \), we can define the left neighborhoods of an element \( x \) in \( U \) as follows:

\[ l_R(x) = \{y \in U : yRx\} \]

**Definition 5.9 ([53]).** For an arbitrary relation \( R \), by substituting equivalence class \([x]_R \) with right neighborhood \( l_R(x) \), we define the operators \( \overline{R} \) and \( R(\text{Lin}, 1992) \) from \( P(U) \) to itself as follows:

\[ \overline{R}(X) = \{x \in U : l_R(x) \cap X \neq \emptyset\} \]

**Lemma 5.10 ([53]).** If \( S \) is another binary relation on \( U \) and \( \overline{R}(X) = \overline{S}(X) \) for any \( X \subseteq U \), then \( R = S \).

**Proposition 5.11.** Let \( S = (\overline{S}, \overline{S}) \) be a pair of subsystems of \( P(U) \) and \( U \) is finite. If \( \overline{apr}_S(X \cup Y) = \overline{apr}_S(X) \cup \overline{apr}_S(Y) \) for any \( X, Y \subseteq U \), then there exists a unique reflexive and transitive relation \( R \) on \( U \) such that \( \overline{apr}_S(X) = \overline{R}(X) \) for any \( X \subseteq U \).

**Proof.** Using \( \overline{S} \) of \( U \), for any \( x \in U \), we choose all \( S \in \overline{S} \) such that \( \{x\} \subseteq S \) which forms the family \( S' \), via left neighborhood of an element \( x \in U \), we construct the binary \( R \) on \( U \) as follows:

\[ l_R(x) = \{y \in U : y \in \cap S'\} \]

It is clear that \( R \) is a reflexive relation. Since \( \overline{apr}_S(\{x\}) = l_R(x) \) and \( \overline{apr}_S(X \cup Y) = \overline{apr}_S(X) \cup \overline{apr}_S(Y) \) for any \( X, Y \subseteq U \), we have \( \overline{apr}_S(X) = \cup_{x \in X} \overline{apr}_S(\{x\}) = \cup_{x \in X} l_R(x) = \overline{R}(X) \).

Since \( \overline{apr}_S(\overline{apr}_S(X)) = \overline{apr}_S(X) \) for any \( X \subseteq U \), this means that \( \overline{R}(\overline{R}(X)) = \overline{R}(X) \). This implies \( R \) is transitive. Thus \( R \) is a reflexive and transitive relation. The unique \( R \) comes from Lemma 5.10.

**Proposition 5.12.** Let \( S = (\overline{S}, \overline{S}) \) be a pair of subsystems of \( P(U) \), then the following are equivalent:

1. \( \overline{S} \) has element \( \emptyset \);
2. \( \overline{apr}_S(\emptyset) = \emptyset \);
3. \( \overline{apr}_S(U) = U \).

**Proof.** (1) \( \Rightarrow \) (2) By the Definition 5.2, it is easy to prove.

(2) \( \Leftrightarrow \) (3) It can be obtained by the duality.

(2) \( \Rightarrow \) (1) If \( \emptyset \notin \overline{S} \), we can choose \( \overline{S} \subseteq \overline{S}, \emptyset \subseteq S \) for any \( S \in \overline{S} \). Thus \( \emptyset \subseteq \cap \overline{S} \). Since \( \overline{S} \) is closed for intersection, then \( \cap \overline{S} \in \overline{S} \) and \( \cap \overline{S} \neq \emptyset \). From the Definition 5.2, we have \( \overline{apr}_S(\emptyset) \neq \emptyset \). It is a contradiction to (2).
Lemma 5.13. For any $X, Y \subseteq U$, $\overline{\text{appr}}_S(X \cup Y) = \overline{\text{appr}}_S(X) \cup \overline{\text{appr}}_S(Y)$ if and only if $A \cup B \in \overline{S}$ for any $A, B \in \overline{S}$.

Proof. ($\Rightarrow$) Suppose there exists $S_1, S_2 \in \overline{S}$, it is easy to prove $S_1 \cup S_2 \in \overline{S}$.

($\Leftarrow$) For any $X, Y \subseteq U$, we have:

$\overline{\text{appr}}_S(X \cup Y) = \cap\{S \in \overline{S} : (X \cup Y) \subseteq S\}$

$\overline{\text{appr}}_S(X) = \cap\{S \in \overline{S} : X \subseteq S\}$

$\overline{\text{appr}}_S(Y) = \cap\{S \in \overline{S} : Y \subseteq S\}$

Thus, we have $X \cup Y \subseteq S_1 \cup S_2 \in \overline{S}$. Hence $\overline{\text{appr}}_S(X \cup Y) \subseteq \overline{\text{appr}}_S(X) \cup \overline{\text{appr}}_S(Y)$. $\overline{\text{appr}}_S(X) \cup \overline{\text{appr}}_S(Y) \subseteq \overline{\text{appr}}_S(X \cup Y)$ is obvious. Therefore $\overline{\text{appr}}_S(X \cup Y) = \overline{\text{appr}}_S(X) \cup \overline{\text{appr}}_S(Y)$.

\[\square\]

Theorem 5.14 (General characterization of coverings $S$ for $\overline{\text{appr}}_S$ being a closure operator). Let $S = (\overline{S}, \overline{S})$ be a pair of subsystems of $\mathcal{P}(U)$. $\overline{\text{appr}}_S$ is a closure operator if and only if $\emptyset \in \overline{S}$ and $S_1 \cup S_2 \in \overline{S}$ for any $S_1, S_2 \in \overline{S}$.

