A class of extensions of Restricted $$(s, t)$$-Wythoff’s game

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Abstract: Restricted $$(s, t)$$-Wythoff’s game, introduced by Liu et al. in 2014, is an impartial combinatorial game. We define and solve a class of games obtained from Restricted $$(s, t)$$-Wythoff’s game by adjoining to it some subsets of its $$P$$-positions as additional moves. The results show that under certain conditions they are equivalent to one case in which only one $$P$$-position is adjoined as an additional move. Furthermore, two winning strategies of exponential and polynomial are provided for the games.

Keywords: Discrete mathematics, Combinatorial games, Wythoff’s game, Winning strategy, Computational complexity

MSC: 91A46

1 Introduction

All our games in this paper are 2-player impartial combinatorial games played on two piles of finitely many tokens. All the information about the games is available to both players; there is no difference between the moves allowed to each player, and there are no chance moves (such as dice); and the game must end at some position with a clear winner, i.e., without ties or draws. In normal play the player who makes the last move wins and the opponent loses.

In general, a position is called a $$P$$-position from which the Previous player can force a win, otherwise it is an $$N$$-position from which the Next player can win regardless of the opponent’s moves. The sets of all $$P$$- and $$N$$-positions of a game are denoted by $$P$$ and $$N$$, respectively. For a position $$u$$, by $$F(u)$$ we denote the set of all its followers, i.e., all positions attainable by a legitimate move from $$u$$. Then it follows from the definition of $$P$$- and $$N$$-positions that the following relationship between $$P$$ and $$N$$ holds (it is well explained in [1, 2]):

$$u \in P \iff F(u) \subseteq N$$ (stability property of $$P$$-positions)

$$u \notin P \iff u \in N \iff F(u) \cap P \neq \emptyset$$ (absorbing property of $$P$$-positions)

The sets of $$P$$- and $$N$$-positions of any game are uniquely determined by these two properties.

For example, Wythoff’s game over a hundred years old is exactly a famous impartial game, in which players can remove any positive number of tokens from a single pile, or the same number of tokens from both piles. In normal play, the player who is first unable to move loses. Various generalizations and results on this game were done by Berlekamp et al. [3], Duchêne and Gravier [4], Fraenkel [5, 6], Fraenkel and Borosh [7], Fraenkel and Zusman [8], Liu and Li [9], Li et al. [10].

We next describe two variants: $$(s, t)$$-Wythoff’s game and Restricted $$(s, t)$$-Wythoff’s game. By $$\mathbb{Z}^0$$ and $$\mathbb{Z}^+$$ we denote the set of nonnegative integers, and positive integers, respectively. Let $$\mathbb{Z}^{even} = \{2n : n \in \mathbb{Z}^0\}$$, $$\mathbb{Z}^{odd} = \{2n + 1 : n \in \mathbb{Z}^0\}$$.
Consider two parameters $s, t \in \mathbb{Z}^+$ and two piles of finitely many tokens. In $(s, t)$-Wythoff’s game [6], a player may either remove any positive number of tokens from one pile or remove tokens from both piles, $k > 0$ from one pile and $\ell > 0$ from the other, say $\ell \geq k$, subject to the constraint

$$0 \leq \ell - k < (s - 1)k + t. \tag{1}$$

Throughout the paper, we consider the games in normal play, i.e., the player making the last move wins. Notice that the case $s = t = 1$ is Wythoff’s game, and the case $s = 1, t \geq 1$ is Generalized Wythoff [5].

In Restricted $(s, t)$-Wythoff [9], two types of moves are as follows:

**Type I** Remove $k$ tokens with $0 < k \in \mathbb{Z}^{even}$ from a single pile;

**Type II** Remove $0 < k \in \mathbb{Z}^{even}$ tokens from one pile and $0 < \ell \in \mathbb{Z}^{even}$ from the other simultaneously, also constrained by the condition (1).

It was proved in [9] that the set of $P$-positions of Restricted $(s, t)$-Wythoff are given by four sequences of pairs of integers $P = \{(A'_n, B'_n), (A'_n, B'_n + 1), (A'_n + 1, B'_n), (A'_n + 1, B'_n + 1)\}_{n \geq 0}$ such that

$$A'_n = \text{mex}\{A'_i, A'_i + 1, B'_i, B'_i + 1 : 0 \leq i < n\}, \quad B'_n = sA'_n + (t + \delta_t)n \tag{2}$$

where \( \text{mex} S = \min(\mathbb{Z}^0 \setminus S) \) is minimal excluded value of the set $S$, in particular, \( \text{mex} \emptyset = 0 \); and $\delta_t = 1$ if $t$ is odd, $\delta_t = 0$ if $t$ is even.

For every $n \geq 0$, we call the pair $(A'_n, B'_n)$ a $P$-generator of $P$-positions, since it generates the set \( \{(A'_n, B'_n), (A'_n + 1, B'_n), (A'_n + 1, B'_n + 1)\} \) of $P$-positions.

In [11], Fraenkel et al. adjoined to Generalized Wythoff some appropriate subsets of its $P$-positions as additional moves, obtaining some interesting games. This idea of “adjoin some $P$-positions as moves” has also been exploited in [12] to examine $(s, t)$-Wythoff’s game, resulting in some generalizations of those in [11]. This observation enables us to examine Restricted $(s, t)$-Wythoff in this paper.

We define four games, $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$, generated from Restricted $(s, t)$-Wythoff by adjoining to it subsets of its $P$-generators $\bigcup_{n \in Q_i} (A'_n, B'_n)$ as additional moves for $i \in \{1, 2, 3, 4\}$. In addition to the moves of Type I and II, a player in $\Gamma_i$ has a **Type III** move, i.e., remove $A'_n$ from one pile and $B'_n$ from the other for some $n \in Q_i$.

More specifically, we let $Q_1 = \{1\}$, i.e., $\Gamma_1$ is obtained from Restricted $(s, t)$-Wythoff by adjoining to it the $P$-generator $(A'_1, B'_1) = (2, 2s + t + \delta_t)$ as the only Type III move. Put $Q_2 = \mathbb{Z}^+ \cup \{0\}$, thus in $\Gamma_2$ all the nonzero $P$-generators of Restricted $(s, t)$-Wythoff are added as Type III moves. In $\Gamma_3$, let $Q_3 = \mathbb{Z}^+ \setminus \{2 + (t + \delta_t)/2\}$. In $\Gamma_4$, we define $Q_4$ with $Q_4 = \mathbb{Z}^+ \setminus \{2 + (t + \delta_t)/2\}$.

Each of our games is an infinite class of games, since they also depend on two parameters $s$ and $t$ as in Restricted $(s, t)$-Wythoff. But for every fixed $s$ and $t$, each $\Gamma_i$ is a single game.

In Section 2, some results useful throughout the paper are proved. Our main results are enunciated in Sections 3 and 4. Theorem 3.1 of Section 3 gives a recursive characterization of the $P$-positions of $\Gamma_1$ for $2s + t + \delta_t > 4$, which provides an exponential time winning strategy for $\Gamma_1$. Theorem 3.5 and Corollary 3.6, together with a special numeration system, provide a polynomial time winning strategy for $\Gamma_1$.

In Section 4, Theorem 4.5 shows that for $(s = 1, t \in \{3, 4\})$ or $(s = 2, t \geq 1)$, $\Gamma_1$ and $\Gamma_2$ are equivalent; Theorem 4.6 shows that for $s = 1$ and $t > 4$, $\Gamma_1$ and $\Gamma_4$ are equivalent; and in Theorem 4.7 we prove that for $s \geq 3$ and $t + \delta_t \geq 2s - 2$, $\Gamma_1$ and $\Gamma_4$ are equivalent. Final section provides the conclusions and some relevant open problems.

## 2 Preliminary

Let $A' = \bigcup_{n=1}^{\infty} A'_n$ and $B' = \bigcup_{n=1}^{\infty} B'_n$, where $A'_n$ and $B'_n$ are defined in Eq. (2). From Lemma 12 of [9], $A' \cup B' = \emptyset$, $A' \cup B' = \mathbb{Z}^{even} \setminus \{0\}$, and for every $n > m \geq 0$, $B'_{n+1} > B'_{n} > A'_{n} > A'_{m}$.

**Lemma 2.1.** For all $n \in \mathbb{Z}^+$, $A'_n - A'_{n-1} \in \{2, 4\}$, $B'_n - B'_{n-1} \in \{2s + t + \delta_t, 4s + t + \delta_t\}$. Moreover, $B'_{n} - B'_{n-1} = 2s + t + \delta_t$ if and only if $A'_{n} - A'_{n-1} = 2$; $B'_{n} - B'_{n-1} = 4s + t + \delta_t$ if and only if $A'_{n} - A'_{n-1} = 4$. 
Proof. Clearly $0 < A'_n - A'_{n-1} < Z^{even}$. Now suppose $A'_n - A'_{n-1} \geq 6$. Then we have $A'_{n-1} + 2 < A'_{n-1} + 4 < A'_n = \max\{A'_i, A'_i + 1, B'_i, B'_i + 1 : 0 \leq i < n\}$. Thus $A'_{n-1} + 2$ and $A'_{n-1} + 4$ are in the set $\{A'_i, A'_i + 1, B'_i, B'_i + 1 : 0 \leq i < n\}$. Now $A'$ and $B'$ are disjoint, thus the only possibility is that both $A'_{n-1} + 2$ and $A'_{n-1} + 4$ are in $B'$, i.e., the gap between the adjacent elements in $B'$ can be 2, which contradicts the fact that $B'_n - B'_{n-1} \notin \{2s + t + \delta_t, 4s + t + \delta_t\}$. Hence, $A'_n > A'_{n-1} = \{2, 4\}$. The remaining part is directly from the definition of $B'_n$ and the above.

