Dynamics for a discrete competition and cooperation model of two enterprises with multiple delays and feedback controls

Lin Lu*, Yi Lian, and Chaoling Li

Abstract: This paper is concerned with a competition and cooperation model of two enterprises with multiple delays and feedback controls. With the aid of the difference inequality theory, we have obtained some sufficient conditions which guarantee the permanence of the model. Under a suitable condition, we prove that the system has global stable periodic solution. The paper ends with brief conclusions.

Keywords: Competition and cooperation model, Permanence, Feedback control, Delay, Enterprise

MSC: 34K20, 34C25, 92D25

1 Introduction

It is known that the coexistence of species has become one of interesting subjects in mathematical ecology. In the past few decades, permanence dynamics of species have received great attention and have been investigated in a number of notable work. For example, Wang and Huang [1] analyzed permanence of a predator-prey model with harvesting predator. Mukherjee [2] addressed the permanence and global attractivity for facultative mutualism predator-prey model, Zhao and Jiang [3] considered the permanence and extinction for Lotka-Volterra model, Teng et al. [4] established the permanence criteria for a delayed discrete species systems, Liu et al. [5] studied the permanence and periodic solutions for reaction-diffusion food-chain system with impulsive effect. For more detailed research about this topic, one can see [6–23]. In real life, the co-existence and stability of enterprise clusters has become one of the most prevalent phenomena in our society. Thus it is important for us to study the permanence and global attractivity of enterprise clusters. However, there are few papers that consider this topic. We think that this study on the dynamics of enterprise clusters has wide application in economic performance and so on.

In 2006, Tian and Nie [24] investigated the following competition and cooperation model of two enterprises

\[
\begin{align*}
\frac{dx_1(t)}{dt} &= r_1(t)x_1(t) \left(1 - \frac{x_1(t)}{K} - \frac{a(x_2(t) - c_2)^2}{K}\right), \\
\frac{dx_2(t)}{dt} &= r_2(t)x_2(t) \left(1 - \frac{x_2(t)}{K} - \frac{\beta(x_1(t) - c_1)^2}{K}\right),
\end{align*}
\]

(1)

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where $x_1(t), x_2(t)$ represent the output of enterprises $A$ and $B$, $r_1, r_2$ are the intrinsic growth rate, $K$ denotes the carrying capacity of marks under nature unlimited conditions, $\alpha, \beta$ are the competitive parameters of two enterprises, $c_1, c_2$ are the initial production of two enterprises. Letting $a_1 = \frac{r_1}{K}, a_2 = \frac{r_2}{K}, b_1 = \frac{r_1a}{K}, b_2 = \frac{r_2a}{K}$, then system (1) becomes

$$\begin{cases}
\frac{dx_1(t)}{dt} = x_1(t)[r_1 - a_1x_1(t) - b_1(x_2(t) - c_2)^2], \\
\frac{dx_2(t)}{dt} = x_2(t)[r_2 - a_2x_2(t) + b_2(x_1(t) - c_1)^2].
\end{cases} \quad (2)$$

Considering the effect of time delay, Liao et al. [25] modified system (2) as follows:

$$\begin{cases}
\frac{dx_1(t)}{dt} = x_1(t)[r_1 - a_1x_1(t - \tau_1) - b_1(x_2(t - \tau_2) - c_2)^2], \\
\frac{dx_2(t)}{dt} = x_2(t)[r_2 - a_2x_2(t - \tau_1) + b_2(x_1(t - \tau_2) - c_1)^2].
\end{cases} \quad (3)$$

where $\tau_1$ is nonnegative constant which stands for the gestation periodic of production for two enterprises, $\tau_2$ in the first equation of the system (3) stands for the block delay of enterprise $B$ to $A$, and $\tau_2$ in the second equation of the system (3) stands for the promoting delay of enterprise $A$ to $B$. By regarding the two delays $\tau_1$ and $\tau_2$ as bifurcation parameters, Liao et al. [25] discussed the effect of different delays on the dynamical behavior of system (3). If $\tau_1 = \tau_2 = \tau$, then system (3) becomes

$$\begin{cases}
\frac{dx_1(t)}{dt} = x_1(t)[r_1 - a_1x_1(t - \tau) - b_1(x_2(t - \tau) - c_2)^2], \\
\frac{dx_2(t)}{dt} = x_2(t)[r_2 - a_2x_2(t - \tau) + b_2(x_1(t - \tau) - c_1)^2].
\end{cases} \quad (4)$$

By choosing the time delay $\tau$ as bifurcation parameter, Liao et al. [26] focused on the stability and Hopf bifurcation properties of system (4).

Li and Zhang [27] focused on (2) with nonconstant coefficients, which takes the following from:

$$\begin{cases}
\frac{dx_1(t)}{dt} = x_1(t)[r_1(t) - a_1(t)x_1(t) - b_1(t)(x_2(t) - c_2)^2], \\
\frac{dx_2(t)}{dt} = x_2(t)[r_2(t) - a_2(t)x_2(t) + b_2(t)(x_1(t) - c_1)^2].
\end{cases} \quad (5)$$

Using the continuation theorem of coincidence degree theory and differential inequality theory, Xu [28] established some sufficient criteria to guarantee the existence of periodic solutions of (5).

