Oscillation and nonoscillation of half-linear Euler type differential equations with different periodic coefficients

1 Introduction

An equation of the form
\[
(r(t) \Phi(x'))' + c(t) \Phi(x) = 0, \quad \Phi(s) = |s|^{p-2} s, \quad p > 1, \tag{1}
\]
is called half-linear differential equation, where \( r, c \) are continuous functions and \( r(t) > 0 \) was introduced for the first time in [1]. During the last decades, these equations have been widely studied in the literature. The name half-linear equation was introduced in [2]. This term is motivated by the fact that the solution space of these equations is homogeneous (likewise in the linear case) but not additive. Since the linear Sturmian theory extends verbatim to half-linear case (for details, we refer to Section 1.2 in [3]), we can classify Eq.(1) as oscillatory or nonoscillatory. It is well known that oscillation theory of Eq.(1) is very similar to that of the linear Sturm-Liouville differential equation, which is the special case of Eq.(1) for \( p = 2 \) (see [4]).

Actually, we are interested in the conditional oscillation of half-linear differential equations with different periodic coefficients. We say that the equation
\[
(r(t) \Phi(x'))' + yc(t) \Phi(x) = 0 \tag{2}
\]
with positive coefficients is conditionally oscillatory if there exists a constant \( \gamma_0 \) such that Eq. (2) is oscillatory for all \( \gamma > \gamma_0 \) and nonoscillatory for all \( \gamma < \gamma_0 \). The constant \( \gamma_0 \) is called an oscillation constant (more precisely, oscillation constant of \( c \) with respect to \( r \)) of this equation.

Considerable effort has been made over the years to extend oscillation constant theory of half linear differential equation (1), see [5–10] and reference therein. According to our knowledge, the first attempt to this problem was made by Kneser in [11], where the oscillation constant for Cauchy-Euler differential equation

\[ x'' + \frac{\gamma}{t^2}x = 0, \tag{3} \]

(which is special case of Eq.(1) for \( p = 2, \ r(t) = 1 \) and \( c(t) = \frac{1}{t^2} \)) has been identified \( \gamma_0 = \frac{1}{4} \) and Eq.(3) is oscillatory if \( \gamma > \frac{1}{4} \), nonoscillatory if \( \gamma < \frac{1}{4} \). Additionally, \( x(t) = c_1 \sqrt{t} + c_2 \sqrt{t} \log t \) is the general solution of Eq.(3) and nonoscillatory for \( \gamma = \frac{1}{4} \).

The conditional oscillation of linear equations is studied, e.g., in [9, 10]. In [9, 12], the oscillation constant was obtained for linear equation with periodic coefficients. Using the notion of the principle solution, the main result of [9] was generalized in [8], where periodic half-linear equations were considered.

In [7] the half-linear Euler differential equation of the form

\[ (\Phi(x'))' + \frac{\gamma}{t^p} \Phi(x) = 0, \tag{4} \]

and the half-linear Riemann-Weber differential equation of the form

\[ (\Phi(x'))' + \frac{1}{t^p} \left( \gamma + \frac{\mu}{\log^2 t} \right) \Phi(x) = 0, \tag{5} \]

was considered and it was shown that Eq.(4) is nonoscillatory if and only if \( \gamma \leq \gamma_p := \left( \frac{p-1}{p} \right)^p \) and Eq.(5) with \( \gamma = \gamma_p \), is nonoscillatory if \( \mu < \mu_p := \frac{1}{2} \left( \frac{p-1}{p} \right)^{p-1} \) and oscillatory if \( \mu > \mu_p \).

In [8], the half-linear differential equation of the form

\[ (r(t) \Phi(x'))' + \frac{\gamma c(t)}{t^p} \Phi(x) = 0, \tag{6} \]

was considered for \( r, c \) being \( \alpha \)--periodic, positive functions and it was shown that Eq.(6) is oscillatory if \( \gamma > K \) and nonoscillatory if \( \gamma < K \), where \( K \) is given by

\[ K = q^{-p} \left( \frac{1}{\alpha} \int_0^\alpha \frac{d\tau}{\tau^{q-1}} \right)^{1-p} \left( \frac{1}{\alpha} \int_0^\alpha c(t) d\tau \right)^{-1} \]

for \( p \) and \( q \) are conjugate numbers, i.e., \( \frac{1}{p} + \frac{1}{q} = 1 \). If the functions \( r, c \) are positive constants, then Eq.(6) is reduced to the half-linear Euler equation (4), whose oscillatory properties were studied in detail [4, 7] and references given therein.

In [6], Eq.(6) and the half-linear differential equation of the form

\[ (r(t) \Phi(x'))' + \frac{\gamma c(t) + \frac{\mu d(t)}{\log^2 t}}{t^p} \Phi(x) = 0, \tag{7} \]

was considered positive, \( \alpha \)--periodic functions \( r, c \) and \( d \), which are defined on \([0, \infty)\) and it was shown that Eq.(6) is oscillatory if and only if \( \gamma \leq \gamma_{rc} \), where \( \gamma_{rc} \) is given by

\[ \gamma_{rc} := \frac{a^p \gamma_p}{\left( \int_0^\alpha r^{1-q}(t) \, dt \right)^{p-1} \int_0^\alpha c(t) \, dt}, \]

and in the limiting case \( \gamma = \gamma_{rc} \) Eq.(7) is nonoscillatory if \( \mu < \mu_{rd} \) and it is oscillatory if \( \mu > \mu_{rd} \), where \( \mu_{rd} \) is given by

\[ \mu_{rd} = \frac{a^p \mu_p}{\left( \int_0^\alpha r^{1-q}(t) \, dt \right)^{p-1} \int_0^\alpha d(t) \, dt}. \]
If the functions \( r, c, \) and \( d \) are positive constants, then Eq. (7) is reduced to the half-linear Rimann-Weber equation (5), whose oscillatory properties are studied in detail [4, 7] and references given therein.