Lemma 2.2. For every $n > m \geq 2$, $2(n - m) \leq A'_n - A'_m \leq 4(n - m)$. In particular, letting $m = 0$, gives $2n \leq A'_n \leq 4n$.

Proof. By Lemma 2.1, $2(n - m) \leq A'_n - A'_m = \sum^{n-1}_{j=m}(A'_j - A'_j) \leq 4(n - m)$.

Lemma 2.3. Let $\phi_n = A'_n - 2n$. For $r \in Z^{0,}$
$$\phi_j = 2r \Leftrightarrow B'_r < A'_j < B'_{r+1} \Leftrightarrow B'_r - 2(r - 1) \leq 2j \leq B'_{r+1} - 2(r + 1). \quad (3)$$

In particular, for fixed integer parameters $s, t$ with $2s + t + \delta_t > 4$, then $A'_j = 2j$ if and only if $2 \leq 2j \leq 2s + t + \delta_t - 2$, and $A'_j = 2j + 2$ if and only if $2s + t + \delta_t \leq 2j \leq 4s + 2t + 2\delta_t - 4$.

Proof. Let $S = \{x : 0 < x < A'_j\}$, and $x \in Z^{even}$. By $\sharp(S)$ we denote the number of elements of the set $S$. Put $I_{A'} = I(A' \cap S)$ and $I_{B'} = I(B' \cap S)$. Since $A' \cap B' = \emptyset$, $A' \cup B' = Z^{even} \setminus \{0\}$, it is easy to derive that $I_{A'} + I_{B'} = I(S) = A'/2$, and $I_{A'} = A'/2 - j$. Assume $r$ is exactly the largest index such that $B'_r < A'_j$. Now $B'_r$ is a strictly increasing sequence. Thus $I_{B'} = r$. Therefore,
$$\phi_j = A'_j - 2j = 2I_{B'} = 2r \Leftrightarrow B'_r < A'_j < B'_{r+1} \Leftrightarrow B'_r + 2 \leq 2j + 2r \leq B'_{r+1} + 2 \leq 2r + 1$$. From $\phi_j = A'_j - 2j = 2I_{B'} = 2r \Leftrightarrow B'_r < A'_j < B'_{r+1} \Leftrightarrow B'_r + 2 \leq 2j + 2r \leq B'_{r+1} + 2 \leq 2r + 1$.

Symmetry of our game rules implies that a game position $(a, b)$ is identical to $(b, a)$. Clearly, the $P$-generators added to Restricted $(s, t)$-Wythoff alter the set of $P$-positions of Restricted $(s, t)$-Wythoff. But the set of $P$-positions of each $\Gamma_i, i \in \{1, 2, 3, 4\}$, is still four sequences of pairs of integers. Denote by $\{A_n, B_n\}, (A_n, B_n + 1), (A_n + 1, B_n), (A_n + 1, B_n + 1)_{n \geq 0}$ the set of $\Gamma_i$, and for any $n \in Z^{0,}$, $A_n, B_n \in Z^{even}, A_n \leq A_{n+1}$ and $A_n \leq B_n$. Like in Restricted $(s, t)$-Wythoff, the pair $(A_n, B_n)$ is a $P$-generator of $P$-positions in each $\Gamma_i$.

Lemma 2.4. For every $n > m \geq 0$, $B_{n+1} > B_n > A_n > A_m$.

Proof. First, by the notations above, $A_n \geq A_m$ and $B_n \geq B_m$. Suppose $A_n = A_m$, this implies $B_n \neq B_m$, or else, $(A_n, B_n) = (A_m, B_m)$, impossible. So if $B_n > B_m$, then there is a move $(A_n, B_n) \rightarrow (A_m, B_m)$ (Type I, and $k = B_n - B_m \in Z^{even}$), but it contradicts stability property of $P$-positions. Now suppose $B_n < B_m$, then $(A_m, B_m) \rightarrow (A_n, B_n)$, resulting in another contradiction. Thus $A_n > A_m$. Similarly, $B_n > B_m$.

We prove below $A_n < B_n$. Again by the notations, $A_n \leq B_n$. If $A_n = B_n$ for some $n \in Z^{+}$, then we can move $(A_n, B_n) \rightarrow (0, 0)$ (Type II, satisfies condition (1)), another contradiction. Therefore, $B_{n+1} > B_n > A_n > A_m$.

Lemma 2.5. Put $A = \bigcup_{i=1}^{\infty} A_i$ and $B = \bigcup_{i=1}^{\infty} B_i$. Then $A \cap B = \emptyset$, and $A \cup B = Z^{even} \setminus \{0\}$.

Proof. We first show $A \cap B = \emptyset$. Suppose $A_n = B_m$ for some $n, m \in Z^{+}$. By Lemma 2.4, $m \neq n$, and $B_n > A_n = B_m > A_m$. Thus there is a move of Type I that $(A_n, B_n) \rightarrow (A_m, B_m)$ with $B_n \rightarrow A_m$, which is impossible.
Clearly $A_0 = B_0 = 0$, and $A \cup B \subseteq \mathbb{Z}^{even} \setminus \{0\}$. Suppose there exists $u \in \mathbb{Z}^{even} \setminus \{0\}$ but $u \notin A \cup B$. Thus for every $v \in \mathbb{Z}^{even}$, $(u, v) \notin \bigcup_{i=1}^{\infty} (A_i, B_i)$, that is, $(u, v)$ is an $N$-position. We show below that there exists some $v_0 \in \mathbb{Z}^{even}$ such that $(u, v_0)$ is not an $N$-position, by proving $F(u, v_0) \cap \mathcal{P} = \emptyset$, i.e., no follower of $(u, v_0)$ is a $P$-position.

Let $n_0$ be the largest index such that $A_{n_0} < u$. Then from $(u, v)$ we can move to $(A_i, B_i)$ only for some $i \leq n_0$. In particular, let $m_0 = j$ be the largest index of Type III move $(A_i', B_i')$ from $(u, v)$. More concretely,

$$m_0 = \begin{cases} 1 & \text{for } \Gamma_1, \\ \max\{i : A_i' \leq u, \} & \text{for } \Gamma_2, \\ \max\{i : A_i' \leq u, i \neq 3 + (t + \delta_i)/2 \} & \text{for } \Gamma_3, \\ \max\{i : A_i' \leq u, i \neq 2 + (t + \delta_i)/2 \} & \text{for } \Gamma_4. \end{cases}$$

Now let $D = \max\{B_i : i \leq n_0\} + 2$, and $v_0 = D + su(t + 1)$. Note that $v_0 \in \mathbb{Z}^{even}$. It is obvious that from $(u, v_0)$ we cannot reach to any position of the form $(A_j + 1, B_j)$ or $(A_j + 1, B_j + 1)$). So it suffices to prove that from $(u, v_0)$ none of Type I, II and III moves leads to $(A_i, B_i)$ with $i \leq n_0$.

For every $i \leq n_0$, $v_0 > \max\{B_i : i \leq n_0\} \geq B_i > A_i$. Then $v_0 \notin \bigcup_{i=1}^{n_0} A_i \cup \bigcup_{i=1}^{n_0} B_i$. Hence, we cannot move from $(u, v_0)$ to any $(A_i, B_i)$ with $i \leq n_0$ by a move of Type I.

For all $i \leq n_0$, we have $k = u - A_i > 0$, $\ell = v_0 - B_i = D - B_i + su(t + 1) > u - A_i = k$, and $\ell - k = D - B_i + (s - 1)u + A_i + su > (s - 1)(u - A_i) + t = (s - 1)k + t$. Therefore, no move of Type II from $(u, v_0)$ leads to $(A_i, B_i)$, $i \leq n_0$.

Finally, we consider the move of Type III. By Lemma 2.2, $A_n' \geq 2n$, and so

$$B_{m_0}' = sA_{m_0}' + (t + \delta_i)m_0 \leq sA_{m_0}' + (t + \delta_i)A_{m_0}'/2 \leq sA_{m_0}' + tA_{m_0}' \leq su(t + 1),$$

by virtue of $(t + \delta_i)/2 \leq t$ and $u \geq A_{m_0}$. By the above, $(A_{m_0}', B_{m_0}')$ is the maximum Type III move from $(u, v_0)$. However, $v_0 - B_{m_0}' = D + su(t + 1) - B_{m_0}' \geq D > B_i > A_i$ for every $i \leq n_0$. Therefore, no move of Type III from $(u, v_0)$ can lead to a $P$-position.