Considering that the change of environment, the output of enterprises $A$ and $B$ usually change rapidly, Xu and Shao [29] considered the existence and global attractivity of periodic solution for the following enterprise clusters model with impulse and varying coefficients

$$\begin{cases}
\frac{dx_1(t)}{dt} = x_1(t)[r_1(t) - a_1(t)x_1(t) - b_1(t)(x_2(t) - c_2)^2], t \neq t_k, \\
\frac{dx_2(t)}{dt} = x_2(t)[r_2(t) - a_2(t)x_2(t) + b_2(t)(x_1(t) - c_1)^2], t \neq t_k, \\
\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-) = -\gamma_i(t_k), i = 1, 2, k = 1, 2, \ldots, q.
\end{cases} \quad (6)$$

where $\Delta_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$ are the impulses at moments $t_k$ and $t_1 < t_2 < \cdots$ is a strictly increasing sequence such that $\lim_{k \to +\infty} t_k = +\infty$ and $q$ is a positive integer. Applying coincidence degree theory, Xu and Shao [29] obtained a set of sufficient criteria to ensure the existence of at least a positive periodic solution, and by constructing a Lyapunov functional, they established a sufficient condition to guarantee the uniqueness and global attractivity of the positive periodic solution for (6).

A lot of researchers [30–41] think that discrete systems are far better in depicting the dynamical behavior than continuous ones. In addition, discrete systems also play an important role in computer simulations for continuous systems. Considering the unpredictable forces, we know that coefficients of competition and cooperation systems of
It is not difficult to see that solutions of (7) and (8) are well defined for all $n$ is the control variable.

\[ x_1(n + 1) = x_1(n) \exp \left\{ r_1(n) - a_1(n)x_1(n - \tau_1(n)) \right\} - b_1(n)(x_2(n - \tau_2(n)) - c_2(n))^2 - \beta_1(n)u_1(n), \]
\[ x_2(n + 1) = x_2(n) \exp \left\{ r_2(n) - a_2(n)x_2(n - \tau_1(n)) \right\} + b_2(n)(x_1(n - \tau_2(n)) - c_1(n))^2 - \beta_2(n)u_2(n), \]
\[ \Delta u_1(n) = -\gamma_1(n)u_1(n) + \eta_1(n)x_1(n), \]
\[ \Delta u_2(n) = -\gamma_2(n)u_2(n) + \eta_2(n)x_2(n), \]

where $x_1(n)$ and $x_2(n)$ denote the output of enterprises $A$ and $B$ at the generation, respectively, and $u_i(n) (i = 1, 2)$ is the control variable. $r_1(n), r_2(n), a_1(n), a_2(n), b_1(n), b_2(n), c_1(n), c_2(n), \tau_1(n)$ and $\tau_2(n)$ are bounded nonnegative sequences. To the authors' knowledge, it is the first time one deals with system (7) with feedback control. For more related work, one can see [42–44].

We assume that

\[(H_1) 0 < r_i^l \leq r_i^u, 0 < a_i^l \leq a_i^u, 0 < b_i^l \leq b_i^u, 0 < c_i^l \leq c_i^u, 0 < \beta_i^l \leq \beta_i^u (i = 1, 2).\]

Here, for any bounded sequence $\{f(n)\}$, $f^u = \sup_{n \in \mathbb{N}} \{f(n)\}$ and $f^l = \inf_{n \in \mathbb{N}} \{f(n)\}$.

Let $\tau = \sup_{n \in \mathbb{Z}} \{\tau_i(n)\}$, $\underline{\tau} = \inf_{n \in \mathbb{Z}} \{\tau_i(n)\}, i = 1, 2$. The initial conditions of (7) are

\[ x_i(\theta) = \varphi_i(\theta) \geq 0, \theta \in \mathbb{N}[\tau, 0] = \{\tau, -\tau + 1, 2, \cdots, 0\}, \varphi_i(0) > 0. \]  

(8)

It is not difficult to see that solutions of (7) and (8) are well defined for all $n \geq 0$ and satisfy

\[ x_i(n) > 0, \text{ for } n \in \mathbb{Z}, i = 1, 2. \]

The remainder of the article is organized as follows: in Section 2, some definitions and lemmas are presented and the permanence of (7) is considered. In Section 3, the existence and stability of a unique globally attractive positive periodic solution of the model are investigated. In Section 4, two examples are given to illustrate correctness of our obtained analytical results in Section 2 and Section 3. Brief conclusions are drawn in Section 5.

## 2 Permanence

In this section, we list several definitions and lemmas.

**Definition 2.1.** System (7) is permanent if there are positive constants $M$ and $m$ such that for each positive solution $(x_1(n), x_2(n), u_1(n), u_2(n))$ of system (7) satisfies

\[ m \leq \lim_{n \to +\infty} \inf_{n \to +\infty} x_i(n) \leq \lim_{n \to +\infty} \sup_{n \to +\infty} x_i(n) \leq M(i = 1, 2). \]
\[ m \leq \lim_{n \to +\infty} \inf_{n \to +\infty} u_i(n) \leq \lim_{n \to +\infty} \sup_{n \to +\infty} u_i(n) \leq M(i = 1, 2). \]

Let us consider the following model:

\[ N(n + 1) = N(n) \exp (a(n) - b(n)N(n)), \]  

where $\{a(n)\}$ and $\{b(n)\}$ are strictly positive sequences of real numbers defined for $n \in \mathbb{N} = \{0, 1, 2, \cdots\}$ and $0 < a_l \leq a_u, 0 < b_l \leq b_u$. Similarly to the proofs of Propositions 1 and 3 in [41], we can obtain the following Lemma 2.2.
Lemma 2.2. Any solution of system (9) with initial condition $N(0) > 0$ satisfies
$$m \leq \lim_{n \to +\infty} \inf N(n) \leq \lim_{n \to +\infty} \sup N(n) \leq M,$$
where
$$M = \frac{1}{b^l} \exp(a^u - 1), m = \frac{a^l}{b^l} \exp(a^l - b^u M).$$

Let consider the following equation
$$y(n + 1) = Ay(n) + B, n = 1, 2, \cdots, \tag{10}$$
where $A$ and $B$ are positive constants. Following Theorem 6.2 of Wang and Wang [45, page 125], we have the following Lemma 2.3.