In [13], the half-linear differential equation of the form
\[
\left( r(t) \Phi \left( \frac{d}{t^p} \Phi (x) \right) \right)' + \frac{c(t)}{t^p} \Phi (x) = 0
\]
was considered for \( r: [a, \infty) \to \mathbb{R}, (a > 0), \) where \( r \) is a continuous function for which mean value \( M \left( r^{1-q} \right) := \lim_{t \to \infty} \frac{1}{t} \int_a^t r^{1-q} (\tau) \, d\tau \) exists and for which
\[
0 < \inf_{t \in (a, \infty)} r(t) \leq \sup_{t \in (a, \infty)} r(t) < \infty \text{ holds,}
\]
and \( c: [a, \infty) \to \mathbb{R}, (a > 0), \) be a continuous function having mean value \( M(c) := \lim_{t \to \infty} \frac{1}{t} \int_a^t c(\tau) \, d\tau \) and it was shown that Eq. (8) is oscillatory if \( M(c) > \Gamma \) and nonoscillatory if \( M(c) > \Gamma', \) where \( \Gamma \) is given by
\[
\Gamma = q^{-p} \left[ M \left( r^{1-q} \right) \right]^{1-p}.
\]

Our goal is to find the explicit oscillation constant for Eq. (7) with periodic coefficients having different periods. We point out that the main motivation of our research comes from the paper [6], where the oscillation constant was computed for Eq. (7) with the periodic coefficients having the same \( \alpha \)-period. But in that paper the oscillation constant wasn’t obtained for the periodic functions having different periods and consequently for the case when the least common multiple of these periodic coefficients is not defined. Thus in this paper we investigate the oscillation constant for Eq. (7) with periodic coefficients having different periods. For the sake of simplicity, we usually use the same notations as in the paper [6].

This paper is organized as follows. In section 2, we recall the concept of half-linear trigonometric functions and their properties. In section 3 we compute the oscillation constant for Eq. (7) with periodic coefficients having different periods. Additionally, we show that if the same periods are taken, then our result compiles with the known result in [4] or if the same period \( \alpha \) given in [6] can be chosen as the least common multiple of periods of the coefficients (which have different periods), then our result coincides with the result of [6]. Thus, our results extend and improve the results of [6]. Finally in the last section, we give an example to illustrate the importance of our result.

## 2 Preliminaries

We start this section with the recalling the concept of half-linear trigonometric functions, see [3] or [4]. Consider the following special half-linear equation of the form
\[
\left( \Phi \left( x' \right) \right)' + (p - 1) \Phi (x) = 0,
\]
and denote by \( x = x(t) \) its solution given by the initial conditions \( x(0) = 0, \ x'(0) = 1. \) We see that the behavior of this solution is very similar to the classical sine function. We denote this solution by \( \sin_p t \) and its derivative as \( \left( \sin_p t \right)' = \cos_p t. \) These functions are \( 2\pi_p \)-periodic, where \( \pi_p := \frac{2\pi}{p \sin \left( \frac{\pi}{p} \right)} \) and satisfy the half-linear Pythagorean identity
\[
| \sin_p t |^p + | \cos_p t |^p = 1, \ t \in \mathbb{R}.
\]
All solutions of Eq. (9) are of the form \( x(t) = C \sin_p (t + \varphi), \) where \( C, \varphi \) are real constants. All these solutions and their derivatives are bounded and periodic with the period \( 2\pi_p. \) The function \( u = \Phi (\cos_p t) \) is a solution of the reciprocal equation to Eq. (9):
\[
\left( \Phi^{-1} \left( u' \right) \right)' + (p - 1)^{q-1} \Phi^{-1} (u) = 0, \ \Phi^{-1} (u) = |u|^{q-2} u, \ q = \frac{p}{p - 1}.
\]
which is an equation of the form as in Eq.(9), so the functions \( u \) and \( u' \) are also bounded.

Let \( x \) be a nontrivial solution of Eq.(1) and we consider the half-linear Prüfer transformation which is introduced using the half-linear trigonometric functions

\[
x(t) = \rho(t) \sin_p \varphi(t), x'(t) = r^{1-q}(t) \frac{\rho(t)}{t} \cos_p \varphi(t),
\]

where \( \rho(t) = \sqrt{|x(t)|^p + r^q(t)|x'(t)|^p} \) and Prüfer angle \( \varphi \) is a continuous function defined at all points where \( x(t) \neq 0 \).

Let \( w(t) = r(t) \Phi \left( \frac{x(t)}{x(t)} \right) \) then \( w(t) \) is the solution of Riccati equation

\[
w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0,
\]

associated with Eq.(1).

Let \( v(t) = t^{p-1}w(t) = t^{p-1}r(t) \Phi \left( \frac{x'(t)}{x(t)} \right) \), then by using Eq.(11) we obtain \( v = \Phi(\cot_p \varphi) \), where \( \cot_p \varphi = \frac{\cos_p \varphi}{\sin_p \varphi} \). If we use the fact that \( \sin_p t \) is a solution of Eq.(9), the function \( v \) satisfies the associated Riccati type equation

\[
v' = \left[ 1 - p + (1 - p)|\Phi(\cot_p \varphi)|^q \right] \varphi'.
\]

At the same time, using the fact that \( w \) solves Eq.(12), we obtain

\[
\begin{align*}
v' &= (p-1)t^{p-2}w - t^{p-1}\left[ c(t) + (p-1) r^{1-q}(t)|w|^q \right] \\
&= \frac{v}{t}\left( p-1 - t^{p-1}c(t) - (p-1)t^{p-1}r^{1-q}(t) \right) t^{1-p}w \notag \\
&= \frac{p-1}{t}\left[ v - \frac{tp}{p-1}c(t) - r^{1-q}(t)|\cot_p \varphi|^p \right].
\end{align*}
\]

Combining the last equation with Eq.(12) we get

\[
(p-1)\left[ 1 + \frac{\cos_p \varphi}{p}\varphi' = \frac{p-1}{t}\left[ \frac{tp}{p-1}c(t) + r^{1-q}(t) \frac{\cos_p \varphi}{p} - \Phi(\cos_p \varphi) \right].
\]

Multiplying both sides of this equation by \( \frac{\sin_p \varphi}{p-1} \) and using the half-linear Pythagorean identity Eq.(10), we obtain the equation

\[
\varphi' = \frac{1}{t}\left[ r^{1-q}(t) |\cos_p \varphi|^p - \Phi(\cos_p \varphi) \sin_p \varphi + \frac{tp}{p-1}c(t) \sin_p \varphi \right],
\]

which will play the fundamental role in our investigation.