In conclusion, $u \in A \cup B$, and then $A \cup B = \mathbb{Z}^{even} \setminus \{0\}$. \hfill \□

3 Two winning strategies for $\Gamma_1$

Obviously, $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$ are $P$-positions in Restricted $(s, t)$-Wythoff, and so are they in $\Gamma_1$. Since now $(A_1', B_1')$ is a legal move of Type III, the $P$-position $(A_1', B_1') = (2, 2s + t + \delta_i)$ in Restricted $(s, t)$-Wythoff is no longer a $P$-position in $\Gamma_1$, nor $(A_1' + 1, B_1')$, $(A_1' + 1, B_1' + 1)$ or $(A_1' + 1, B_1' + 1)$. It is easily seen that this additional move for Restricted $(s, t)$-Wythoff alters the original $P$-positions.

Theorem 3.1. Given $s, t \in \mathbb{Z}^+$ with $2s + t + \delta_i > 4$. For $\Gamma_1$, $\mathcal{P} = \bigcup_{n=0}^{\infty} \{ (A_n, B_n), (A_n, B_n + 1), (A_n + 1, B_n), (A_n + 1, B_n + 1) \}$, where for $n \geq 0$,

$$\begin{cases} A_n = \max\{ A_i, A_i + 1, B_i, B_i + 1 : 0 \leq i < n \}, \\ B_n = (s - 1)A_n + (t + \delta_i + 4)n. \end{cases} \tag{4}$$

Example 3.2. For $s = 2$, $t \in \{3, 4\}$, we display the first few $P$-generators of $\Gamma_1$ in the table below, which show us how to determine $\mathcal{P}$ by using Eq. (4).

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>18</td>
<td>22</td>
<td>24</td>
<td>26</td>
<td>28</td>
<td>32</td>
<td>34</td>
</tr>
<tr>
<td>$B_n$</td>
<td>0</td>
<td>10</td>
<td>20</td>
<td>30</td>
<td>40</td>
<td>52</td>
<td>62</td>
<td>72</td>
<td>82</td>
<td>94</td>
<td>104</td>
<td>114</td>
<td>124</td>
<td>136</td>
<td>146</td>
</tr>
</tbody>
</table>
Proof. Notice first that the condition $2s + t + \delta_t > 4$ implies that the case $s = 1, t \in \{1, 2\}$ is not covered. Thus we have either $(s = 1, t > 2)$ or $(s > 1, t \geq 1)$.

For $s = 1$ and $t > 2$, the game was solved in [13], where Theorem 10 is precisely our assertion. But the structure of $P$-positions of the game is of algebraic form, which provides a poly-time winning strategy for our game. In addition, the methods of proof in [13] are totally different from those for the case $s > 1$ and $t \geq 1$ in this paper.

For $s > 1$ and $t \geq 1$, before we give the proof, some useful properties of the sequences $A_n$ and $B_n$ are required. Analysis similar to that in the proof of Lemmas 2.1 and 2.2 shows that for every $n > m \geq 0$, we have (i) $2(n - m) \leq A_n - A_m \leq 4(n - m)$, (ii) $A_n - A_{n-1} \in \{2, 4\}$, (iii) $B_n - B_{n-1} \in \{2s + t + \delta_t + 2, 4s + t + \delta_t\}$.

Particularly, $B_n - B_{n-1} = 2s + t + \delta_t + 2$ if and only if $A_n - A_{n-1} = 2$, $B_n - B_{n-1} = 4s + t + \delta_t$ if and only if $A_n - A_{n-1} = 4$.

Let $A = \bigcup_{i=1}^{\infty} A_i$ and $B = \bigcup_{i=1}^{\infty} B_i$, then by Lemma 2.5 we have the fact that $A \cap B = \emptyset$ and $A \cup B = \mathbb{Z}^{even} \setminus \{0\}$.

Now it suffices to show two things:

I. Every move from a position $u \in \mathcal{P}$ cannot terminate in $\mathcal{P}$.

II. From every position $v \notin \mathcal{P}$, there exists a legal move $v \rightarrow u \in \mathcal{P}$.

Proof of I. Let $(x, y)$ with $x \leq y$ be a position in $\mathcal{P}$. Assume that a move from $(x, y)$ leads to another position in $\mathcal{P}$. Since $A_n, B_n \in \mathbb{Z}^{even}$ and so $A_n + 1, B_n + 1 \in \mathbb{Z}^{odd}$, considering our game rules, we have eight possible moves for $n > m \geq 0$:

\[
\begin{align*}
(A_n, B_n) &\rightarrow (A_{m}, B_{m}), \\
(A_n, B_n + 1) &\rightarrow (A_{m}, B_{m} + 1), \\
(A_n, B_n) &\rightarrow (B_{m}, A_{m}), \\
(A_n, B_n + 1) &\rightarrow (B_{m}, A_{m} + 1).
\end{align*}
\]

Now none of them could be a Type I move, since $A_n$ and $B_n$ are strictly increasing sequences.

They cannot be moves of Type II either. Indeed, for the first four cases, clearly $n > m \geq 0$, and we have

\[
\ell - k = (B_n - B_m) - (A_n - A_m) = (s - 2)(A_n - A_m) + (t + \delta_t + 4)(n - m) \geq (s - 1)(A_n - A_m) + (4(n - m) - (A_n - A_m)) + t + \delta_t \geq (s - 1)(n - m) + t + \delta_t,
\]

which contradicts Eq. (1). Next consider the last four cases. For every $s > 1$, $t \in \mathbb{Z}^+$ and every $n \in \mathbb{Z}^{0}$, we have $B_n = (s - 1)A_n + (t + \delta_t + 4)n > A_n$. And by above, for every $n > m \geq 0$, $B_n - B_m \geq 4s(A_n - A_m) + 2 > A_n - A_m$. It follows that $B_n - A_m \geq B_n - B_m > A_n - A_m \geq A_n - B_m$. Thus for these four cases, we have $k' = A_n - B_m \leq A_n - A_m = k, \ell' = B_m - A_m \geq B_n - B_m = \ell$, and $\ell' - k' \geq \ell - k \geq (s - 1)k + t \geq (s - 1)k' + t$, a contradiction.

It remains to consider the one and only Type III move $(A'_1, B'_1) = (2, 2s + t + \delta_t)$. For the first four cases, note that $A_n - A_m = 2$ implies $n = m$. Then we get

\[B'_1 = B_n - B_m = (s - 1)(A_n - A_m) + (t + \delta_t + 4)(n - m) = 2s + t + \delta_t + 2 > B'_1,
\]

a contradiction. For the remaining four cases, if $A_n - B_m = A'_1 = 2$ and $B_n - A_m = B'_1 = 2s + t + \delta_t$. Now $A_n - A_m \geq 2(n - m) \geq 2$. Then $A'_1 + B'_1 = B_n - B_m + A_n - A_m = s(A_n - A_m) + (t + \delta_t + 4)(n - m) \geq 2s + t + \delta_t + 4$, which is impossible. Hence, any one of the hypothetical moves above cannot be a Type III move, ending the proof of I.

Proof of II. Without loss of generality, let $(x, y)$ with $x \leq y$ be a position not in $\mathcal{P}$.

If $x \in \{0, 1\}$, then we move $y \rightarrow \delta_y$, since clearly $y - \delta_y$ is even. In what follows, suppose $x \geq 2$. Now $A \cap B = \emptyset$ and $A \cup B = \mathbb{Z}^{even} \setminus \{0\}$. So $x$ appears exactly once in exactly one of $\bigcup_{n=1}^{\infty} \{A_n\} \cup \bigcup_{n=1}^{\infty} \{A_n + 1\}$ and $\bigcup_{n=1}^{\infty} \{B_n\} \cup \bigcup_{n=1}^{\infty} \{B_n + 1\}$. Thus we have one of the following two cases:

Case (i) $x = B_n$ or $B_n + 1$, for some $n \in \mathbb{Z}^+$. Then $y \geq x \geq B_n > A_n + 1 \geq A_n + \delta_y$, thus move $y \rightarrow A_n + \delta_y$ since $y - A_n - \delta_y$ is even.
Case (ii) $x = A_n$ or $A_n + 1$, for some $n \in \mathbb{Z}^+$. In this case, we have $y > B_n + 1$ or $x \leq y < B_n$. If $y > B_n + 1$, then move $y \to B_n + \delta_y$ since $0 < y - B_n - \delta_y \in \mathbb{Z}^{even}$. If $x \leq y < B_n$, we distinguish the following three subcases:

(ii.1) $y = B_n - 1$ or $B_n - 2$. We know $A_n - A_{n-1} \in \{2, 4\}$ for all $n \in \mathbb{Z}^+$. If $A_n - A_{n-1} = 2$, then $B_n - B_{n-1} = 2s + t + \delta_t + 2$. We move $(x, y) \to (x - 2, y - 2s - t - \delta_t)$ by a move of Type III, with $k = A'_1 = 2$, $\ell = B'_1 = 2s + t + \delta_t$. Notice that $x - 2 = A_{n-1}$ or $A_{n-1} + 1$, and $y - 2s - t - \delta_t = B_{n-1}$ or $B_{n-1} + 1$. Hence, $(x - 2, y - 2s - t - \delta_t) \in \mathcal{P}$.