**Lemma 2.3** ([45]). If $|A| < 1$, then for any initial value $y(0)$, there exists a unique solution $y(n)$ of (10) which can be expressed as follows:
$$y(n) = A^n (y(0) - y^*) + y^*,$$
where $y^* = \frac{B}{1 - A}$. Thus, for any solution $\{y(n)\}$ of system (10), $\lim_{n \to +\infty} y(n) = y^*$.

**Lemma 2.4** ([45]). Let $n \in N_0^+ = \{n_0, n_0 + 1, \cdots, n_0 + l, \cdots\}, r \geq 0$. For any fixed $n$, $g(n, r)$ is a nondecreasing function with respect to $r$, and for $n \geq n_0$, then
$$y(n + 1) \leq g(n, y(n)), u(n + 1) \geq g(n, u(n)).$$
If $y(n_0) \leq u(n_0)$, then $y(n) \leq u(n)$ for all $n \geq n_0$.

**Proposition 2.5.** Let $\varepsilon > 0$ be any constant. If (H1) holds, then
$$\lim_{n \to +\infty} \sup x_i(n) \leq M_i, \lim_{n \to +\infty} \sup u_i(n) \leq U_i, i = 1, 2,$$
where
$$M_1 = \frac{1}{r_1^u} \exp(r_1^u (\tau + 1) - 1), U_i = \frac{\eta_i^u M_i}{\gamma_i^u} (i = 1, 2),$$
$$M_2 = \frac{1}{d_2} \exp[(r_2^u + b_2^u (M_1 + \varepsilon + c_1^u)^2)(\tau + 1) - 1].$$

**Proof.** Let $(x_1(n), x_2(n), u_1(n), u_2(n))$ be any positive solution of system (7) with the initial condition $(x_1(0), x_2(0), u_1(0), u_2(0))$. It follows from the first equation of system (7) that
$$x_1(n + 1) \leq x_1(n) \exp \{r_1(n)\}. \tag{11}$$
Let $x_1(n) = \exp \{y_1(n)\}$, then (11) is equivalent to
$$y_1(n + 1) - y_1(n) \leq r_1(n). \tag{12}$$
Summing both sides of (12) from $n - \tau_1(n)$ to $n - 1$, we have
$$\sum_{j = n - \tau_1(n)}^{n-1} (y_1(j + 1) - y_1(j)) \leq \sum_{j = n - \tau_1(n)}^{n-1} r_1(j) \leq r_1^u \tau,$$
which leads to
$$y_1(n - \tau_1(n)) \geq y_1(n) - r_1^u \tau. \tag{13}$$
Then
$$x_1(n - \tau_1(n)) \geq x_1(n) \exp \{-r_1^u \tau\}. \tag{14}$$
Substituting (14) into the first equation of system (7) gives
\[ x_1(n + 1) \leq x_1(n) \exp \left\{ r_1(n) - r_1(n) \exp \{-r_1^u \tau \} x_1(n) \right\}. \tag{15} \]

It follows from (15) and Lemma 2.2 that
\[ \lim_{n \to +\infty} \sup x_1(n) \leq \frac{1}{r_1^u} \exp \{r_1^u (\tau + 1) - 1\} := M_1. \tag{16} \]

For any positive constant \( \varepsilon > 0 \), it follows (16) that there exists a \( N_1 > 0 \) such that for all \( n > N_1 + \tau \)
\[ x_1(n) \leq M_1 + \varepsilon. \tag{17} \]

For any positive constant \( \varepsilon > 0 \) and for all \( n > N_1 + \tau \), by the second equation of system (7), we get
\[ x_2(n + 1) \leq x_2(n) \exp \left\{ r_2(n) + b_2(n)(M_1 + \varepsilon + c_1(n))^2 \right\}. \tag{18} \]

Let \( x_2(n) = \exp \{y_2(n)\} \), then (18) is equivalent to
\[ x_2(n + 1) - x_2(n) \leq r_2(n) + b_2(n)(M_1 + \varepsilon + c_1(n))^2. \tag{19} \]

Summing both sides of (19) from \( n - \tau_1(n) \) to \( n - 1 \), we have
\[ \sum_{j=n-\tau_1(n)}^{n-1} (x_2(j + 1) - x_2(j)) \leq \sum_{j=n-\tau_1(n)}^{n-1} \left[ r_2(n) + b_2(n)(M_1 + \varepsilon + c_1(n))^2 \right] \]
\[ \leq [r_2^u + b_2^u(M_1 + \varepsilon + c_1^n)^2] \tau, \]
which leads to
\[ x_2(n - \tau_1(n)) \geq x_2(n) - [r_2^u + b_2^u(M_1 + \varepsilon + c_1^n)^2] \tau. \tag{20} \]

Then
\[ x_2(n - \tau_1(n)) \geq x_2(n) \exp \{-[r_2^u + b_2^u(M_1 + \varepsilon + c_1^n)^2] \tau\}. \tag{21} \]

By substituting (21) into the second equation of system (7) we have
\[ x_2(n + 1) \leq x_2(n) \exp \left\{ -[r_2^u + b_2^u(M_1 + \varepsilon + c_1^n)^2] \tau \right\} x_2(n). \tag{22} \]

It follows from (22) and Lemma 2.2 that
\[ \lim_{n \to +\infty} \sup x_2(n) \leq \frac{1}{a_2^u} \exp \{[r_2^u + b_2^u(M_1 + \varepsilon + c_1^n)^2](\tau + 1) - 1\} := M_2. \tag{23} \]