It is well-known that the nonoscillation of Eq.(7) is equivalent to the boundedness from above of the Prüfer angle \( \varphi \) given by Eq.(11) (see [6, 10]). Next, we briefly mention the principal solution of nonoscillatory equation Eq.(1) in [5], which is defined via the minimal solution of the associated Riccati equation Eq.(12). Nonoscillation of Eq.(1) implies that there exist \( T \in \mathbb{R} \) and a solution \( \tilde{w} \) of Eq.(12) defined on some interval \([T_\infty, \infty)\). \( \tilde{w} \) is called the minimal solution of all solutions of Eq.(12) and it satisfies the inequality \( w(t) > \tilde{w}(t) \) where \( w \) is any other solution of Eq.(12) defined on some interval \([T_\infty, \infty)\) and then \( \tilde{w} \) is the principal solution of Eq.(1) via the formula

\[
\tilde{w}(t) = r(t) \Phi \left( \frac{x(t)}{\tilde{x}(t)} \right).
\]

We finish this section with a lemma without proof, to be used in the next section.

**Lemma 2.1.**

(i) Eq.(1) is non-oscillatory if and only if there exists a positive differentiable function \( h \) such that

\[
(r(t) \Phi(h))' + c(t) \Phi(h) \leq 0
\]

for large \( t \).
(ii) Let \( \mu > \frac{1}{4} \), then for every positive \( \epsilon \), the linear differential equation

\[
(t \epsilon)'' + \frac{\epsilon}{\log^2 t} x' + \frac{\mu}{t \log^2 t} x = 0
\]

is oscillatory.

(iii) Suppose that \( \int r^{1-q} (t) \, dt = \infty \), \( c(t) > 0 \) for large \( t \) in Eq. (1) and this equation is non-oscillatory. Then the minimal solution of associate Riccati equation (12) is positive for large \( t \) [6].

3 Main results

To prove the main result, we need the following lemmas.

Lemma 3.1. Let \( \varphi = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 + M \), \( M \) is a suitable constant) be a solution of the equation

\[
\varphi'(t) = \varphi_1'(t) + \varphi_2'(t) + \varphi_3'(t) + \varphi_4'(t)
\]

where

\[
\varphi_1'(t) = \frac{1}{t} r^{1-q} (t) | \cos_p \varphi (t) |^p,
\]

\[
\varphi_2'(t) = - \frac{1}{t} \Phi (\cos_p \varphi (t)) \sin_p \varphi (t),
\]

\[
\varphi_3'(t) = \frac{c(t)}{(p-1)t} | \sin_p \varphi (t) |^p,
\]

\[
\varphi_4'(t) = \frac{d(t)}{(p-1)t \log^2 t} | \sin_p \varphi (t) |^p.
\]

with \( r \), \( c \) and \( d \) are positive defined functions having different periods \( \tilde{\beta}_1, \tilde{\beta}_2, \) and \( \tilde{\beta}_3 \) respectively and let

\[
\theta (t) = \frac{1}{\tilde{\beta}_1} \int_{\xi}^{t+\beta_1} \varphi_1 (s) \, ds + \frac{1}{\tilde{\beta}_1} \int_{\xi}^{\xi+\beta_1} \varphi_2 (s) \, ds + \frac{1}{\tilde{\beta}_2} \int_{\xi}^{\xi+\beta_2} \varphi_3 (s) \, ds + \frac{1}{\tilde{\beta}_3} \int_{\xi}^{\xi+\beta_3} \varphi_4 (s) \, ds,
\]

where \( \xi \) is one of the periods \( \tilde{\beta}_1, \tilde{\beta}_2, \) or \( \tilde{\beta}_3 \). Then \( \theta \) is a solution of

\[
\theta'' = \frac{1}{t} \left[ R | \cos_p \theta |^p - \Phi (\cos_p \theta) \sin_p \theta + \frac{1}{(p-1) \log^2 t} | \sin_p \theta |^p \right] + O \left( \frac{1}{t \log^2 t} \right),
\]

where

\[
R = \frac{1}{\tilde{\beta}_1} \int_0^{\beta_1} r^{1-q} (\tau) \, d\tau, C = \frac{1}{(p-1) \tilde{\beta}_2} \int_0^{\beta_2} c(\tau) \, d\tau \text{ and } D = \frac{1}{(p-1) \tilde{\beta}_3} \int_0^{\beta_3} d(\tau) \, d\tau
\]

and \( \varphi (\tau) - \theta (\tau) = o (1) \) as \( t \to \infty \).

Proof. Taking derivative of \( \theta (t) \), we have

\[
\theta' (t) = \frac{1}{\tilde{\beta}_1} \int_{\xi}^{t+\beta_1} \varphi_1' (s) \, ds + \frac{1}{\tilde{\beta}_1} \int_{\xi}^{\xi+\beta_1} \varphi_2' (s) \, ds + \frac{1}{\tilde{\beta}_2} \int_{\xi}^{\xi+\beta_2} \varphi_3' (s) \, ds + \frac{1}{\tilde{\beta}_3} \int_{\xi}^{\xi+\beta_3} \varphi_4' (s) \, ds
\]

\[
= \frac{1}{\tilde{\beta}_1} \int_{\xi}^{t+\beta_1} r^{1-q} (s) | \cos_p \varphi (s) |^p \, ds
\]
\[
-\frac{1}{\xi} \int_{t}^{t+\xi} \frac{1}{s} \Phi(\cos \phi(s)) \sin \phi(s) \, ds \\
+ \frac{1}{\beta_2} \int_{t}^{t+\beta_2} \frac{c(s)}{(p-1)s} |\sin \phi(s)|^p \, ds \\
+ \frac{1}{\beta_3} \int_{t}^{t+\beta_3} \frac{d(s)}{(p-1)s \log^2 s} |\sin \phi(s)|^p \, ds.
\]