If $A_n - A_{n-1} = 4$, then $B_n - B_{n-1} = 4s + t + \delta_t$. We move $(x, y) \to (x - 4, y - 4s - t - \delta_t + 2)$, which is a legal move of Type II: First, it is easy to see that $x - 4 = A_{n-1}$ or $A_{n-1} + 1$, and $y - 4s - t - \delta_t + 2 = B_{n-1}$ or $B_{n-1} + 1$. Thus $(x - 4, y - 4s - t - \delta_t + 2) \in \mathcal{P}$. Secondly, $k = 4$, $\ell = 4s + t + \delta_t - 2$, and $0 < k \leq \ell = 4s + t + \delta_t - 2 < sk + t$, which satisfies Eq. (1).

(ii.2) $x \leq y < sA_n + t + \delta_t$. In this subcase, we move $(x, y) \to (x - A_n, \delta_y) \in \mathcal{P}$. This is a move of Type II, with $k = A_n$ and $\ell = y - \delta_y$. Indeed, both $k$ and $\ell$ are even, then $0 < k = A_n \leq y - \delta_y = \ell$. And $sA_n + t + \delta_t$ is even, which implies $y - \delta_y \leq sA_n + t + \delta_t - 2$, and so $\ell - k = y - \delta_y - A_n \leq (s - 1)A_n + t + \delta_t - 2 < (s - 1)k + t$.

(ii.3) $sA_n + t + \delta_t \leq y < B_n - 2$. Put

$$m = \frac{[y - (s - 1)A_n - 2n + 2 - \delta_y]}{t + \delta_t + 2},$$

where $[x]$ denotes the largest integer $\leq x$. Then move $(x, y) \to (x - A_n + A_m, sA_m + \delta_y) \in \mathcal{P}$ by a move of Type II, which is legal:

First, $k = A_n - A_m$ and $\ell = y - B_m - \delta_y$ are even. Next we have (a) $k > 0$, (b) $\ell \geq k > 0$, (c) $\ell < sk + t$. Indeed,

(a) Since $A_n \geq 2n$, we have $y - (s - 1)A_n - 2n + 2 - \delta_y \geq sA_n + t + \delta_t - (s - 1)A_n - 2n + 2 > 0$, thereby $m \geq 0$. On the other hand, $y - (s - 1)A_n - 2n + 2 - \delta_y < B_n - (s - 1)A_n - 2n = (t + \delta_t + 2)n$, so $0 \leq m < n$. Thus $k = A_n - A_m > 0$.

(b) By the definition of $m$, we have

$$m \leq \frac{y - (s - 1)A_n - 2n + 2 - \delta_y}{t + \delta_t + 2},$$

so $y \geq (t + \delta_t + 2)m + (s - 1)A_n + 2n - 2 + \delta_y$. By (a) $n > m$ and $s > 1$, we get

$$\ell = y - B_m - \delta_y \geq (t + \delta_t + 2)m + (s - 1)A_n + 2n - 2 - (s - 1)A_n - (t + \delta_t + 4)m \geq (s - 1)(A_n - A_m) + 2(n - m) - 2 \geq (s - 1)k \geq k > 0.$$

(c) By the definition of $m$, we have

$$\frac{y - (s - 1)A_n - 2n + 2 - \delta_y}{t + \delta_t + 2} - 1 < m,$$

i.e., $y - \delta_y < (t + \delta_t + 2)(m + 1) + (s - 1)A_n + 2n - 2$. Further, note that both sides of this inequality are even, then $y - \delta_y \leq (t + \delta_t + 2)(m + 1) + (s - 1)A_n + 2n - 4$. It follows from $2(n - m) \leq (A_n - A_m)$ that

$$\ell = y - B_m - \delta_y \leq (t + \delta_t + 2)(m + 1) + (s - 1)A_n + 2n - 4 - (s - 1)A_n - (t + \delta_t + 4)m \geq (s - 1)(A_n - A_m) + t + \delta_t + 2(n - m) - 2 \leq s(A_n - A_m) + t + \delta_t - 2 < sk + t.$$

The proof is completed. 

Theorem 3.1 provides a recursive winning strategy in terms of the mex function, which is exponential in the input size $\log xy$ of any game position $(x, y) \in \mathbb{Z}^0 \times \mathbb{Z}^0$. For more theory of computing complexity of heap games, see [6]. Next, the central question we address here is whether our game has a better strategy, such as a polynomial time winning strategy.
A class of extensions of Restricted \((s, t)\)-Wythoff’s game

Table 2. Representations \(R(N)\) with \(N \in \mathbb{Z}^{\text{even}}\) over \(U\).

| \(N\) | \(1\) | \(2\) | \(3\) | \(4\) | \(5\) | \(6\) | \(7\) | \(8\) | \(9\) | \(10\) | \(11\) | \(12\) | \(13\) | \(14\) | \(15\) | \(16\) | \(17\) | \(18\) | \(19\) | \(20\) | \(21\) | \(22\) | \(23\) | \(24\) | \(25\) | \(26\) | \(27\) | \(28\) | \(29\) | \(30\) | \(31\) | \(32\) | \(33\) | \(34\) | \(35\) | \(36\) | \(37\) | \(38\) | \(39\) | \(40\) | \(41\) | \(42\) | \(43\) | \(44\) | \(45\) | \(46\) | \(47\) | \(48\) | \(49\) | \(50\) | \(51\) | \(52\) | \(53\) | \(54\) | \(55\) | \(56\) | \(57\) | \(58\) | \(59\) | \(60\) | \(61\) | \(62\) | \(63\) | \(64\) | \(65\) | \(66\) | \(67\) | \(68\) | \(69\) | \(70\) | \(71\) | \(72\) | \(73\) | \(74\) | \(75\) | \(76\) | \(77\) | \(78\) | \(79\) | \(80\) | \(81\) | \(82\) | \(83\) | \(84\) | \(85\) | \(86\) | \(87\) | \(88\) | \(89\) | \(90\) | \(91\) | \(92\) | \(93\) | \(94\) | \(95\) | \(96\) | \(97\) | \(98\) | \(99\) | \(100\) | \(101\) | \(102\) | \(103\) | \(104\) | \(105\) | \(106\) | \(107\) | \(108\) | \(109\) | \(110\) | \(111\) | \(112\) | \(113\) | \(114\) | \(115\) | \(116\) | \(117\) | \(118\) | \(119\) | \(120\) | \(121\) | \(122\) | \(123\) | \(124\) | \(125\) | \(126\) | \(127\) | \(128\) | \(129\) | \(130\) | \(131\) | \(132\) | \(133\) | \(134\) | \(135\) | \(136\) | \(137\) | \(138\) | \(139\) | \(140\) | \(141\) | \(142\) | \(143\) | \(144\) | \(145\) | \(146\) | \(147\) | \(148\) | \(149\) | \(150\) | \(151\) | \(152\) | \(153\) | \(154\) | \(155\) | \(156\) | \(157\) | \(158\) | \(159\) | \(160\) | \(161\) | \(162\) | \(163\) | \(164\) | \(165\) | \(166\) | \(167\) | \(168\) | \(169\) | \(170\) | \(171\) | \(172\) | \(173\) | \(174\) | \(175\) | \(176\) | \(177\) | \(178\) | \(179\) | \(180\) |

We introduce a numeration system that turns out to be relevant to our game \(\Gamma_1\). For fixed integers \(s > 1\) and \(t \geq 1\), let \(u_{-1} = 2/(s-1)\), \(u_0 = 2\), and put \(u_n = (s + \lceil t/2 \rceil)u_{n-1} + (s-1)u_{n-2}\) (\(n \geq 1\)). Here and subsequently, \([x]\) stands for the smallest integer \(\geq x\). Denote by \(U\) the numeration system with bases \(u_0, u_1, \ldots\) and digits \(d_i \in \{0, 1, \ldots, s\} \setminus \{ \lceil t/2 \rceil \}\). Note that an integer such as \(u_n\) has two representations: \(u_n\) itself and \((s + \lceil t/2 \rceil)u_{n-1} + (s-1)u_{n-2}\). Since we would like to have uniqueness of representation, it is natural to stipulate that \(d_i < s\) \((i \geq 1)\). Then we claim two facts:

(i) For every \(n \geq 1\), \(u_n \in \mathbb{Z}^{\text{even}}\), clearly by induction on \(n\).

(ii) Every decimal number \(N \in \mathbb{Z}^{\text{even}}\) has a unique representation \(R(N)\) over \(U\). This is a special case of Theorem 3 in \([14]\). Note that the greedy algorithm of repeatedly dividing \(N\) or its remainder by the largest \(u_i\) not exceeding this remainder gives the unique representation.