For any positive constant \( \varepsilon > 0 \), it follows (23) that there exists a \( N_2 > N_1 + \tau \) such that for all \( n > N_2 + \tau \)
\[ x_2(n) \leq M_2 + \varepsilon. \tag{24} \]

In view of the third and fourth equations of the system (7), we can obtain
\[ \Delta u_1(n) \leq -\gamma_1(n)u_1(n) + \eta_1(n)(M_1 + \varepsilon). \tag{25} \]

\[ \Delta u_2(n) \leq -\gamma_2(n)u_2(n) + \eta_2(n)(M_2 + \varepsilon). \tag{26} \]

Then
\[ u_1(n + 1) \leq (1 - \gamma_1^u)u_1(n) + \eta_1^u(M_1 + \varepsilon). \tag{27} \]

\[ u_2(n + 1) \leq (1 - \gamma_2^u)u_2(n) + \eta_2^u(M_2 + \varepsilon). \tag{28} \]
Applying Lemmas 2.2 and 2.4, we immediately get
\[
\lim_{n \to +\infty} \sup u_1(n) \leq \frac{\eta_1^u(M_1 + \varepsilon)}{\gamma_1^l}, \\
\lim_{n \to +\infty} \sup u_2(n) \leq \frac{\eta_2^u(M_2 + \varepsilon)}{\gamma_2^l}.
\]
(29) (30)

By setting \( \varepsilon \to 0 \), we have
\[
\lim_{n \to +\infty} \sup u_1(n) \leq \frac{\eta_1^u M_1}{\gamma_1^l} := U_1, \\
\lim_{n \to +\infty} \sup u_2(n) \leq \frac{\eta_2^u M_2}{\gamma_2^l} := U_2.
\]
(31) (32)

This completes the proof of Proposition 2.5. \( \square \)

**Theorem 2.6.** Let \( M_1, M_2, U_1 \) and \( U_2 \) be defined by (16), (23), (31) and (32), respectively. Assume that (H1) and (H2) \( r_1^1 > b_1^u(M_2 + c_2^M)^2 + \beta_1 U_1, r_1^2 > \beta_2 U_2 \) hold, then system (7) is permanent.

**Proof.** By applying Proposition 2.5, we easily see that to end the proof of Theorem 2.6, it is enough to show that under the conditions of Theorem 2.6,
\[
\lim_{n \to +\infty} \inf x_1(n) \geq m_1, \quad \lim_{n \to +\infty} \inf x_2(n) \geq m_2, \\
\lim_{n \to +\infty} \inf u_1(n) \geq v_1, \quad \lim_{n \to +\infty} \inf u_2(n) \geq v_2.
\]

In view of Proposition 2.5, for all \( \varepsilon > 0 \), there exists a \( N_3 > 0, N_3 \in \mathbb{N} \), for all \( n > N_3 \),
\[
x_1(n) \leq M_1 + \varepsilon, x_2(n) \leq M_2 + \varepsilon, u_1(n) \leq U_1 + \varepsilon, u_2(n) \leq U_2 + \varepsilon.
\]
(33)

It follows from the first equation of system (7) and (33) that
\[
x_1(n + 1) \geq x_1(n) \exp \left\{ r_1^1 - a_1^u(M_1 + \varepsilon) - b_1^u(M_2 + \varepsilon + c_2^M)^2 - \beta_1^u(U_1 + \varepsilon) \right\},
\]
for all \( n > N_3 + \tau \).

Let \( x_1(n) = \exp\{y_1(n)\} \), then (34) is equivalent to
\[
y_1(n + 1) - y_1(n) \geq r_1^1 - a_1^u(M_1 + \varepsilon) - b_1^u(M_2 + \varepsilon + c_2^M)^2 - \beta_1^u(U_1 + \varepsilon).
\]
(34)

Summing both sides of (34) from \( n - \tau_1(n) \) to \( n - 1 \) leads to
\[
\sum_{j=n-\tau_1(n)}^{n-1} (y_1(j + 1) - y_1(j)) \geq \sum_{j=n-\tau_1(n)}^{n-1} \left[ r_1^1 - a_1^u(M_1 + \varepsilon) - b_1^u(M_2 + \varepsilon + c_2^M)^2 - \beta_1^u(U_1 + \varepsilon) \right]
\]
\[
\geq \left[ r_1^1 - a_1^u(M_1 + \varepsilon) - b_1^u(M_2 + \varepsilon + c_2^M)^2 - \beta_1^u(U_1 + \varepsilon) \right] \tilde{c}.
\]
Then
\[
y_1(n - \tau_1(n)) \leq y_1(n) - \left[ r_1^1 - a_1^u(M_1 + \varepsilon) - b_1^u(M_2 + \varepsilon + c_2^M)^2 - \beta_1^u(U_1 + \varepsilon) \right] \tilde{c}.
\]
Thus
\[
x_1(n - \tau_1(n)) \leq x_1(n) \exp \left\{ - \left[ r_1^1 - a_1^u(M_1 + \varepsilon) - b_1^u(M_2 + \varepsilon + c_2^M)^2 - \beta_1^u(U_1 + \varepsilon) \right] \tilde{c} \right\}.
\]
(35)

Substituting (33) and (35) into the first equation of (7), we have
\[
x_1(n + 1) \geq x_1(n) \exp \left\{ r_1^1 - b_1^u(M_2 + \varepsilon + c_2^M)^2 - \beta_1^u(U_1 + \varepsilon) - a_1^u x_1(n) \right\}
\]
for all \( n > N_3 + \tau \).