Using integration by parts, we get

\[
\theta'(t) = \frac{1}{\beta_1 t} \int_{t}^{t+\beta_1} r^{1-q} (\tau) |\cos \phi(\tau)|^p \, d\tau \\
- \frac{1}{\xi t} \int_{t}^{t+\xi} \Phi(\cos \phi(\tau)) \sin \phi(\tau) \, d\tau \\
+ \frac{1}{\beta_2 t} \int_{t}^{t+\beta_2} c(\tau) (p-1) |\sin \phi(\tau)|^p \, d\tau \\
+ \frac{1}{\beta_3 t} \int_{t}^{t+\beta_3} d(\tau) (p-1) \log^2 \tau |\sin \phi(\tau)|^p \, d\tau \\
- \frac{1}{\beta_1 t} \int_{t}^{t+\beta_1} \frac{1}{s^2} \int_{s}^{r^{1-q}(\tau)} |\cos \phi(\tau)|^p \, d\tau \, ds \\
+ \frac{1}{\xi t} \int_{t}^{t+\xi} \frac{1}{s^2} \int_{s}^{r^{1-q}(\tau)} \Phi(\cos \phi(\tau)) \sin \phi(\tau) \, d\tau \, ds \\
- \frac{1}{\beta_2 t} \int_{t}^{t+\beta_2} \frac{1}{s^2} \int_{s}^{r^{1-q}(\tau)} \frac{c(\tau)}{(p-1)} |\sin \phi(\tau)|^p \, d\tau \, ds \\
- \frac{1}{\beta_3 t} \int_{t}^{t+\beta_3} \frac{1}{s^2} \int_{s}^{r^{1-q}(\tau)} \frac{d(\tau)}{(p-1) \log^2 \tau} |\sin \phi(\tau)|^p \, d\tau \, ds.
\]

By the fact that, \( \int_{t}^{t+T} f(s) \, ds = \int_{0}^{T} f(s) \, ds \) for any \( T \)-periodic function and half-linear Pythagorean identity, the expressions

\[ r^{1-q}(\tau) |\cos \phi|^p, -\Phi(\cos \phi) \sin \phi, \frac{c(\tau)}{p-1} |\sin \phi|^p \text{ and } \frac{d(\tau)}{p-1} |\sin \phi|^p \]

are bounded. Thus we get

\[
\theta'(t) = \frac{1}{\beta_1 t} \int_{t}^{t+\beta_1} r^{1-q} (\tau) |\cos \phi(\tau)|^p \, d\tau \\
- \frac{1}{\xi t} \int_{t}^{t+\xi} \Phi(\cos \phi(\tau)) \sin \phi(\tau) \, d\tau \\
+ \frac{1}{\beta_2 t} \int_{t}^{t+\beta_2} \frac{c(\tau)}{(p-1)} |\sin \phi(\tau)|^p \, d\tau 
\]
We can rewrite this equation as

\[
\theta' (t) = \frac{1}{\beta_1 t} \int_t^{t+\xi_1} r^{1-q} (\tau) \left| \cos_p \theta (t) \right|^p d \tau
\]

\[
- \frac{1}{\xi t} \int_t^{t+\xi} \Phi (\cos_p \theta (t)) \sin_p \theta (t) d \tau
\]

\[
+ \frac{1}{\beta_2 t} \int_t^{t+\xi_2} \frac{c (\tau)}{(p-1)} \left| \sin_p \theta (t) \right|^p d \tau
\]

\[
+ \frac{1}{\beta_3 t} \int_t^{t+\xi_3} \frac{d (\tau)}{(p-1) \log^2 \tau} \left| \sin_p \theta (t) \right|^p d \tau
\]

\[
+ \frac{1}{\beta_1 t} \int_t^{t+\xi_1} r^{1-q} (\tau) \left\{ \left| \cos_p \varphi (\tau) \right| - \left| \cos_p \theta (t) \right| \right\} d \tau
\]

\[
- \frac{1}{\xi t} \int_t^{t+\xi} \left\{ \Phi (\cos_p \varphi (\tau)) \sin_p \varphi (\tau) - \Phi (\cos_p \theta (t)) \sin \theta (t) \right\} d \tau
\]

\[
+ \frac{1}{\beta_2 t} \int_t^{t+\xi_2} \frac{c (\tau)}{(p-1)} \left\{ \left| \sin_p \varphi (\tau) \right| - \left| \sin_p \theta (t) \right| \right\} d \tau
\]

\[
+ \frac{1}{\beta_3 t} \int_t^{t+\xi_3} \frac{d (\tau)}{(p-1) \log^2 \tau} \left\{ \left| \sin_p \varphi (\tau) \right| - \left| \sin_p \theta (t) \right| \right\} d \tau
\]

\[
+ O \left( \frac{1}{t \log^2 t} \right).
\]

Similarly as in [6] if we use integration by parts, we get

\[
\int_t^{t+\xi} \frac{d(s)}{\log^2 s} ds = \frac{(p-1) \beta_3 D}{\log^2 t} + O \left( \frac{1}{t \log^3 t} \right)
\]

and by using the definition of \( R, C, \) and \( D \) we get

\[
\theta' (t) = \frac{1}{t} \left[ R \left| \cos_p \theta (t) \right|^p - \Phi (\cos_p \theta (t)) \sin_p \theta (t) \right]
\]

\[
+ \frac{1}{p-1} \left[ C + \frac{D}{\log^2 t} \right] \left| \sin_p \theta (t) \right|^p
\]

\[
+ \frac{1}{\beta_1 t} \int_t^{t+\xi_1} r^{1-q} (\tau) \left\{ \left| \cos_p \varphi (\tau) \right| - \left| \cos_p \theta (t) \right| \right\} d \tau
\]

\[
- \frac{1}{\xi t} \int_t^{t+\xi} \left\{ \Phi (\cos_p \varphi (\tau)) \sin_p \varphi (\tau) - \Phi (\cos_p \theta (t)) \sin \theta (t) \right\} d \tau
\]

\[
+ \frac{1}{\beta_2 t} \int_t^{t+\xi_2} \frac{c (\tau)}{(p-1)} \left\{ \left| \sin_p \varphi (\tau) \right| - \left| \sin_p \theta (t) \right| \right\} d \tau
\]

\[
+ \frac{1}{\beta_3 t} \int_t^{t+\xi_3} \frac{d (\tau)}{(p-1) \log^2 \tau} \left\{ \left| \sin_p \varphi (\tau) \right| - \left| \sin_p \theta (t) \right| \right\} d \tau
\]

\[
+ O \left( \frac{1}{t \log^2 t} \right).
\]
This implies that
\[ t \rightarrow \beta_3 t \int \frac{d(\tau)}{(p-1)\log^2 \tau} \{ |\sin_p \varphi(t)|^P - |\sin_p \theta(t)|^P \} \, d\tau \]
\[ + O \left( \frac{1}{t \log^2 t} \right). \]