Those whose representations \(R(N)\) end in an even number of 0s are called \(vile\) numbers, and those whose representations \(R(N)\) end in an odd number of 0s are called \(dopey\) numbers (for an etymology of the terms vile, dopey, see \([15]\)). In addition, by \(L R(N)\) we denote the “left shift” of \(R(N)\), i.e., \(L R(N)\) is obtained from \(R(N)\) by adjoining 0 to the right end of \(R(N)\).

Example 3.3. We consider \(\Gamma_1\) of Example 3.2, where \(s = 2\), \(t \in \{3, 4\}\), and so \(\lceil t/2 \rceil = 2\). Thus, \(u_{-1} = 2, u_0 = 2, u_1 = 10, u_2 = 42, u_3 = 178, \ldots\). The representations \(R(N)\) over \(U\) of the first few numbers \(N \in \mathbb{Z}^{\text{even}}\) appear in Table 2.
A question we just might ask at this point is what the connection is between Tables 1 and 2. By scanning the first few entries of both tables, we may be tempted to conclude that all $A_n$s in Table 1 are vile, also it seems that all $B_n$s are dopey. Moreover, we consider a $P$-generator say $(12, 52)$ in $\Gamma_1$ with representations $R(12, 52) = (11, 110)$. It is obvious that $LR(12) = L(11) = 110 = R(52)$. Now consider an $N$-position $(20, 84)$ in $\Gamma_1$, whose representations are $R(20, 84) = (20, 200)$, we also have $LR(20) = R(84)$. Therefore, it appears that there is no causal relation between the facts that $(x, y) \in P$ in $\Gamma_1$ and the condition $R(y) = LR(x)$. We next show how to determine $P$-positions of $\Gamma_1$ via $\mathcal{U}$.

Lemma 3.4. Let $\{V_m\}_{m \geq 0}$ denote the set of all vile numbers in $\mathcal{U}$ with $0 = V_0 < V_1 < V_2 < \cdots$, and put $R(D_m) = LR(V_m)$ for all $m$. Then

$$D_m - (s - 1)V_m = (2\lceil t/2 \rceil + 4)m$$

for all $m$.

Proof. Induction on $m$. Clearly for $m = 0$, $D_m - (s - 1)V_m = (2\lceil t/2 \rceil + 4)m$ for arbitrary but fixed $m$. It suffices to prove that the assertion holds for $m + 1$. Let $V_m = \sum_{i=0}^{n} d_i u_i$. Since $R(D_m) = LR(V_m)$, $D_m = \sum_{i=0}^{n} d_i u_i + 1$. So we have

$$D_m - (s - 1)V_m = (2\lceil t/2 \rceil + 4)m = \sum_{i=0}^{n} d_i (u_{i+1} - (s - 1)u_i).$$

Let $q = s - 2$ and $r = s + \lceil t/2 \rceil$. Then the linear recurrence of $\mathcal{U}$ has the form $u_n = ru_{n-1} + (q + 1)u_{n-2}$ ($n \geq 1$), with the digits $d_i \in \{0, 1, \ldots, r\}$ such that $d_{i+1} = r \implies d_i \geq q$ ($i \geq 0$). We proceed by distinguish three cases, because the tail of $R(V_m)$ must be one of the following three forms:

(i) The tail of $R(V_m)$ has digits

$$d_{2k}d_{2k-1}d_{2k-2} \ldots d_2 d_1 d_0 = d_2 r q \ldots r q r q,$$

for some $k \in \mathbb{Z}^0$, where $d_{2k} \in \{0, 1, \ldots, q\}$ and $d_{2k} = q \implies d_{2k+1} < r$. Then it is not hard to see that $V_m + 2 = (d_{2k} + 1)u_{2k} + \sum_{i=2k+1}^{n} d_i u_i$, and so $V_m + 2$ is vile. Thus $V_{m+1} = V_m + 2$, since $V_i$ is even for every $i \in \mathbb{Z}^0$.

<table>
<thead>
<tr>
<th>(u_n)</th>
<th>(u_{2k})</th>
<th>(u_{3})</th>
<th>(u_2)</th>
<th>(u_1)</th>
<th>(u_0)</th>
<th>(N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d_n)</td>
<td>(d_{2k})</td>
<td>(r)</td>
<td>(q)</td>
<td>(r)</td>
<td>(q)</td>
<td>(V_m)</td>
</tr>
<tr>
<td>(d_n)</td>
<td>(d_{2k} + 1)</td>
<td>(0)</td>
<td>(0)</td>
<td>(0)</td>
<td>(V_m + 2)</td>
<td></td>
</tr>
</tbody>
</table>

Thus

$$D_{m+1} - (s - 1)V_{m+1} = (d_{2k} + 1)(u_{2k+1} - (s - 1)u_{2k}) + \sum_{i=2k+1}^{n} d_i (u_{i+1} - (s - 1)u_i)$$

$$= u_{2k+1} - (s - 1)u_{2k} + \sum_{i=2k}^{n} d_i (u_{i+1} - (s - 1)u_i).$$

On the other hand, by Eq. (6) and the recurrence $ru_{n-1} + (q + 1)u_{n-2} = u_n$, we have

$$(2\lceil t/2 \rceil + 4)m = q(u_1 - (s - 1)u_0) + ru_{2} - (s - 1)u_1 + q(u_3 - (s - 1)u_2)$$

$$+ \ldots + q(u_{2k+1} - (s - 1)u_{2k}) + ru_{2k} - (s - 1)u_{2k+1}$$

$$+ \sum_{i=2k}^{n} d_i (u_{i+1} - (s - 1)u_i)$$

$$= u_{2k+1} - u_1 - (s - 1)u_{2k} + (s - 1)u_0 + \sum_{i=2k}^{n} d_i (u_{i+1} - (s - 1)u_i)$$

$$= u_{2k+1} - (s - 1)u_{2k} - (2\lceil t/2 \rceil + 4) + \sum_{i=2k}^{n} d_i (u_{i+1} - (s - 1)u_i).$$

Therefore, it follows from Eq. (7) that

$$(2\lceil t/2 \rceil + 4)(m + 1) = u_{2k+1} - (s - 1)u_{2k} + \sum_{i=2k}^{n} d_i (u_{i+1} - (s - 1)u_i)$$

$$= D_{m+1} - (s - 1)V_{m+1}.$$
(ii) The tail of $R(V_m)$ has digits
\[ d_{2k+1}d_{2k}d_{2k-1} \ldots d_2d_1d_0 = d_{2k+1}r_1 \ldots r_qrqr, \]
for some $k \in \mathbb{Z}^0$, where $d_{2k+1} \in \{0, 1, \ldots, q \}$ and $d_{2k+1} = q \implies d_{2k+2} < r$. Then $V_m + 2 = (d_{2k+1} + 1)u_{2k+1} + \sum_{i=2k+2}^{n} d_i u_i$, and so $V_m + 2$ is dopey. But $V_m + 4$ is vile, since $R(V_m + 4)$ ends in 1. Thus $V_{m+1} = V_m + 4 = u_0 + (d_{2k+1} + 1)u_{2k+1} + \sum_{i=2k+2}^{n} d_i u_i$.

<table>
<thead>
<tr>
<th>$u_0$</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_{2k+1}$</th>
<th>$N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_0$</td>
<td>$d_{2k+1}$</td>
<td>$d_{2k+1}$</td>
<td>$r$</td>
<td>$V_m$</td>
</tr>
<tr>
<td>$d_n$</td>
<td>$d_{2k+1} + 1$</td>
<td>$d_{2k+1} + 1$</td>
<td>$0$</td>
<td>$V_m + 2$</td>
</tr>
<tr>
<td>$d_n$</td>
<td>$d_{2k+1} + 1$</td>
<td>$d_{2k+1} + 1$</td>
<td>$0$</td>
<td>$V_m + 4$</td>
</tr>
</tbody>
</table>

Hence,
\[
D_{m+1} - (s-1)V_{m+1} = (u_1 - (s-1)u_0) + (d_{2k+1} + 1)(u_{2k+2} - (s-1)u_{2k+1}) + \sum_{i=2k+2}^{n} d_i (u_{i+1} - (s-1)u_i) \\
= (2\lceil t/2 \rceil + 4) + u_{2k+2} - (s-1)u_{2k+1} + \sum_{i=2k+1}^{n} d_i (u_{i+1} - (s-1)u_i). \tag{8}
\]

On the other hand, by Eq. (6),
\[
(2\lceil t/2 \rceil + 4)m = r(u_1 - (s-1)u_0) + q(u_2 - (s-1)u_1) + \ldots + q(u_{2k} - (s-1)u_{2k-1}) + r(u_{2k+1} - (s-1)u_{2k}) + \sum_{i=2k+1}^{n} d_i (u_{i+1} - (s-1)u_i) \\
= u_{2k+2} - (s-1)u_{2k+1} + \sum_{i=2k+1}^{n} d_i (u_{i+1} - (s-1)u_i).
\]

The last equality follows from that summing the positive terms, adding and subtracting $(s-1)u_0$, leads to $u_{2k+2} - 2(s-1)$, and that summing the negative terms, subtracting and adding $(s-1)^2u_1 = 2(s-1)$, leads to $-(s-1)u_{2k+1} + 2(s-1)$. Thus, by Eq. (8), $D_{m+1} - (s-1)V_{m+1} = (2\lceil t/2 \rceil + 4)(m+1)$, as required.