Proof. ($\Rightarrow$) Assume $\overline{\text{appr}}_S$ is a closure operator. By the Proposition 5.4 and Lemma 5.13, we have $\emptyset \in \overline{S}$ and $S_1 \cup S_2 \in \overline{S}$ for any $S_1, S_2 \in \overline{S}$.

($\Leftarrow$) Assume $\emptyset \in \overline{S}$ and $S_1 \cup S_2 \in \overline{S}$ for any $S_1, S_2 \in \overline{S}$. We need to prove $\overline{\text{appr}}_S$ is a closure operator. By the Proposition 5.12, we have $\overline{\text{appr}}_S(\emptyset) = \emptyset$. Hence, (H3) is satisfied. By the Lemma 5.13, we have $\overline{\text{appr}}_S(X \cup Y) = \overline{\text{appr}}_S(X) \cup \overline{\text{appr}}_S(Y)$ for any $X, Y \subseteq U$. Thus, (H1) is satisfied. By the Definition 5.2, it is easy to prove $X \subseteq \overline{\text{appr}}_S(X)$ for any $X \subseteq U$.

Then we only need to prove (H4) holds. For any $X \subseteq U$, $\overline{\text{appr}}_S(X) \subseteq \overline{\text{appr}}_S(\overline{\text{appr}}_S(X))$ is obvious, we prove $\overline{\text{appr}}_S(\overline{\text{appr}}_S(X)) \subseteq \overline{\text{appr}}_S(X)$ for any $X \subseteq U$. By the Definition 5.2, we have $\overline{\text{appr}}_S(X) = \cap\{S \in \overline{S} : X \subseteq S\}$.
Denote $S_1 = \cap\{S \in \overline{S} : X \subseteq S\}$. Since $\overline{S}$ is closed for intersection, so $\overline{\text{appr}}_S(\overline{\text{appr}}_S(X)) = S_1$, thus $S_1 \in \overline{S}$. Therefore $\overline{\text{appr}}_S(\overline{\text{appr}}_S(X)) = \overline{\text{appr}}_S(X)$ for any $X \subseteq U$. Hence $\overline{\text{appr}}_S$ is a closure operator.

\[\square\]

Lemma 5.15. Let $S = (\overline{S}, \overline{S})$ be a pair of subsystems of $\mathcal{P}(U)$. $\emptyset \in \overline{S}$ and $\emptyset \in \overline{S}$ for any $A, B \in \overline{S}$ if and only if $\hat{S} = \{U \setminus S : S \in \overline{S}\}$ is a topology on $U$.

Proof. ($\Rightarrow$) Assume $\overline{S}$ satisfies the condition, we need to prove $\hat{S}$ is a topology on $U$. Since $\emptyset, U \in \overline{S}$, so $\emptyset, U \in \hat{S}$.

For any $X, Y \in \hat{S}$, we prove $X \cap Y \in \hat{S}$. There exists $S_1, S_2 \in \overline{S}$ such that $X = U \setminus S_1$ and $Y = U \setminus S_2$. Thus $X \cap Y = (U \setminus S_1) \cap (U \setminus S_2) = U \setminus (S_1 \cup S_2)$. By the condition, we obtain $(S_1 \cup S_2) \in \overline{S}$. From the definition of $\overline{S}$, we have $U \setminus (S_1 \cup S_2) \in \hat{S}$.

Let $\{A_i : i \in I\} \subseteq \hat{S}$, there exists $\{S_i : i \in I\} \subseteq \overline{S}$ such that $A_i = U \setminus S_i$ for each $i \in I$, thus $\bigcup_{i \in I} A_i = \bigcup_{i \in I} (U \setminus S_i) = U \setminus \bigcup_{i \in I} S_i$. Since $\overline{S}$ is closed for intersection, so $\bigcup_{i \in I} S_i \in \overline{S}$, therefore $U \setminus \bigcup_{i \in I} S_i \in \hat{S}$. Thus $\hat{S} = \{U \setminus S : S \in \overline{S}\}$ is a topology on $U$.

($\Leftarrow$) Similarly, we can prove the converse. 

From the Lemma 5.15, we have the following conclusion:

Theorem 5.16 (Topological characterization of coverings $S$ for $\overline{\text{appr}}_S$ being a closure operator). Let $S = (\overline{S}, \overline{S})$ be a pair of subsystems of $\mathcal{P}(U)$. $\overline{\text{appr}}_S$ is a closure operator if and only if $\hat{S} = \{U \setminus S : S \in \overline{S}\}$ is a topology on $U$.

6 Conclusions

In this paper, we not only obtained the properties of $NS(U)$ and $S$, but also investigated two type approximation operators. We give general characterization of covering $S$ for covering-based upper approximation operator $\overline{\text{appr}}_S$ being a closure operator. Besides this, We obtain topological characterizations of two types of upper approximation operators $\overline{\text{appr}}_{NS}$ and $\overline{\text{appr}}_S$ to be a closure operators. In our future work, we will investigate the intuitive characterizations of covering $S$ for $\overline{\text{appr}}_S$ or $\overline{\text{appr}}_{NS}$ to be a closure operator and describe covering-based
approximation space as some special types of information exchange systems when $\overline{\text{apr}}_S$ or $\overline{\text{apr}}_{NS}$ is a closure operator respectively.

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