And using the half-linear trigonometric functions, we have
\[ ||\cos_p \varphi(t)|^P - |\cos_p \theta(t)|^P| \leq P \int_{\theta(t)}^{\varphi(t)} \left| \Phi(\cos_p s) (\cos_p s)' \right| ds \]
\[ \leq \text{const} |\varphi(t) - \theta(t)|. \]

\[ |\Phi(\cos_p \varphi(t)) \sin_p \varphi(t) - \Phi(\cos_p \theta(t)) \sin_p \theta(t)| \leq \int_{\theta(t)}^{\varphi(t)} \left| (\Phi(\cos_p s) \sin_p s)' \right| ds \]
\[ \leq \text{const} |\varphi(t) - \theta(t)| \]
and
\[ ||\sin_p \varphi(t)|^P - |\sin_p \theta(t)|^P| \leq \text{const} |\varphi(t) - \theta(t)|. \]

By the mean value theorem we can write
\[ \theta(t) = \varphi_1(t_1) + \varphi_2(t_2) + \varphi_3(t_3) + \varphi_4(t_4) \]
for \( t_1 \in [t, t + \beta_1], t_2 \in [t, t + \beta_2], t_3 \in [t, t + \beta_2], t_4 \in [t, t + \beta_3] \). Thus
\[ |\varphi(t) - \theta(t)| \leq |\varphi_1(t) - \varphi_1(t_1)| + |\varphi_2(t) - \varphi_2(t_2)| + |\varphi_3(t) - \varphi_3(t_3)| + |\varphi_4(t) - \varphi_4(t_4)|. \]

This implies that
\[ |\varphi(t) - \theta(t)| \leq o \left( \frac{1}{t} \right) \text{ as } t \rightarrow \infty, \varphi(t) - \theta(t) = o(1). \]

Hence we get
\[ \theta' = \frac{1}{t} \left[ R|\cos_p \theta|^P - \Phi(\cos_p \theta) \sin_p \theta + \frac{1}{p-1}(C + D \log^2 t)|\sin_p \theta|^P \right] + O \left( \frac{1}{t \log^2 t} \right). \]

The computation of oscillation constant \( \mu \) in Eq. (7) is based on the following lemma

**Lemma 3.2.** Let \( \int_{0}^{\beta_1} r^{1-q}(t) \, dt \) \( \int_{0}^{\beta_2} c(t) \, dt = \gamma \beta_1^{p-1} \beta_2 \) and \( \theta \) is a solution of the differential equation
\[ \theta' = \frac{1}{t} \left[ R|\cos_p \theta|^P - \Phi(\cos_p \theta) \sin_p \theta + \frac{1}{p-1}(C + D \log^2 t)|\sin_p \theta|^P \right] + o \left( \frac{1}{t \log^2 t} \right). \quad (13) \]

where \( R, C, D \) are as in Lemma 3.1.

If \( \int_{0}^{\beta_1} r^{1-q}(t) \, dt \) \( \int_{0}^{\beta_3} d(t) \, dt > \mu \beta_1^{p-1} \beta_3 \), then \( \theta(t) \rightarrow \infty \) as \( t \rightarrow \infty \)

and
\[ \int_{0}^{\beta_1} r^{1-q}(t) \, dt \int_{0}^{\beta_3} d(t) \, dt < \mu \beta_1^{p-1} \beta_3, \text{ then } \theta(t) \text{ is bounded.} \]

**Proof.** We rewrite Eq. (13) in the form
\[ \theta' = \frac{1}{t} \left[ R|\cos_p \theta|^P - \Phi(\cos_p \theta) \sin_p \theta + \frac{1}{p-1}(C + D \log^2 t)|\sin_p \theta|^P \right] \]
This is the equation for Prüfer angle \( \theta \), which corresponds to the differential equation
\[
\left( \left( 1 - \frac{o(1)}{\log^2 t} \right) \phi \left( x' \right) \right)' + \frac{p-1}{t^p} \left( R^{p-1} C + \frac{R^{p-1} D + o(1)}{\log^2 t} \right) \phi (x) = 0,
\]
which is the same (using the formula \((1 + x)^\alpha = 1 + \alpha x + o(x)\) as \(x \to 0\)) as the equation
\[
\left( \left( 1 - \frac{o(1)}{\log^2 t} \right) \phi \left( x' \right) \right)' + \frac{p-1}{t^p} \left( \left( p - p_1 \right) R^{p-1} D + o(1) \right) \phi (x) = 0,
\]
i.e., the same as
\[
\left( \left( 1 - \frac{o(1)}{\log^2 t} \right) \phi \left( x' \right) \right)' + \frac{1}{tp} \left( \gamma_\nu + \frac{(p-1) R^{p-1} D + o(1)}{\log^2 t} \right) \phi (x) = 0.
\]

First, suppose that
\[
\frac{1}{\beta_1^{p-1}} \left( \int_0^\beta_1 t^{1-q} (t) dt \right) \int_0^{\beta_3} d(t) dt < \mu_\nu
\]
and denote
\[
\delta := \frac{1}{2} \left[ \mu_\nu - \frac{1}{\beta_1^{p-1}} \left( \int_0^\beta_1 t^{1-q} (t) dt \right) \int_0^{\beta_3} d(t) dt \right].
\]

Let \( \varepsilon > 0 \) be sufficiently small and let \( T \) be so large that
\[
|o(1)| < \varepsilon \quad \text{and} \quad \mu_\nu \geq \frac{1}{\beta_1^{p-1}} \left( \int_0^\beta_1 t^{1-q} (t) dt \right) \int_0^{\beta_3} d(t) dt + o(1) > \delta
\]
for \( t \geq T \). Then, the equation
\[
\left( \left( 1 - \frac{\varepsilon}{\log^2 t} \right) \phi \left( x' \right) \right)' + \frac{1}{tp} \left( \gamma_\nu + \frac{\mu_\nu - \delta}{\log^2 t} \right) \phi (x) = 0
\]
is a Sturmian majorant of Eq.(14), i.e., nonoscillation of Eq.(15) implies nonoscillation of Eq.(14).