By applying Lemmas 2.2 and 2.4, we immediately obtain

\[
\lim_{n \to +\infty} \inf x_1(n) \geq m_1^e, \tag{37}
\]

where

\[
m_1^e = \frac{r^1_1 - b^1_u(M_2 + \varepsilon + c^u_2)^2 - \beta^1_u(U_1 + \varepsilon)}{a^u Q_1 \exp \left\{ - \left[ r^1_1 - a^u_1(M_1 + \varepsilon) - b^1_u(M_2 + \varepsilon + c^u_2)^2 - \beta^1_u(U_1 + \varepsilon) \right] \bar{\tau} \right\}} \times \exp \left\{ r^1_1 - b^1_u(M_2 + \varepsilon + c^u_2)^2 - \beta^1_u(U_1 + \varepsilon) - a^u_1 \exp \left\{ - \left( r^1_1 - a^u_1(M_1 + \varepsilon) \right) \right\} \right\}.
\]

Setting \( \varepsilon \to 0 \) in (37), then

\[
\lim_{n \to +\infty} \inf x_1(n) \geq m_1, \tag{38}
\]

where

\[
m_1 = \frac{r^1_1 - b^1_u(M_2 + c^u_2)^2 - \beta^1_u U_1}{a^u Q_1 \exp \left\{ - \left[ r^1_1 - a^u_1 M_1 - b^1_u(M_2 + c^u_2)^2 - \beta^1_u U_1 \right] \bar{\tau} \right\}} \times \exp \left\{ r^1_1 - b^1_u(M_2 + c^u_2)^2 - \beta^1_u U_1 - a^u_1 \exp \left\{ - \left( r^1_1 - a^u_1 M_1 \right) \right\} \right\}.
\]

By the second equation of system (7) and (33), we can obtain

\[
x_2(n + 1) \geq x_2(n) \exp \left\{ r^2_2(n) - a^u_2(M_2 + \varepsilon) - \beta^u_2(U_2 + \varepsilon) \right\}
\]

\[
\geq x_2(n) \exp \left\{ r^2_2 - a^u_2(M_2 + \varepsilon) - \beta^u_2(U_2 + \varepsilon) \right\} \tag{39}
\]

for all \( n > N_3 + \tau \).

Let \( x_2(n) = \exp\{y_2(n)\} \), then (39) is equivalent to

\[
y_2(n + 1) - y_2(n) \geq r^2_2 - a^u_2(M_2 + \varepsilon) - \beta^u_2(U_2 + \varepsilon). \tag{40}
\]

Summing both sides of (40) from \( n = \tau_1(n) \) to \( n - 1 \) leads to

\[
\sum_{j=\tau_1(n)}^{n-1} (y_2(n + 1) - y_2(n)) \geq \sum_{j=\tau_1(n)}^{n-1} \left[ r^2_2 - a^u_2(M_2 + \varepsilon) - \beta^u_2(U_2 + \varepsilon) \right] \bar{\tau}.
\]

Then

\[
y_2(n - \tau_1(n)) \leq y_2(n) - \left[ r^2_2 - a^u_2(M_2 + \varepsilon) - \beta^u_2(U_2 + \varepsilon) \right] \bar{\tau}.
\]

Thus

\[
x_2(n - \tau_1(n)) \leq x_2(n) \exp \left\{ - \left[ r^2_2 - a^u_2(M_2 + \varepsilon) - \beta^u_2(U_2 + \varepsilon) \right] \bar{\tau} \right\}, \tag{41}
\]

Substituting (33) and (41) into the second equation of (7), we have

\[
x_2(n + 1) \geq x_2(n) \exp \left\{ r^2_2 - \beta^u_2(U_2 + \varepsilon) - a^u_2 x_2(n) \right\}
\]
\[
\times \exp \left\{ - \left[ r_2^l - a_2^u (M_2 + \varepsilon) - \beta_2^u (U_2 + \varepsilon) \right] \right\} \right\}
\]

for all \( n > N_3 + \tau \).

By applying Lemmas 2.2 and 2.4, we immediately get

\[
\lim_{n \to +\infty} \inf x_2(n) \geq m_2^\varepsilon,
\]

where

\[
\begin{align*}
m_2^\varepsilon &= \frac{r_2^l - \beta_2^u (U_2 + \varepsilon)}{a_2^u \exp \left\{ - \left[ r_2^l - a_2^u (M_2 + \varepsilon) - \beta_2^u (U_2 + \varepsilon) \right] \right\}} \\
&\times \exp \left\{ r_2^l - \beta_2^u U_2 - a_2^u \exp \left\{ - \left[ r_2^l - a_2^u (M_2 + \varepsilon) - \beta_2^u (U_2 + \varepsilon) \right] \right\} M_2 \right\}.
\end{align*}
\]

Setting \( \varepsilon \to 0 \) in (43), then

\[
\lim_{n \to +\infty} \inf x_2(n) \geq m_2,
\]

where

\[
\begin{align*}
m_2 &= \frac{r_2^l - \beta_2^u U_2}{a_2^u \exp \left\{ - \left[ r_2^l - a_2^u M_2 - \beta_2^u U_2 \right] \right\}} \\
&\times \exp \left\{ r_2^l - \beta_2^u U_2 - a_2^u \exp \left\{ - \left[ r_2^l - a_2^u M_2 - \beta_2^u U_2 \right] \right\} M_2 \right\}.
\end{align*}
\]

Without the loss of generality, we assume that \( \varepsilon < \frac{1}{2} \min\{m_1, m_2\} \). For any positive constant \( \varepsilon \) small enough, it follows from (38) and (44) that there exists large enough \( N_4 > N_3 + \tau \) such that

\[
x_1(n) \geq m_1 - \varepsilon, x_2(n) \geq m_2 - \varepsilon
\]

for any \( n \geq N_4 \).