(iii) The digit $d_0$ satisfies $q < d_0 < r$. By the definition of $\mathcal{U}, d_1 < r$, and so $V_{m+1} = V_m + 2$. Thus it follows from Eq. (6) that
\[
D_{m+1} - (s-1)V_{m+1} = (d_0 + 1)(u_1 - (s-1)u_0) + \sum_{i=1}^{n} d_i (u_{i+1} - (s-1)u_i) \\
= (2\lceil t/2 \rceil + 4) + \sum_{i=0}^{n} d_i (u_{i+1} - (s-1)u_i) \\
= (2\lceil t/2 \rceil + 4)(m+1).
\]

Finally, note that if $V_m$ ends in an even number of 0s, it is of the form (i). The proof is completed.

\[\Box\]

**Theorem 3.5.** For all $n \in \mathbb{Z}^0$, $(V_n, D_n) = (A_n, B_n)$.

**Proof.** Obviously, $(V_0, D_0) = (A_0, B_0) = (0, 0)$. For $n \geq 1$, we have two facts:

(i) $A_n, B_n, V_n, D_n \in \mathbb{Z}^{even}$;
(ii) For all $t \in \mathbb{Z}^+, t + \delta_t = 2\lceil t/2 \rceil$.

By Eqs. (4) and (5), together with Fact (ii), $B_n$ and $D_n$ have the same formation law. It remains only to show that $A_n$ and $V_n$ have the same inductive formation rule, i.e., $V_n = \text{mex}\{V_i, V_{i+1}, D_i, D_{i+1} : 0 \leq i < n\}$.

Now for every $n \geq 1$, $D_n$ is dopey, since $V_n$ is vile and $R(D_n) = LR(V_n)$. By Fact (i), $D_n > V_n + 1 > V_n$. It follows that $\bigcup_{i=1}^{\infty} V_i$ and $\bigcup_{i=1}^{\infty} D_i$ are complementary with respect to $\mathbb{Z}^{even} (> 0)$, since every positive integer has
precisely one representation in $U_1$, either vile as $V_2$s do or dopey as the $D_n$s do. Let $S = \{V_i, V_j + 1, D_i, D_j + 1 : 0 \leq i < n \}$. Suppose $\text{max} S = D_j$ for some $j \geq n$. This implies that $V_j \in S$ on account of $V_j < D_j$, and then $j < n$, a contradiction. Therefore, $\text{max} S = V_n$, the assertion follows.

\(\square\)

**Corollary 3.6.** Given a position $(x, y) \in \Gamma_1$ with $s > 1$, $(x, y)$ is a P-position if and only if there exists some $n \in \mathbb{Z}^0$ such that $([x/2], [y/2]) = (V_n, D_n)$.

**Proof.** If $(x, y)$ is a P-position of $\Gamma_1$ with its P-generator $([x/2], [y/2]) = (A_n, B_n)$ for some $n \in \mathbb{Z}^0$, then by Theorem 3.5, $(A_n, B_n) = (V_n, D_n)$, and vice versa.

Given any game position $(x, y)$ with $0 < x \leq y$, compute $R(x)$ using the greedy algorithm mentioned above. If $R(x)$ ends in an odd number of $0$s, then $x = B_n$ for some $n \in \mathbb{Z}^+$. Then we move $y \rightarrow A_n$, where $R(B_n) = LR(A_n)$. If $R(x)$ ends in an even number of $0$s, then $x = A_n$ for some $n \in \mathbb{Z}^+$. The relative size of $y$ and $B_n$ can also be tested, since $R(B_n) = LR(A_n)$. Thus the complexity of this computation, up to a multiplication constant, is that of computing $R(x)$, which is linear in $\log x$. Hence, this winning strategy bases on $U$ is polynomial in the input size $\log x$.

4 Analyzing the equivalences

**Lemma 4.1.** Let $A_n$ and $B_n$ be determined by Eq. (4). Then for every $n > m \geq 0$, $A_n - A_m = 2(n - m) + 2r$. If $r \geq 1$, then $A_n - A_m \geq (r - 1)(2s + t + \delta_t + 2) + 4$.

**Proof.** Let $S = \{x : A_m < x \leq A_n, x \in \mathbb{Z}^{even}\}$. As in the proof of Lemma 2.3, $[S] = (A_n - A_m)/2$, $[A] = [S \cap S] = n - m$, and $[A] = [S \setminus S] = m = r$.

By $B_{j+1}, B_{j+2}, \ldots, B_{j+r}$ we denote the $r$ elements of $B \cap S$. From the properties of $A_n$ and $B_n$ in the proof of Theorem 3.1, we have $A_n + 2 \leq B_{j+1} + B_{j+2} \leq \cdots \leq B_{j+r} + 2.4s + t + \delta_t$, thus $A_n - A_m \geq B_{j+1} + B_{j+2} + 4 = 4 + \sum_{i = j+1}^{j+r-1} (B_{i+1} + B_i) \geq (r - 1)(2s + t + \delta_t + 2) + 4$. \(\square\)

**Lemma 4.2.** Given two parameters $s, t \in \mathbb{Z}^+$. For every $n > m > 0$, there is no $i \in \mathbb{Z}^+$ such that $A'_i = A_n - B_m$ and $B'_i = B_n - A_m$, where $A'_i$ and $B'_i$ are determined in Eq. (4). $A_1$ and $B_1$ are determined in Eq. (4).

**Proof.** Suppose that there exists some $i_0 \in \mathbb{Z}^+$ such that $A'_i = A_n - B_m$ and $B'_i = B_n - A_m$. Write $A_n - A_m = 2(n - m) + 2r$. According to the proof of Theorem 3.1, we have $2(n - m) \leq A_n - A_m \leq 4(n - m)$ for every $n > m \geq 0$, thus $r \geq 0$.

If $r = 0$, From Lemma 2.2, for all $n \in \mathbb{Z}^0$, $2n \leq A'_n \leq 4n$. Thus $2(n - m) = A_n - A_m > A_n - B_m = A'_i \geq 2i_0$, i.e., $n - m > i_0$. Hence,

$$B'_i = B_n - A_m \geq B_n - B_m - (s - 1)A'_i + (t + \delta_t + 4)i_0 = B'_i + 4i_0 - A'_i \geq B'_i,$$

a contradiction.

If $r \geq 1$, then $2(n - m) + 2r = A_n - A_m > A_n - B_m = A'_i$. More accurately, $2(n - m) \geq A'_i - 2r + 2$. By Lemma 4.1, we have $A_n - A_m \geq (r - 1)(2s + t + \delta_t + 2) + 4 > (r - 1)(t + \delta_t + 4)$. Then

$$B'_i + A'_i = B_n - B_m + A_n - A_m$$

$$= s(A_n - A_m) + (t + \delta_t + 4)(n - m)$$

$$\geq s(A'_i + B_m - A_m) + (t + \delta_t + 4)(A'_i/2 - r + 1)$$

$$= (s + 1)A'_i + A'_i + s(B_m - A_m) + (t + \delta_t)A'_i/2 - (r - 1)(t + \delta_t + 4)$$

$$> (s + 1)A'_i + (t + \delta_t)i_0 + A_n - A_m - (r - 1)(t + \delta_t + 4)$$

$$> (s + 1)A'_i + (t + \delta_t)i_0 = B'_i + A'_i. $$

another contradiction. \(\square\)
Lemma 4.3. For $s > 1$ and $t \geq 1$, Suppose that $(A_n, B_n) \longrightarrow (A_m, B_m)$ with $A'_j = A_n - A_m$ and $B'_j = B_n - B_m$ for some $n > m \geq 0$. Then

$$A'_j \leq 2(j + 1 + \frac{2s - 2}{t + \delta_t}) \quad (9)$$

Proof. Let $S = \{x : A_m < x \leq A_n, \text{ and } x \in \mathbb{Z}^{even}\}$. As the above, $\#(S) = A'_j / 2, \#A = \#(A \cap S) = n - m, \#B = \#(B \cap S) = \#(S) - \#A$. Then $2(n - m) = A'_j - 2\#B$. The minimum value of $2(n - m)$ is reached at the maximum value of $\#B$, that is, when there is a smallest distance of $2s + t + \delta_t + 2$ between consecutive elements of $B \cap S$.