We will show that the function \( h(t) = t^{\frac{p-1}{p}} \log^\frac{1}{p} \frac{1}{t} \) satisfies
\[
\left( \left( 1 - \frac{\varepsilon}{\log^2 t} \right) \phi \left( h' \right) \right)' + \frac{1}{tp} \left( \gamma_\nu + \frac{\mu_\nu - \delta}{\log^2 t} \right) \phi (h) \leq 0
\]
for large \( t \), then nonoscillation of Eq.(15) follows from Lemma 2.1. According to [5], we have for \( h(t) = t^{\frac{p-1}{p}} \log^\frac{1}{p} \frac{1}{t} \)
\[
h(t) \left\{ \phi \left( h' \right) \right\}' + \frac{1}{tp} \left( \gamma_\nu + \frac{\mu_\nu - \delta}{\log^2 t} \right) \phi (h) \sim \frac{H}{t \log^2 t}
\]
for \( t \to \infty \), where \( H \) is a real constant. At the same time, by a direct computation,
\[
-h \left( \phi \left( h' \right) \right)' \frac{\varepsilon \phi (h)}{\log^2 t} \sim \frac{\varepsilon \gamma_\nu - \delta}{t \log t} < 0
\]
for large \( t \), if \( \varepsilon < \frac{\delta}{\gamma_\nu} \), so we see that Eq.(16) really holds, hence Eq.(13) is non-oscillatory, i.e., the "Prüfer angle " of its solution is bounded.
Now suppose that
\[
\frac{1}{\beta_1} \left( \int_0^1 r^{1-q}(t) \, dt \right)^{p-1} \frac{1}{\beta_3} \int_0^1 d(t) > \mu_p
\]
and denote
\[
\delta := \frac{1}{2} \left[ \frac{1}{\beta_1} \left( \int_0^1 r^{1-q}(t) \, dt \right)^{p-1} \frac{1}{\beta_3} \int_0^1 d(t) \, dt - \mu_p \right].
\]
Let \( \varepsilon > 0 \) be again sufficiently small and let \( T \) be so large that
\[
o(1) < \varepsilon \text{ and } \frac{1}{\beta_1} \left( \int_0^1 r^{1-q}(t) \, dt \right)^{p-1} \frac{1}{\beta_3} \int_0^1 d(t) \, dt - \mu_p + o(1) > \delta
\]
for \( t \geq T \). Then the equation
\[
\left( \left( 1 + \frac{\varepsilon}{\log t} \right) \Phi \left( x' \right) \right)' + \frac{1}{t^p} \left( \gamma_p + \frac{\mu_p + \delta}{\log^2 t} \right) \Phi(x) = 0
\]
(17)
is a Sturmian minorant of Eq.(14), i.e., oscillation of Eq.(17) implies oscillation of Eq.(14).

Suppose, by contradiction, that Eq.(17) is nonoscillatory and let \( w \) be the minimal solution of its associated Riccati equation
\[
w' + \frac{1}{t^p} \left( \gamma_p + \frac{\mu_p + \delta}{\log^2 t} \right) + (p-1) (1 + R(t))^{1-q} |w|^q = 0,
\]
where we denote \( R(t) := \frac{\varepsilon}{\log t} \).

Let \( h(t) = t^{\frac{p-1}{p}} \), \( w_h = \Phi\left( \frac{h'}{h} \right) = \left( \frac{p-1}{p} \right)^{p-1} t^{1-p} \) and denote \( \lambda_p = \left( \frac{p-1}{p} \right)^{p-1} \) and \( v = h^p (w - w_h) \).

Then by a direct computation \( v \) is a solution of the equation
\[
v' + \frac{\mu_p + \delta}{t \log^2 t} + \frac{(p-1)}{t} (1 + R(t))^{1-q} \left[ |v + \lambda_p|^q - q (1 + R(t))^{q-1} \Phi^{-1}(\lambda_p) v - \lambda_q^p \right] - q (1 + R(t))^q \Phi^{-1}(\lambda_p) v - \lambda_q^p \left[ 1 - (1 + R(t))^{1-q} \right] = 0.
\]
(18)
Denote now \( f(t) := (1 + R(t))^{q-1} \), and \( F(v) := |v + \lambda_p|^q - q \Phi^{-1}(\lambda_p) f v - \lambda_q^p \). Then we have
\[
F'(v) = q \left[ \Phi^{-1}(v + \lambda_p) - \Phi^{-1}(\lambda_p) f \right] = 0 \iff v = v^* \equiv \lambda_p \left( \Phi(f) - 1 \right).
\]

At this extremal point
\[
F \left( v^* \right) = \lambda_p^q \left[ f^p \left(1 - q \right) +fq -1 \right]
\]
Substituting \( f(t) = (1 + R(t))^{q-1} \) into \( F(v^*) \) and using again the formula \( (1 + x)^q = 1 + \alpha x + o(x) \) as \( x \to 0 \), we obtain
\[
F \left( v^* \right) = \lambda_p^q \left[ \left(1 + R(t) \right)^{p(q-1)} (1-q) + q (1 + R(t))^{q-1} -1 \right]
\]
\[
= \lambda_p^q o \left( R \right) = o \left( R \right), \text{ as } t \to \infty.
\]
Consequently, \( F(v) \geq -\frac{\eta}{\log t} \) for sufficiently large \( t \), where \( \eta \) is any positive constant. Using this estimate in Eq.(18) we get
\[
v' + \frac{\mu_p + \delta}{t \log^2 t} + \frac{(p-1)}{t} (1 + \frac{\varepsilon}{\log^2 t})^{1-q} \left[ |v + \lambda_p|^q - q \left(1 + \frac{\varepsilon}{\log^2 t} \right)^{q-1} \Phi^{-1}(\lambda_p) v - \lambda_q^p \right]
\]
\[
+ \frac{(p-1)}{t} \lambda_q^p \left[ 1 - 1 + \frac{q-1}{2 \log^2 t} + o(1) \right] = 0
\]
and we see that
\[
v' + \frac{\mu_p + \delta}{t \log^2 t} - \frac{\xi}{\log^2 t} < 0,
\]
for large $t$, where the constant $\zeta > 0$ depends on $\varepsilon$ and $\eta$ and $\mu_p + \delta - \zeta > 0$ if $\varepsilon, \eta$ are sufficiently small. Hence $v'(t) < 0$ for large $t$, which means that there exists the limit $v_0 = \lim_{t \to \infty} v(t)$. This limit is finite, because of Lemma 2.1, $w(t) > 0$, i.e., $v(t) = h_p(t)(w(t) - w_h(t)) \geq -h_p(t)w_h(t) = -\lambda_p$.