From the third and fourth equations of system (7) and (45), we can derive that

\[
\Delta u_1(n) \geq -\gamma_1(n) u_1(n) + \eta_1(n)(m_1 - \varepsilon),
\]

(46)

\[
\Delta u_2(n) \geq -\gamma_2(n) u_2(n) + \eta_2(n)(m_2 - \varepsilon),
\]

(47)

Hence

\[
u_1(n + 1) \geq (1 - \gamma_1^u) u_1(n) + \eta_1^l (m_1 - \varepsilon),
\]

(48)

\[
u_2(n + 1) \geq (1 - \gamma_2^u) u_2(n) + \eta_2^l (m_2 - \varepsilon).
\]

(49)

By applying Lemmas 2.2 and 2.3, we immediately get

\[
\lim_{n \to +\infty} \inf u_1(n) \geq \frac{\eta_1^l (m_1 - \varepsilon)}{\gamma_1^u},
\]

(50)

\[
\lim_{n \to +\infty} \inf u_2(n) \geq \frac{\eta_2^l (m_2 - \varepsilon)}{\gamma_2^u}.
\]

(51)

Setting \( \varepsilon \to 0 \) in the above inequality leads to

\[
\lim_{n \to +\infty} \inf u_1(n) \geq \frac{\eta_1^l m_1}{\gamma_1^u} := U_1^l,
\]

(52)

\[
\lim_{n \to +\infty} \inf u_2(n) \geq \frac{\eta_2^l m_2}{\gamma_2^u} := U_2^l.
\]

(53)

This completes the proof of Theorem 2.6. \( \square \)
3 Existence and stability of periodic solution

In this section, we will study the stability of (7) under the assumption \( \tau_i(n) = 0 \; (i = 1, 2) \), namely, we consider the following system

\[
\begin{align*}
    x_1(n + 1) &= x_1(n) \exp \left\{ r_1(n) - a_1(n)x_1(n) \right\} - b_1(n)(x_2(n) - c_2(n))^2 - \beta_1(n)u_1(n) \\
    x_2(n + 1) &= x_2(n) \exp \left\{ r_2(n) - a_2(n)x_2(n) \right\} + b_2(n)(x_1(n) - c_1(n))^2 - \beta_2(n)u_2(n),
\end{align*}
\]

(54)

Throughout this section we always assume that \( r_i(n), a_i(n), b_i(n), c_i(n), \gamma_i(n) \) and \( \eta_1(n) \) are all bounded nonnegative periodic sequences with a common period \( \omega \) and satisfy

\[0 < \gamma_i(n) < 1, \; n \in N \cap [0, \omega], \; i = 1, 2.\] (55)

Also it is assumed that the initial conditions of (54) are of the form

\[x_i(0) > 0, \; u_i(0) > 0, \; i = 1, 2.\] (56)

In similar way, we can derive the permanence of (54). We still let \( M_i \) and \( U_i \) be the upper bound of \( \{x_i(n)\} \) and \( \{u_i(n)\} \), and \( m_i \) and \( U_i^0 \) be the lower bound of \( \{x_i(n)\} \) and \( \{u_i(n)\} \).

**Theorem 3.1.** In addition to (55), assume that (H1) and (H2) \( r_1^0 > b_1^0(M_2 + c_2^0)^2 + \beta_1(U_1), \; r_2^0 > \beta_2^0(U_2) \) hold, then system (54) has a periodic \( \omega \) solution denoted by \( \{\hat{x}_1(n), \hat{x}_2(n), \hat{u}_1(n), \hat{u}_2(n)\} \).

**Proof.** Let \( \Omega = \{(x_1, x_2, u_1, u_2) | m_i \leq x_i \leq M_i, \; U_i^0 \leq u_i \leq U_i, \; i = 1, 2\} \). It is easy to see that \( \Omega \) is an invariant set of system (54). Then we can define a mapping \( F \) on \( \Omega \) by

\[
F(x_1(0), x_2(0), u_1(0), u_2(0)) = (x_1(\omega), x_2(\omega), u_1(\omega), u_2(\omega))
\]

(57)

for \((x_1(0), x_2(0), u_1(0), u_2(0)) \in \Omega \). Obviously, \( F \) depends continuously on \((x_1(0), x_2(0), u_1(0), u_2(0)) \). Thus \( F \) is continuous and maps a compact set \( \Omega \) into itself. Therefore, \( F \) has a fixed point \((\hat{x}_1(n), \hat{x}_2(n), \hat{u}_1(n), \hat{u}_2(n)) \). So we can conclude that the solution \((\tilde{x}_1(n), \tilde{x}_2(n), \tilde{u}_1(n), \tilde{u}_2(n)) \) passing through \((\hat{x}_1, \hat{x}_2, \hat{u}_1, \hat{u}_2) \) is a periodic solution of system (54). The proof of Theorem 3.1 is complete.

Next, we investigate the global stability property of the periodic solution obtained in Theorem 3.1.

**Theorem 3.2.** In addition to the conditions of Theorem 3.1, assume that the following condition (H3) hold,

\[
\begin{align*}
    \chi_1 &= \max \left\{1 - a_1^0m_1, |1 - a_1^0M_1| + 2b_1^0c_2^0[M_2^2 + M_2] + \beta_1^0 < 1 \right\} \\
    \chi_2 &= \max \left\{1 - a_2^0m_2, |1 - a_2^0M_2| + 2b_2^0c_1^0[M_1^2 + M_1] + \beta_2^0 < 1 \right\} \\
    \chi_3 &= (1 - \gamma_1^0) + \eta_1^0M_1 < 1, \\
    \chi_4 &= (1 - \gamma_2^0) + \eta_2^0M_2 < 1.
\end{align*}
\]

(H3)

then the \( \omega \) periodic solution \((\hat{x}_1(n), \hat{x}_2(n), \hat{u}_1(n), \hat{u}_2(n)) \) obtained in Theorem 3.1 is globally attractive.

