On sequences not enjoying Schur’s property

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Abstract: Here we proved the existence of a closed vector space of sequences - any nonzero element of which does not comply with Schur’s property, that is, it is weakly convergent but not norm convergent. This allows us to find similar algebraic structures in some subsets of functions.

Keywords: Schur property, Weakly convergent sequence, Analytic function

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1 Introduction and preliminary results

In [12], the authors give a characterization for a complex Banach space $E$ having the Schur property in terms of the algebra of holomorphic functions from the unit disk into $E$ which admits a continuous extension to the boundary. In particular, the authors in [12] proved that if a space $E$ does not satisfy the Schur property, then there exists an analytic function $f : \mathbb{D} \rightarrow E$ so that, for every $y^* \in E^*$, $y^* \circ f$ admits a continuous extension to $\partial \mathbb{D}$, but $f$ does not. The authors proved that in fact one can find such a function in the more general setting where the function is defined from $B_X$, where $X$ is a Banach space, and not just when it is defined from $\mathbb{D}$.

One natural question that arises is then how many of such functions exist. Since the beginning of the 21st century a new trend in mathematics began to attract the attention of many mathematicians. This new topic was constructed around concepts such as lineability and spaceability, which we define below:

Definition 1.1. Let $X$ be a topological vector space and $M$ a subset of $X$. Let $\mu$ be a cardinal number.

1. $M$ is said to be $\mu$-lineable ($\mu$-spaceable) if $M \cup \{0\}$ contains a vector space (resp. a closed vector space) of dimension $\mu$. At times, we shall be referring to the set $M$ as simply lineable or spaceable if the existing subspace is infinite dimensional.

2. We also let $\lambda(M)$ be the maximum cardinality (if it exists) of such a vector space.

3. When the above linear space can be chosen to be dense in $X$ we shall say that $M$ is $\mu$-dense-lineable.

The first examples for the above properties dealt with “natural” functions: in [13], V. I. Gurariy studied continuous functions on $[0,1]$ which are analytic on $(0,1)$, and in [17], Levine and Milman focused on continuous functions on $[0,1]$ with bounded variation. After the definitions from 1.1 appeared, mathematicians started to study pathological and anti-intuitive properties of elements of topological vector spaces, and this topic has proved to be greatly fruitful. The interested reader can consult the works in ([3–9, 11, 13–18, 20]) and access an exhaustive overview of many of the results published in this trend in [1]. Let us recall now the main definitions that we will use:

Definition 1.2. A Banach space is Schur, or has the Schur property, if every weakly convergent sequence is norm-convergent.
In fact, a Banach space has the Schur property if and only if for every \( \varepsilon > 0 \), every \( \varepsilon \)--separated bounded sequence (that is, a bounded sequence whose elements are at distance at least \( \varepsilon \) from one another) has a subsequence which is equivalent to the canonical basis of \( \ell_1 \).

We will also make use of the algebra of analytic functions:

**Definition 1.3.** Let \( E \) and \( X \) be complex Banach spaces. We define

\[
C(\overline{B}_X, E) = \{ f : \overline{B}_X \to E : f \text{ is continuous on } \overline{B}_X \},
\]

\[
A_u(B_X, E) = \{ f : \overline{B}_X \to E : f \text{ is analytic on } B_X \text{ and uniformly continuous on } \overline{B}_X \},
\]

\[
A_u(B_X, E_{\omega}) = \{ f : \overline{B}_X \to E : y^* \circ f \in A_u(B_X, \mathbb{C}) \text{ for every } y^* \in E^* \}.
\]

Recall that a function \( f : X \to Y \) between two complex Banach spaces \((X, \| \cdot \|_X)\) and \((Y, \| \cdot \|_Y)\) is said to be analytic if for every \( x \in X \) there exists a linear mapping \( A_x : X \to Y \) so that

\[
\lim_{\tilde{x} \to x} \frac{f(\tilde{x}) - f(x) - A(x - x)}{\| \tilde{x} - x \|_X} = 0.
\]

We refer the interested reader to [19] for a complete survey on this topic.

Due to a classic result of Dunford ([10], Theorem 76, p. 354), a function \( f : B_X \to E \) is analytic if and only if \( y^* \circ f \) is analytic for every \( y^* \in E^* \). The question that motivated some of the results that appeared in [12] was whether an analogous result was true for the setting stated in the definition 1.3, that is, if the equality of sets

\[
A_u(\overline{B}_X, E) = A_u(\overline{B}_X, E_{\omega}),
\]

also held for every pair of complex Banach spaces \( X \) and \( E \). The following result was proved:

**Theorem 1.4 ([12]).** Let \( X \) and \( E \) be complex Banach spaces. The following are equivalent.

(i) \( E \) has the Schur property.

(ii) \( A_u(\overline{B}_X, E) = A_u(\overline{B}_X, E_{\omega}).\)

To prove the direction \((i) \Rightarrow (ii)\) in the previous theorem, the authors managed to find a function \( f \) which belongs to \( A_u(\overline{B}_X, E_{\omega}) \setminus A_u(\overline{B}_X, E) \), provided \( E \) does not have the Schur property. In particular, the authors remarked that the function they constructed was not continuous over \( \partial B_X \).