Next, we show that $v_0 = 0$. If $v_0 \neq 0$, then $|v_0 + \lambda_p|^q - q\Phi^{-1}(\lambda_p)v_0 - \lambda_p^q > \frac{\omega}{2}$ for large $t$. From Eq.(18), using the fact that

$$F(v) = |v + \lambda_p|^q - q\Phi^{-1}(\lambda_p)v - \lambda_p^q + \left[\frac{q(q - 1)\Phi^{-1}(\lambda_p)\varepsilon}{\log^2 t} + \frac{o(1)}{\log^2 t}\right]v$$

we have

$$v(t) - v(T) + \int_T^t \frac{\mu_p + \delta}{s \log^2 s} ds + \int_T^t \frac{(p - 1)\omega}{2s} ds + \int_T^t \frac{o(1)}{s \log^2 s} \leq 0$$

i.e., since $v(t) > -\lambda_p$

$$\int_T^t \frac{\mu_p + \delta}{s \log^2 s} ds + \int_T^t \frac{(p - 1)\omega}{2s} ds + \int_T^t \frac{o(1)}{s \log^2 s} \leq \lambda_p + v(T)$$

for sufficiently large $T$. When $t \to \infty$, we obtain the convergence of the integral $\int_0^\infty \frac{1}{t} dt$ which is a contradiction, so necessarily $\omega$ must be equal to zero, which means that $v_0 = 0$. Using second order Taylor’s expansion, we have

$$(p - 1)\left| |v + \lambda_p|^q - q\Phi^{-1}(\lambda_p)v - \lambda_p^q\right| \geq (1 - \varepsilon)\frac{q}{2}\lambda_p^{q - 2}v^2$$

for $|v|$ sufficiently small. Hence, for $t$ sufficiently large

$$v' + \frac{\mu_p + \delta}{t \log^2 t} + \frac{(1 - \varepsilon)q}{2t}\lambda_p^{q - 2}v^2 + \frac{o(1)}{t \log^2 t}v \leq 0.$$ 

The last inequality is the Riccati type inequality associated with the linear second order differential equation

$$(tx')' + \frac{o(1)}{\log^2 t}x' + (1 - \varepsilon)\frac{q}{2t}\lambda_p^{q - 2}tx + \frac{(1 + \delta - q\lambda_p^{q - 2})(\mu_p + \delta)}{2t \log^2 t}x = 0, \quad (19)$$

which means that Eq.(19) is nonoscillatory. However, if we compute the value of the coefficient by $x$ in Eq.(19), we get the value

$$\frac{q}{2} \left( \frac{p - 1}{p} \right)^{(p - 1)(q - 2)} \frac{1}{2} \left( \frac{p - 1}{p} \right)^{(p - 1)} \left( \frac{p - 1 + \delta - q\lambda_p^{q - 2}}{p - 1} \right) (1 - \varepsilon) = \frac{1}{4} \left( 1 + \frac{\delta}{\mu_p} \right) (1 - \varepsilon)$$

and the value of this constant is greater than $\frac{1}{4}$ if $\varepsilon$ is sufficiently small, i.e., Eq.(19) is oscillatory by Lemma 2.1 which is the required contradiction implying that Eq.(13) is oscillatory.

The main results of this paper are as follows

**Theorem 3.3.**

(i) Let $r$, $c$ and $d$ be positive defined functions in Eq.(6) and Eq.(7) having different periods $\beta_1$, $\beta_2$ and $\beta_3$ respectively. Then Eq.(6) is nonoscillatory if and only if

$$\gamma \leq \gamma^* := \frac{\beta_1^{p - 1}\beta_2\gamma_p}{\left(\int_0^{\beta_1} r^{1-q}(t) \, dt\right)^{p-1} \left(\int_0^{\beta_2} c(t) \, dt\right)}.$$

where $\gamma_p := \left(\frac{p-1}{p}\right)^p$. 

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(ii) In the limiting case $\gamma = \gamma_*$, Eq.(7) is nonoscillatory if

$$\mu < \mu_* := \frac{\beta_1^{p-1}\beta_3\mu p}{\left(\int_0^1 r^{1-q}(t)dt\right)^{p-1} \int_0^1 d(t)dt}$$

and it is oscillatory if $\mu > \mu_*$, where $\mu_p := \frac{1}{2} \left(\frac{p-1}{p}\right)^{p-1}$.

Proof. The statement (i) is proved in [13] when $\gamma \neq \gamma_*$.

(ii) We consider Eq.(7), let $x$ be the nontrivial solution of Eq.(7) and $\varphi$ is the Prüfer angle of Eq.(7) given by Eq.(11). Then $\varphi$ is a solution of

$$\varphi' = \frac{1}{t} \left[ r^{1-q}(t) \cos \varphi \right]^p - \Phi (\cos \varphi \sin \varphi + \frac{1}{(p-1)} \left( \gamma c(t) + \frac{\mu d(t)}{\log^2 t} \right) \sin \varphi \right]^p]$$

By the help of Lemma 3.1, $\theta$ is a solution of

$$\theta' = \frac{1}{t} \left[ R \cos \theta \right]^p - \Phi (\cos \theta \sin \theta + \left( \gamma C + \frac{\mu D}{\log^2 t} \right) \sin \theta \right]^p] + a \left( \frac{1}{t \log^2 t} \right)$$

where $R, C$ and $D$ are as in Lemma 3.1. By the help of Lemma 3.2 if

$$\mu < \mu_* := \frac{\mu_p \beta_1^{p-1}\beta_3}{\left(\int_0^1 r^{1-q}(t)dt\right)^{p-1} \int_0^1 d(t)dt}$$

holds, then $\theta$ is bounded and by the help of Lemma 3.1, $\varphi$ is bounded. Then from Eq.(11), Eq.(7) is nonoscillatory. Again by the help of Lemma 3.2, if

$$\mu > \mu_* := \frac{\mu_p \beta_1^{p-1}\beta_3}{\left(\int_0^1 r^{1-q}(t)dt\right)^{p-1} \int_0^1 d(t)dt}$$

holds, then $\theta(t) \to \infty$ as $t \to \infty$ and by the help of Lemma 3.1, $\varphi$ is unbounded. Then from Eq.(11), Eq.(7) is oscillatory. This result also shows that Eq.(6) is nonoscillatory in the limiting case $\gamma = \gamma_*$ since this case corresponds to Eq.(7) with $\mu = 0 < \mu_*$. The proof is now complete. \qed