**Proof.** Assume that \((x_1(n), x_2(n), u_1(n), u_2(n)) \) is any positive solution of system (54). Let

\[x_i(n) = \hat{x}_i(n) \exp \{\gamma_i(n)\}, \; u_i(n) = \hat{u}_i(n) + v_i(n), \; i = 1, 2.\] (58)
To complete the proof, it suffices to show
\[ \lim_{n \to \infty} y_i(n) = 0, \quad \lim_{n \to \infty} v_i(n) = 0, \quad i = 1, 2. \tag{59} \]

Since
\[
y_1(n + 1) = y_1(n) - a_1(n)\bar{x}_1(n)[\exp(y_1(n)) - 1] \\
- b_1(n)[(c_2(n) - c_2(n))^2 - (\bar{x}_2(n) - c_2(n))^2] - \beta_1(n)v_1(n) \\
= y_1(n) - a_1(n)\bar{x}_1(n)\exp(\theta_1(n)y_1(n))y_1(n) \\
- b_1(n)\bar{x}_2(n)\exp(2\theta_2(n)y_2(n))y_2(n) \\
+ 2b_1(n)c_2(n)\bar{x}_2(n)\exp(\theta_3(n)y_2(n))y_2(n) - \beta_1(n)v_1(n), \tag{60}
\]

where \( \theta_i(n) \in (0, 1), i = 1, 2, 3 \). In a similar way, we get
\[
y_2(n + 1) = y_2(n) - a_2(n)\bar{x}_2(n)\exp(\theta_4(n)y_2(n))y_2(n) \\
+ b_2(n)\bar{x}_2(n)\exp(2\theta_5(n)y_1(n))y_1(n) \\
+ 2b_2(n)c_1(n)\bar{x}_2(n)\exp(\theta_6(n)y_1(n))y_1(n) - \beta_2(n)v_2(n). \tag{61}
\]

Also, one has
\[
v_1(n + 1) = (1 - y_1(n))v_1(n) + \eta_1(n)\bar{x}_1(n)\exp(\gamma_1(n) - 1) \\
= (1 - y_1(n))v_1(n) + \eta_1(n)\bar{x}_1(n)\exp(\theta_7(n)y_1(n))y_1(n), \tag{62}
\]
\[
v_2(n + 1) = (1 - y_2(n))v_2(n) + \eta_2(n)\bar{x}_2(n)\exp(\gamma_2(n) - 1) \\
= (1 - y_2(n))v_2(n) + \eta_2(n)\bar{x}_2(n)\exp(\theta_8(n)y_2(n))y_2(n). \tag{63}
\]

By (H3), we can choose a \( \epsilon > 0 \) such that
\[
\begin{align*}
\chi_1^c &= \max\{|1 - a_1^c(m_1 - \epsilon)|, |1 - a_1^u(M_1 + \epsilon)|\} + 2b_1^c c_2^u[(M_2 + \epsilon)^2 + M_2 + \epsilon] + \beta_1^u < 1, \\
\chi_2^c &= \max\{|1 - a_2^c(m_2 - \epsilon)|, |1 - a_2^u(M_2 + \epsilon)|\} + 2b_2^c c_1^u[(M_1 + \epsilon)^2 + M_1 + \epsilon] + \beta_2^u < 1, \\
\chi_3^c &= (1 - \gamma_1^c) + \eta_1^u(M_1 + \epsilon) < 1, \\
\chi_4^c &= (1 - \gamma_2^c) + \eta_2^u(M_2 + \epsilon) < 1.
\end{align*}
\tag{64}
\]

In view of Proposition 2.5 and Theorem 2.6, there exists \( N_5 > N_4 \) such that
\[ m_i - \epsilon \leq x_i(n), \bar{x}_i(n) \leq M_i + \epsilon, \text{ for } n \geq N_5, i = 1, 2. \tag{65} \]

It follows from (60) and (61) that
\[
y_1(n + 1) \leq \max\{|1 - a_1^c(m_1 - \epsilon)|, |1 - a_1^u(M_1 + \epsilon)|\}|y_1(n)| \\
+ 2b_1^c c_2^u[(M_2 + \epsilon)^2 + M_2 + \epsilon]|y_2(n)| + \beta_1^u|v_1(n)|, \tag{66}
\]
\[
y_2(n + 1) \leq \max\{|1 - a_2^c(m_2 - \epsilon)|, |1 - a_2^u(M_2 + \epsilon)|\}|y_2(n)| \\
+ 2b_2^c c_1^u[(M_1 + \epsilon)^2 + M_1 + \epsilon]|y_1(n)| + \beta_2^u|v_2(n)|. \tag{67}
\]

Also, for \( n > N_5 \), one has
\[
v_1(n + 1) \leq (1 - \gamma_1^c)|v_1(n)| + \eta_1^u(M_1 + \epsilon)|y_1(n)|, \tag{68}
\]
\[
v_2(n + 1) \leq (1 - \gamma_2^c)|v_2(n)| + \eta_2^u(M_2 + \epsilon)|y_2(n)|. \tag{69}
\]

Let \( \chi = \max\{\chi_1^c, \chi_2^c, \chi_3^c, \chi_4^c\} \), then \( 0 < \chi < 1 \). It follows from (66)-(69) that
\[
\max\{|y_1(n + 1)|, |y_2(n + 1)|, |v_1(n + 1)|, |v_2(n + 1)|\} \leq \chi \max\{|y_1(n)|, |y_2(n)|, |v_1(n)|, |v_2(n)|\}. \tag{70}
\]
for $n > N_5$. Then we get
\[
\max\{|y_1(n)|, |y_2(n)|, |v_1(n)|, |v_2(n)|\} \leq \chi^{n-N_5} \max\{|y_1(N_5)|, |y_2(N_5)|, |v_1(N_5)|, |v_2(N_5)|\}. \tag{71}
\]
Thus
\[
\lim_{n \to \infty} y_i(n) = 0, \quad \lim_{n \to \infty} v_i(n) = 0, \quad i = 1, 2. \tag{72}
\]
This completes the proof.