On the other hand, if we consider the case of functions \( f : \mathbb{D} \to E \), then it must be true that \( A_u(\overline{S}, E_{\omega}) \cap C(\overline{S}, E) = A_u(\overline{S}, E) \), taking into account that, due to the result of Dunford, \( f : \mathbb{D} \to E \) is analytic if and only if \( y^* \circ f \) is analytic for every \( y^* \in E^* \). Following this line of work, the authors in [12] wondered if then the equality of sets

\[
A_u(\overline{B}_X, E_{\omega}) \cap C(\overline{B}_X, E) = A_u(\overline{B}_X, E)
\]

was true for every pair of complex Banach spaces \( X \) and \( E \). The authors gave a negative answer to this question in the result below:

**Theorem 1.5 ([12]).** If \( E \) is a complex Banach space which does not have the Schur property, then

\[
A_u(\overline{B}_\ell_2, E_{\omega}) \cap C(\overline{B}_\ell_2, E) \setminus A_u(\overline{B}, E) \neq \emptyset.
\]

What we will study in the next section is how “big” the sets that have been considered in the previous theorems are. To this end, we will need to study first the space of weakly-null sequences which are not norm-convergent to 0.

### 2 The main result

**Theorem 2.1.** Let \( E \) be a Banach space which does not have the Schur property. Then, the set of weakly convergent sequences with entries in \( E \) which are not norm-convergent is spaceable in the space of norm-bounded sequences, with the supremum norm.
Proof. Since $E$ does not satisfy the Schur property, we can find $(x_n)_{n=1}^\infty \subseteq B_E$ and $\varepsilon > 0$ so that $x_n \xrightarrow{n \to \infty} 0$ but $\|x_n\|_E \geq \varepsilon$ for every $n \in \mathbb{N}$. Let us denote the set of all weakly null sequences in $E$ by $E^N_\omega$. Then, by the uniform boundedness principle, every element in $E^N_\omega$ is bounded (in norm) and therefore, $(E^N_\omega, \| \cdot \|_\infty)$ is a normed space.

Let us also denote
$$\mathcal{E} = \left\{ (y_n)_{n=1}^\infty \in E^N_\omega : \left\| y_n \right\|_\infty \xrightarrow{n \to \infty} 0 \right\}.$$ 

Next, we enumerate the set of prime numbers as $\{p_k : k \geq 1\}$ in the usual way (in increasing order), and consider
$$F : \mathbb{N} \setminus \{1\} \rightarrow \mathbb{N}$$
$$m \mapsto F(m) = l \text{ with } p_l = \min\{ p \text{ prime} : p|m \}.$$ 

Notice that, in particular, $F(p_n) = n$ and hence $F$ is surjective. Define now
$$T : (\ell_\infty, \| \cdot \|_\infty) \rightarrow (E^N_\omega, \| \cdot \|_\infty)$$
$$(a_n)_{n=1}^\infty \mapsto \left\{ T \left( (a_n)_{n=1}^\infty \right) (k) \right\}_{k=1}^\infty,$$
where
$$T \left( (a_n)_{n=1}^\infty \right) (k) = a_{F(k+1)} x_k,$$
and $(x_k)_{k=1}^\infty$ is the sequence we considered at the beginning of this proof.

$T$ is obviously linear. Also, for every natural number $n$,
$$\varepsilon |a_n| = \varepsilon |a_{F(p_n)}| \leq |a_{F(p_n)}| \|x_{p_n-1}\|_E = \left\| T((a_n)_{n=1}^\infty) (p_n - 1) \right\|_E \leq \left\| T((a_n)_{n=1}^\infty) \right\|_\infty$$
$$= \sup \left\{ \left\| T((a_n)_{n=1}^\infty) (l) \right\|_E : l \in \mathbb{N} \right\} \leq \max \left\{ |a_{F(l+1)}| \|x_l\|_E : l \in \mathbb{N} \right\}$$
$$\leq \max \{ |a_l| : l \in \mathbb{N} \} = \left\| (a_k)_{k=1}^\infty \right\|_\infty.$$ 

Hence,
$$\varepsilon \left\| (a_n)_{n=1}^\infty \right\|_\infty \leq \left\| T((a_n)_{n=1}^\infty) \right\|_\infty \leq \left\| (a_n)_{n=1}^\infty \right\|_\infty,$$
and therefore $T$ is an isomorphism.

To finish, let us show that if $(y_k)_{k=1}^\infty \in T(\ell_\infty)$, then $(y_k)_{k=1}^\infty \in \mathcal{E}$.