Since $A'_j$ is even, and also $2s + t + \delta_t + 2$ is even, dividing $A'_j$ by $2s + t + \delta_t + 2$ gives $A'_j = (2s + t + \delta_t + 2)q + 2r, r \in \{1, 2, 3, \ldots, s + (t + \delta_t) / 2\}$. Note that for $r \geq 2$, there may be an additional element in $B \cap S$. Thus

$$2(n - m) \geq (2s + t + \delta_t)q + \begin{cases} 2r & \text{if } r \in \{0, 1\}, \\ 2(r - 1) & \text{if } r \in \{3, 4, \ldots, s + (t + \delta_t) / 2\}. \end{cases}$$

Now

$$B'_j = B_n - B_m = \sum_{i=m}^{n-1} (B_{i+1} - B_i) \geq (2s + t + \delta_t + 2)(n - m).$$

If $r \in \{0, 1\}$,

$$B'_j \geq (2s + t + \delta_t + 2)(n - m) \geq (s + (t + \delta_t) / 2 + 1)((2s + t + \delta_t)q + 2r) \geq (s + (t + \delta_t) / 2)A'_j + 2r \geq (s + (t + \delta_t) / 2)A'_j.$$

If $r \in \{2, 3, \ldots, s + (t + \delta_t) / 2\}$,

$$B'_j \geq (2s + t + \delta_t + 2)(n - m) \geq (s + (t + \delta_t) / 2)(2s + t + \delta_t)q + (2s + t + \delta_t + 2)(r - 1) \geq (s + (t + \delta_t) / 2)A'_j + 2r - (2s + t + \delta_t + 2) \geq (s + (t + \delta_t) / 2)A'_j + 2(2s + t + \delta_t).$$

In either case, the following inequality holds:

$$sA'_j + (t + \delta_t)j = B'_j \geq (s + (t + \delta_t) / 2)A'_j + 2 - (2s + t + \delta_t)$$

Therefore,

$$A'_j \leq 2(j + 1 + \frac{2s - 2}{t + \delta_t}). \quad \Box$$

Lemma 4.4. Given $s, t \in \mathbb{Z}^+$. For $2s + t + \delta_t > 4$ and $t + \delta_t \geq 2s - 2$, suppose that there is a move $(A_n, B_n) \longrightarrow (A_m, B_m)$ with $A'_j = A_n - A_m$ and $B'_j = B_n - B_m$ for some $n > m \geq 0$. Then

1. If $s = 1$ and $t \in \{3, 4\}$, this is impossible for any $j \geq 1$. If $s = 1$ and $t > 4$, the only possibility is $j = 3 + (t + \delta_t) / 2$.
2. If $s = 2$ and $t \geq 1$, this is impossible for any $j \geq 1$.
3. If $s \geq 3$ and $t + \delta_t \geq 2s - 2$, the only possibility is $j = 2 + (t + \delta_t) / 2$.

Proof. Again let $S = \{x : A_m < x \leq A_n, \text{ and } x \in \mathbb{Z}^{even}\}$. From the proof of Lemma 4.3 we have $2(n - m) = A'_j - 2\#B$. By Lemma 2.2, $A'_j \geq 2j$ for all $j \in \mathbb{Z}^+$. Thus Eq. (9) implies that there are three possibilities:

1. $A'_j = 2j$,
2. $A'_j = 2j + 2$ and

Case (a) $A'_j = 2j$. By definition, $B'_j = (2s + t + \delta_t)j$. By Lemma 2.3, $A'_j = 2j$ means that $2 \leq A_n - A_m = 2j \leq 2s + t + \delta_t - 2$.

There must be up to one $B_i \in S$, i.e., $\#B \in \{0, 1\}$. If there are more than two elements in $B \cap S$, then there is a gap less than $2s + t + \delta_t - 2$ between the adjacent elements of $B \cap S$, but it contradicts with the fact that $B_{i+1} - B_i \in \{2s + t + \delta_t + 2, 4s + t + \delta_t\}$.
So suppose \( \#_B = 0 \). Then \( n - m = A'_j/2 = j \), this implies that \( A_j - A_{j-1} = 2 \) for all \( m < i \leq n \). Accordingly, \( B_l - B_{l-1} = 2s + t + \delta t + 2 \) for \( m < i \leq n \). Thus \( B_n - B_m = (2s + t + \delta t + 2)(n - m) > (2s + t + \delta t)j = B'_j \). This contradicts our assumption.

Now suppose \( \#_B = 1 \). Then \( n - m = (A'_j - 2)/2 = j - 1 \). Let \( B_{l_0} \) denote the only element of \( B \cap S \) with \( A_{\lambda} < B_{l_0} < A_{\lambda + 1} \), where \( m < \lambda < n \).

\[
\begin{align*}
\frac{2(\lambda - m)}{4} & , \ldots , \frac{2(n - \lambda - 1)}{4} , \frac{2(\lambda - m)}{4} & , \ldots , \frac{2(n - \lambda - 1)}{4} , \frac{2(\lambda - m)}{4} & , \ldots , \frac{2(n - \lambda - 1)}{4} , \ldots
\end{align*}
\]

Therefore,

\[
\begin{align*}
B_n - B_m &= \sum_{\lambda \leq i \leq \lambda_n} (B_{i+1} - B_i) + (B_{\lambda+1} - B_{\lambda}) + \sum_{\lambda < i \leq n-1} (B_{i+1} - B_i) \\
&= (2s + t + \delta t + 2)(\lambda - m) + (4s + t + \delta t) + (2s + t + \delta t + 2)(n - \lambda - 1) \\
&= (2s + t + \delta t + 2)(n - m - 1) + 4s + t + \delta t \\
&= (2s + t + \delta t + 2)(j - 2) + 4s + t + \delta t \\
&= (2s + t + \delta t)j + 2j - (t + \delta t + 4) \\
&= B'_j + (2j - (t + \delta t + 4)) .
\end{align*}
\]

If \( s \in \{1, 2\} \), then \( 2j - (t + \delta t + 4) \leq 2s + t + \delta t - 2 - (t + \delta t + 4) = 2s - 6 < 0 \). Thus for any \( t \in \mathbb{Z}^+ \), \( B'_j = B_n - B_m \) is impossible.

If \( s \geq 3 \), then \( B'_j = B_n - B_m \) if and only if \( 2 \leq 2j = t + \delta t + 4 \leq 2s + t + \delta t - 2 \), which is true for any \( t \in \mathbb{Z}^+ \). Thus the only possibility is \( j = (t + \delta t)/2 + 2 \).

Case (b) \( A'_j = 2j + 2 \). In this case, \( B'_j = sA'_j + (t + \delta t)j = (2s + t + \delta t)j + 2s \). By Lemma 2.3, \( A'_j = 2j + 2 \) means that

\[
2s + t + \delta t \leq A_n - A_m - 2 = 2j \leq 4s + 2t + 2\delta t - 4 .
\]

We claim that \( \#_B \in \{1, 2\} \). Indeed, if \( \#_B = 0 \), then \( n - m = A'_j/2 = j + 1 \). By the same argument as for the case (a), we get \( B'_j = B_n - B_m = (2s + t + \delta t + 2)(n - m - 1) = (2s + t + \delta t + 2)(j + 1) > B'_j \), a contradiction.

If \( \#_B \geq 3 \), then there is a gap less than \( (A_n - A_m)_{\max}/2 \) between the neighbouring elements of \( B \cap S \). But \( (A_n - A_m)/2 \leq 2s + t + \delta t - 1 \), so this also cannot happen by reason of \( B_{j+1} - B_j \in \{2s + t + \delta t + 2, 4s + t + \delta t\} \).

We first suppose \( \#_B = 1 \). Then \( n - m = (A'_j - 2)/2 = j \). In the same manner of case (a), we can see that

\[
\begin{align*}
B_n - B_m &= (2s + t + \delta t + 2)(n - m - 1) + 4s + t + \delta t \\
&= (2s + t + \delta t + 2)(j - 1) + 4s + t + \delta t \\
&= (2s + t + \delta t + 2)j + 2s + 2j - 2 \\
&= B'_j + 2j - 2 \\
&> B'_j ,
\end{align*}
\]

a contradiction.

Next suppose \( \#_B = 2 \). Then \( n - m = (A'_j - 4)/2 = j - 1 \). Let \( B_{l_0} , B_{l_0 + 1} \in B \cap S \) with \( A_{\lambda_0} < B_{l_0} < A_{\lambda_0 + 1} \), and \( A_{\lambda_1} < B_{l_0 + 1} < A_{\lambda_1 + 1} \), where \( m < \lambda_0 < \lambda_1 < n \).

\[
\begin{align*}
\frac{2(\lambda_0 - m)}{4} & , \ldots , \frac{2(\lambda_1 - \lambda_0 - 1)}{4} & , \ldots , \frac{2(\lambda_0 - m)}{4} & , \ldots , \frac{2(\lambda_1 - \lambda_0 - 1)}{4} , \ldots
\end{align*}
\]

Similarly, we have

\[
\begin{align*}
B_n - B_m &= (2s + t + \delta t + 2)(n - m - 2) + 2(4s + t + \delta t) \\
&= (2s + t + \delta t + 2)(j - 3) + 2(4s + t + \delta t) \\
&= (2s + t + \delta t)j + 2s + 2j - (t + \delta t + 6) \\
&= B'_j + 2j - (t + \delta t + 6) .
\end{align*}
\]
It is worth to mention that $B_{t_0} \geq A_m + 2$ and $B_{t_0+1} \leq A_n - 2$. Consequently, $2j + 2 = A'_j = A_n - A_m \geq B_{t_0+1} - B_{t_0} + 4 \geq 2s + t + \delta t + 6$. It follows Eq. (10) that

$$2s + t + \delta t + 4 \leq 2j \leq 4s + 2t + 2\delta t - 4.$$  \(\tag{12}\)

If $s = 1, t \in \{3, 4\}$, there is no such $j$ satisfying Eq. (12).