Corollary 3.4. If the periods of the functions $r, c$ and $d$ in Eq.(7) coincide with $\alpha$-period, which is given in [6] we get for $\beta_1 = \beta_2 = \beta_3 = \alpha$

$$\mu_* := \frac{\beta_1^{p-1}\beta_3\mu p}{\left(\int_0^1 r^{1-q}(t)dt\right)^{p-1} \int_0^1 d(t)dt} = \frac{\alpha^p \mu p}{\left(\int_0^1 r^{1-q}(t)dt\right)^{p-1} \int_0^1 d(t)dt} = \mu \alpha \beta_1.$$

Thus in this case our oscillation constant $\mu_*$ reduces to $\mu_{rd}$ given in [6] and the main result compiles with the result given by [6].

Corollary 3.5. In a similar way it is easy to see that if there exists a lcm ($\beta_1, \beta_2, \beta_3$) and the period $\alpha$, given in [6] is chosen as the number lcm ($\beta_1, \beta_2, \beta_3$), then there exist some natural numbers $m, l$, and $s$ such as $\alpha = m \beta_1 = \ldots$.
$l \beta_2 = s \beta_3$ and by the help of periodic functions properties we get

$$\mu_* = \frac{\beta_1^{p-1} \beta_3 \mu_p}{\left( \int_{0}^{\alpha} r^{1-q}(t) \, dt \right)^{p-1} \beta_3 \left( \int_{0}^{\alpha} d(t) \, dt \right)}$$

$$= \frac{\left( \frac{\alpha}{m} \right)^{p-1} \frac{\mu_p}{p-1}}{\left( \int_{0}^{\alpha} r^{1-q}(t) \, dt \right)^{p-1} \frac{\alpha}{2} \left( \frac{d(t)}{dt} \right)}$$

$$= \mu_{rd}.$$

Thus in this case our oscillation constant $\mu_*$ reduces to $\mu_{rd}$ given in [6] and the main result compiles with the result given by [6].

**Remark 3.6.** If $\text{lcm}(\beta_1, \beta_2, \beta_3)$ is not defined, then we can not use the results of Remark 3.4 and Remark 3.5. However, only our result can be applied while the result given in [6, 8] cannot be applied.

**Example 3.7.** Consider the equation

$$\left( \left( \frac{1}{2 + \cos 6t} \right) \Phi(x) \right)' + \frac{1}{t^p} \left[ \gamma (2 + \cos 8t) + \frac{\mu (2 + \cos (ax + b))}{\log^2 t} \right] \Phi(x) = 0. \quad (20)$$

which is Eq.(7) for $q = 3$, $r(t) = \frac{1}{2 + \cos 6t}$, $c(t) = 2 + \cos 8t$ and $d(t) = 2 + \cos (ax + b), (a, b \in \mathbb{R})$. In this case $r(t) = \frac{1}{2 + \cos 6t}$ is positive defined for all $t \in \mathbb{R}$ with period $\frac{\pi}{4}$, $c(t) = 2 + \cos 8t$ is positive defined for all $t \in \mathbb{R}$ with period $\frac{\pi}{4}$, and $d(t) = 2 + \cos (ax + b) > 0$ can be considered as positive defined function with period $\frac{2\pi}{|a|}$.

Thus we can apply Theorem 3.3 for all $a \neq 0$ and we obtain an oscillation constant for the differential equation Eq.(20)

$$\mu_* = \frac{\beta_1^{p-1} \beta_3 \mu_p}{\left( \int_{0}^{\alpha} r^{1-q}(t) \, dt \right)^{p-1} \beta_3 \left( \int_{0}^{\alpha} d(t) \, dt \right)}$$

$$= \frac{\left( \frac{\alpha}{m} \right)^{p-1} \frac{\mu_p}{p-1}}{\left( \int_{0}^{\alpha} r^{1-q}(t) \, dt \right)^{p-1} \frac{\alpha}{2} \left( \frac{d(t)}{dt} \right)}$$

$$= \frac{1}{12 \sqrt{2}}.$$

and equation Eq.(20) is nonoscillatory if and only if $\mu < \frac{1}{12 \sqrt{2}}$. This computation shows that we can compute the oscillation constant $\mu_*$ for every period $\frac{2\pi}{|a|} > 0$. If the functions $r(t), c(t),$ and $d(t)$ are having an $\alpha = \text{lcm} \left( \frac{\pi}{4}, \frac{\pi}{4}, \frac{2\pi}{|a|} \right)$ period it is well known that there exists some $m, l$ and $s$ natural numbers such as $\alpha = m \frac{\pi}{4} = l \frac{\pi}{4} = s \frac{2\pi}{|a|}$. In this case we use the fact of the remark 3.4 and we can apply Theorem 3.1 in [6] to the above example and we get oscillation constant as

$$\mu_{rd} = \frac{\alpha^p \mu_p}{\left( \int_{0}^{\alpha} r^{1-q}(t) \, dt \right)^{p-1} \int_{0}^{\alpha} d(t) \, dt}$$

$$= \frac{\frac{1}{2 \sqrt{3}} \left( \frac{2\pi}{|a|} \right)^{\frac{3}{2}}}{\left( \int_{0}^{\alpha} (2 + \cos 6t)^2 \, dt \right)^{\frac{1}{2}} \left( \int_{0}^{\alpha} (2 + \cos (ax + b)) \, dt \right)^{\frac{1}{2}}}.$$
Here the important point to note is that while we cannot apply the Theorem 3.1 in [6] for this example if we choose $a = \sqrt{3}$ then $\text{lcm} \left( \frac{\pi}{3}, \frac{\pi}{4}, \frac{2\pi}{101} \right)$ is not defined, we can apply our theorem Theorem 3.3.

Finally, as a future work this paper can be improved if we replace the periodic coefficient functions, having different periods, with asymptotically almost periodic coefficients or having different mean values coefficients functions. The conditional oscillation of half-linear equations with asymptotically almost periodic coefficient or coefficients having mean values are studied in [13–15].

Conflict of interests
The authors declare that there is no conflict of interest regarding the publication of this paper.

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