Indeed, consider $(a^{(n)}) \subseteq \ell_\infty$ so that
$$\left\{ a^{(n)}_{F(k+1)} x_k \right\}_{k=1}^\infty \xrightarrow{n \to \infty} (y_k)_{k=1}^\infty.$$ 

Then, given $\tilde{\varepsilon} > 0$, there exists $N \in \mathbb{N}$ so that, if $n > m \geq N$ and $k \in \mathbb{N}$,
$$\tilde{\varepsilon} > \left\| a^{(n)}_{F(k+1)} - a^{(m)}_{F(k+1)} \right\|_E \geq \varepsilon \left| a^{(n)}_{F(k+1)} - a^{(m)}_{F(k+1)} \right|.$$ 

so that
$$\left| a^{(n)}_{F(k+1)} - a^{(m)}_{F(k+1)} \right| < \frac{\tilde{\varepsilon}}{\varepsilon},$$
for every $k \in \mathbb{N}$. Therefore, $(a^{(n)})_{n=1}^\infty$ is Cauchy in $(\ell_\infty, \| \cdot \|_\infty)$ and then we can find $a = (a_k)_{k=1}^\infty \in \ell_\infty$ with $a^{(n)} \xrightarrow{n \to \infty} a$.

We claim that $(y_k)_{k=1}^\infty = (a_{F(k+1)} x_k)_{k=1}^\infty$. Indeed, if $r > 0$ we can find $\tilde{N} \in \mathbb{N}$ so that, for every $k \in \mathbb{N}$ and $n \geq \tilde{N}$,
$$|a^{(n)}_k - a_k| < \frac{r}{2M},$$
$$\|a^{(n)}_{F(k+1)} x_k - y_k\|_E < \frac{r}{2},$$
where $M = \sup \{ \|x_l\|_E : l \in \mathbb{N} \}$. Then,
$$\left\| a_{F(k+1)} x_k - y_k \right\|_E \leq \left\| a^{(n)}_{F(k+1)} x_k - a_{F(k+1)} x_k \right\|_E + \left\| a^{(n)}_{F(k+1)} x_k - y_k \right\|_E \leq \frac{r}{2} + \frac{r}{2} = r.$$
Let \( x = \sum_{k=1}^{\infty} x_k e_k \) be an element of \( C^* \), where \( \{e_k\} \) is a sequence of elements of \( E \) that is not uniformly convergent to \( 0 \) but is norm-convergent to \( 0 \) in \( \ell_1 \). Then, \( x \) cannot be \( \ell^* \)-convergent to \( 0 \) in \( C^* \).

Proof. Suppose that \( x = \sum_{k=1}^{\infty} x_k e_k \) is \( \ell^* \)-convergent to \( 0 \) in \( C^* \). Then, \( \langle x_k, e_n \rangle = x_k \) for all \( n \geq 1 \) and \( \sum_{n=1}^{\infty} \langle x_k, e_n \rangle = 0 \) for all \( k \geq 1 \). This implies that \( x_k = 0 \) for all \( k \geq 1 \), and hence \( x = 0 \) in \( C^* \). This contradicts our assumption that \( x \) is not \( \ell^* \)-convergent to \( 0 \) in \( C^* \).

Let \( \{f_n\} \) be a sequence of auxiliary functions that do not depend on the sequence \( \{e_n\} \) and are auxiliary functions that do not satisfy the Schur property, the authors managed to prove that \( f \) does not satisfy the Schur property, the authors managed to prove that \( f \) is not uniformly convergent to \( 0 \) in \( \ell_1 \). Using the fact that \( f \) does not satisfy the Schur property, the authors managed to prove that \( f \) is not uniformly convergent to \( 0 \) in \( \ell_1 \).

Proof. In Theorem 1.4, the authors considered the function

\[
f(x) = \sum_{n=1}^{\infty} f_n(\phi(x)) e_n,
\]

where \( \{e_n\} \) is a sequence that prevents \( E \) from being a Schur space (in particular, it is weakly null but uniformly bounded below in norm) and \( f_n, \phi \) are auxiliary functions that do not depend on the sequence \( \{e_n\} \). Using the fact that \( \{e_n\} \) does not satisfy the Schur property, the authors managed to prove that \( f \neq 0 \) in \( \ell_1 \). Using the fact that \( f_n \) and \( \phi \) are independent of the sequence not satisfying the Schur property, it is easy to see that \( T_1 \) is injective.

Corollary 2.4. Let \( E \) be a Banach space that does not satisfy the Schur property. Then, \( \ell_1 \) is isomorphic to a subspace of functions in \( A_u(\overline{B}_X, E_\omega) \) all of whose nonzero elements are continuous on \( \overline{B}_X \) (we would like to remark that, due to the result by Dunford in [10], those functions must be analytic).

Proof. Following the same idea as in Corollary 2.3, we may just define

\[
T_2 : \ell_1 \to A_u(\overline{B}_\ell_2, E_\omega)
\]

where \( T_1(\{x_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} z_n T(\{x_k\}_{k=1}^{\infty})_n \).

This function is injective and, if we follow the proofs from Theorems 2.1 and 1.5, one can prove that \( T_2 \) provides the desired isomorphism.
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References

[7] Botelho G., Fávaro V.V., Pellegrino D., Seoane-Sepúlveda J.B., \( L_p[0, 1] \setminus \cup_{q<p} L_q[0, 1] \) is spaceable for every \( p > 0 \), Linear Algebra Appl. 436 (2012), no. 9, 2978–2985.