If $s = 1, t > 4$, by Eq. (11), $B_n - B_m = B'_j$ if and only if $j = (t + \delta t)/2 + 3$, which satisfies Eq. (12).

If $s \geq 2$, then $2j \geq 2s + t + \delta t + 4 \geq t + \delta t + 8 > t + \delta t + 6$, thus $B'_j = B_n - B_m$ is impossible.

Case (c) $A'_j = 2j + 4$. On account of Eq. (9) this case can happen only when $t + \delta t = 2s - 2$. Thus we have

$$B'_j = sA'_j + (t + \delta t)j = (4s - 2j) + 4s.$$  \(\tag{c-i} \)

And the condition $2s + t + \delta t = 4s - 2 > 4$ implies $s \geq 2$. We below consider $s = 2$ and $s \geq 3$, respectively.

(c-ii) $s = 2$ and also $t + \delta t = 2$. Now $B'_j = 6j + 8$. By Lemma 2.3, $A'_j = 2j + 4$ implies that $B'_2 - 2 \leq A_n - A_m - 4 = 2j \leq B'_3 - 6$. Recall that $A'_0 = B'_0 = 0$, $A'_1 = \max\{0, 1\} = 2$, and $B'_1 = 2s + t + \delta t = 6$. Thereby, $A'_2 = \max\{0, 1, 2, 3, 6\} = 4$ and then $B'_2 = 4s + 2t + 2\delta t$ = 12. Also $A'_3 = \max\{0, 1, 2, 3, 4, 5, 6, 7, 12, 13\} = 8$, and so $B'_3 = 22$. Thus $14 \leq A_n - A_m \leq 24$, and $5 \leq j \leq 8$. In this case, we also have the fact that $B_{t_0+1} - B_t \in \{2s + t + \delta t + 2, 4s + t + \delta t\} = \{8, 10\}$.

Then we claim that $\xi_B \in \{1, 2, 3\}$. Indeed, if $\xi_B = 0$, then $n - m = A'_j/2 = j + 2$. By the same argument as for the cases (a) and (b), we have $B'_j = B_n - B_m = 8(n - m - 1) + 10 = 8j + 10$, contradicting the assumption.

If $\xi_B = 2$, we know $n - m = (A'_j - 2)/2 = j + 1$. As in the proof of case (a), $B_n - B_m = 8(n - m - 2) + 20 = 8(j - 2) + 20 > B'_j$, a contradiction.

If $\xi_B = 3$, then $n - m = (A'_j - 6)/2 = j - 1$. Assume the three elements in $B \cap S$ are $B_{t_0}, B_{t_0+1}, B_{t_0+2}$ for some $i_0 \in \mathbb{Z}^+$. If we have $B_{t_0} \leq A_m + 2$ and $B_{t_0+2} \leq A_n - 2$, thus $A'_j = A_n - A_m \geq B_{t_0+2} - B_{t_0} + 4 \geq 20$, i.e., $j \geq 8$. In the same manner we can see that

$$B_n - B_m = 8(n - m - 3) + 30 = 8(j - 4) + 30 > 6j + 8 = B'_j$$

since $j \geq 8$ (by above $5 \leq j \leq 8$, the only possibility in fact is $j = 8$), giving a contradiction.

First suppose $\xi_B = 1$. As the above, we get $B_n - B_m = 4s(n - m - 1) + (6s - 2) = 4sj + 6s - 2 = B'_j + 2j + 2s - 2 \geq B'_j$ by reason of $j \geq 4s + 3$ and $s \geq 3$, a contradiction.

Next suppose $\xi_B = 2$. Similarly, $B_n - B_m = 4s(n - m - 2) + (6s - 2) = 4sj + 12s - 4 = B'_j + 2j - 4 \geq B'_j$, a contradiction.

Finally suppose $\xi_B = 3$. Analogously, we have $B_n - B_m = 4s(n - m - 3) + 3(6s - 2) = 4sj - 4 + 18s - 6 = B'_j + 2j - 6 < B'_j$ since $j \geq 4s - 3$ and $s \geq 3$.

Therefore, if $A'_j = 2j + 4$, for any $s \geq 2$ and $t \geq 1$ with $t + \delta t = 2s - 2$, the hypothesis $B'_j = B_n - B_m$ is impossible.

Given two games, they have the same game positions but with possibly different move rules. We call them equivalent if their $P$-positions are the same. The two equivalent games certainly have also the same $N$-positions, as well as winning strategy.

**Theorem 4.5.** For $(s = 1$ and $t \in \{3, 4\})$ or $(s = 2$ and $t \geq 1)$, $\Gamma_1$ and $\Gamma_2$ are equivalent.
Proof. Recall that $\Gamma_2$ can be viewed as $\Gamma_1$ to which the moves $\bigcup_{i \geq 2} (A'_i, B'_i)$ have been adjoined. By Lemmas 4.2 and 4.4, for $(s = 1$ and $t \in \{3, 4\})$ or $(s = 2$ and $t \geq 1)$, there is no $i \geq 2$ such that $(A'_i, B'_i) = (A_n - A_m, B_n - B_m)$ or $(A_n - B_m, B_n - A_m)$ for any $n > m \geq 0$. In other words, adding $\bigcup_{i \geq 2} (A'_i, B'_i)$ to $\Gamma_1$ makes the set of its $P$-positions $P$ unaltered. Thus $\Gamma_1$ and $\Gamma_2$ have the same set of $P$-positions.

**Theorem 4.6.** For $s = 1$ and $t \geq 4$, $\Gamma_1$ and $\Gamma_3$ are equivalent.

**Proof.** $\Gamma_3$ can be viewed as $\Gamma_1$ to which infinitely many of its nonzero $P$-positions are adjoined, but except one, $(A_{3+(t+\delta_1)}', B_{3+(t+\delta_1)}')$. By Lemmas 4.2 and 4.4, for $s = 1$ and $t \geq 4$, there is only one $j = 3 + (t + \delta_1)/2$ such that $(A'_j, B'_j) = (A_n - A_m, B_n - B_m)$ or $(A_n - B_m, B_n - A_m)$ for some $n > m \geq 0$. Thus those additional moves to $\Gamma_1$ do not change the set of its $P$-positions $P$, that is, $\Gamma_1$ and $\Gamma_3$ have the same set of $P$-positions.

**Theorem 4.7.** For $s \geq 3$ and $t + \delta_1 \geq 2s - 2$, $\Gamma_1$ and $\Gamma_4$ are equivalent.

**Proof.** Similarly to $\Gamma_3$, $\Gamma_4$ is obtained from $\Gamma_1$ by adjoining to it infinitely many of its nonzero $P$-positions, but except $(A'_{2+(t+\delta_1)/2}, B'_{2+(t+\delta_1)/2})$. Again by Lemmas 4.2 and 4.4, for $s \geq 3$ and $t + \delta_1 \geq 2s - 2$, there is only one $j = 2 + (t + \delta_1)/2$ such that $(A'_j, B'_j) = (A_n - A_m, B_n - B_m)$ or $(A_n - B_m, B_n - A_m)$ for some $n > m \geq 0$. Thus the additional moves to $\Gamma_1$ do not change the set of its $P$-positions $P$. Therefore, $\Gamma_1$ and $\Gamma_4$ are equivalent.

5 Conclusions

Both exponential and polynomial time winning strategies for $\Gamma_1$ are obtained when $2s + t + \delta_1 > 4$, and under certain conditions, $\Gamma_2$, $\Gamma_3$, $\Gamma_4$ have the same winning strategy with $\Gamma_1$. However, the special case $s = 1$ and $t \in \{1, 2\}$ is not covered, in which the $P$-positions are actually too irregular to be described explicitly. Similarly, the special case $s = t = 1$ is not involved in [12], nor the special case $a = 1$ is covered in [5].

We mentioned a modular type game in [10], where a player may have to remove a multiple of $K$ ($K$ is a fixed positive integer) tokens in each move, and the move rules are the same as in $(s, t)$-Wythoff’s game. Thus the case $K = 1$ is exactly $(s, t)$-Wythoff’s game, and the case $K = 2$ is our Restricted $(s, t)$-Wythoff. Another question of interest is to consider what the results might be if we exploit the idea “adding $P$-positions as moves” to examine this modular type game, which is worth further studying and it is in our agenda.

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