Remark 3.3. Although Zhi et al. [46] have investigated the permanence and almost periodic solution for an enterprise cluster model based on ecology theory with feedback controls on time scales, they also do not consider the enterprise cluster model with time delays, moreover, they do not investigate the periodic solution of this model. In this paper, we consider the permanence and periodic solutions of two enterprises with multiple delays, which is more general than those models in [46]. Thus our results complement the previous work of [46].

4 Examples

Example 4.1. Consider the following system
\[
\begin{cases}
x_1(n+1) = x_1(n) \exp \left\{ r_1(n) - a_1(n)x_1(n - \tau_1(n)) - b_1(n)(x_2(n - \tau_2(n)) - c_2(n))^2 - \beta_1(n)u_1(n) \right\}, \\
x_2(n+1) = x_2(n) \exp \left\{ r_2(n) - a_2(n)x_2(n - \tau_1(n)) + b_2(n)(x_1(n - \tau_2(n)) - c_1(n))^2 - \beta_2(n)u_2(n) \right\}, \\
\Delta u_1(n) = -\gamma_1(n)u_1(n) + \eta_1(n)x_1(n), \\
\Delta u_2(n) = -\gamma_2(n)u_2(n) + \eta_2(n)x_2(n),
\end{cases}
\]
where $r_1(n) = 0.4 + \sin(n), r_2(n) = 0.5 + \cos(n), a_1(n) = a_2(n) = 1, b_1(n) = 0.4 + \cos(n), b_2(n) = 0.3 + \cos(n), c_1(n) = 0.4 + \cos(n), c_2(n) = 0.4 + \cos(n), \beta_1(n) = 0.2 + \cos(n), \beta_2(n) = 0.4 + \sin(n), \gamma_1(n) = \cos(n) + 0.3, \gamma_2(n) = \sin(n) + 0.4, \eta_1(n) = 0.5 + \cos(n), \eta_2(n) = 0.7 + \cos(n), \tau_1(n) = 1$. We can verify that all the assumptions in Theorem 2.6 are fulfilled. Then (73) is permanent which is shown in Figures 1-2.

Fig. 1. Times series of $x_1$ and $x_2$ for system (73), where the blue line stands for $x_1$ and the red line stands for $x_2$. 
Fig. 2. Times series of $u_1$ and $u_2$ for system (73), where the blue line stands for $u_1$ and the red line stands for $u_2$.

Example 4.2. Consider the following system

$$\begin{cases} x_1(n + 1) = x_1(n) \exp \left\{ r_1(n) - a_1(n)x_1(n) - b_1(n)(x_2(n) - c_2(n))^2 - \beta_1(n)u_1(n) \right\}, \\ x_2(n + 1) = x_2(n) \exp \left\{ r_2(n) - a_2(n)x_2(n) + b_2(n)(x_1(n) - c_1(n))^2 - \beta_2(n)u_2(n) \right\}, \\ \Delta u_1(n) = -\gamma_1(n)u_1(n) + \eta_1(n)x_1(n), \\ \Delta u_2(n) = -\gamma_2(n)u_2(n) + \eta_2(n)x_2(n), \end{cases}$$

where $r_1(n) = 0.4 + \sin(n)$, $r_2(n) = 0.5 + \sin(n)$, $a_1(n) = a_2(n) = 1$, $b_1(n) = 0.4 - \cos(n)$, $b_2(n) = 0.3 - \cos(n)$, $c_1(n) = 0.4 + \sin(n)$, $c_2(n) = 0.3 + \sin(n)$, $\beta_1(n) = 0.3 + \cos(n)$, $\beta_2(n) = 0.2 + \sin(n)$, $\gamma_1(n) = \cos(n) + 0.2$, $\gamma_2(n) = \sin(n) + 0.3$, $\eta_1(n) = 0.2 + \cos(n)$, $\eta_2(n) = 0.2 + \cos(n)$. We can verify that all the assumptions in Theorem 3.1 are fulfilled. Then we know that the periodic solution of system (74) is globally attractive which is illustrated in Figures 3-3.

Fig. 3. Times series of $x_1$ and $x_2$ for system (74), where the blue line stands for $x_1$ and the red line stands for $x_2$. 
5 Conclusions

In the present article, a discrete system with competition and cooperation model of two enterprises is proposed. Applying the difference inequality theory, we have established some sufficient conditions which guarantee the permanence of the system. We find that under some suitable conditions, the competition and cooperation of enterprises cluster can remain balanced. This shows that feedback control effect and time delays play a key role in deciding the survival of enterprises. The sufficient conditions which ensure the existence and stability of unique globally attractive periodic solution of the system without time delays are established. The obtained results are completely new and complement the published works of [25–29, 46].

Acknowledgement: The first and second authors were supported by the Key Research Institute of Philosophies and Social Sciences in Guangxi Universities and Colleges(16YC001, 16YC002). The third author was supported by Key Research Institute of Philosophies and Social Sciences in Guangxi Universities and Colleges(16YC001,16YC002) and Key Project of Science and Technology Research in Guangxi Universities and Colleges(ZD2014058). The authors would like to thank the referees and the editor for helpful suggestions incorporated into this paper.

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