Binomials transformation formulae for scaled Fibonacci numbers

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Abstract: The aim of the paper is to present the binomial transformation formulae of Fibonacci numbers scaled by complex multipliers. Many of these new and nontrivial relations follow from the fundamental properties of the so-called delta-Fibonacci numbers defined by Wituła and Słota. The paper contains some original relations connecting the values of delta-Fibonacci numbers with the respective values of Chebyshev polynomials of the first and second kind.

Keywords: δ-Fibonacci numbers, δ-Lucas numbers, Binomial transformation

MSC: 11B39, 11B83

1 Introduction

Wituła and Słota in [1] defined the δ-Fibonacci numbers \(a_n(\delta), b_n(\delta), n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\) in the following way

\[
\left(1 - \frac{1 + \sqrt{5}}{2} \delta\right)^n = a_n(\delta) - \frac{1 + \sqrt{5}}{2} b_n(\delta), \quad \delta \in \mathbb{C}, \quad n \in \mathbb{N}_0.
\]  

or equivalently by the relation

\[
\left(1 + \frac{\sqrt{5} - 1}{2} \delta\right)^n = a_n(\delta) + \frac{\sqrt{5} - 1}{2} b_n(\delta), \quad \delta \in \mathbb{C}, \quad n \in \mathbb{N}_0.
\]

From this, one can easily conclude that

\[
a_0(\delta) = 1, \quad b_0(\delta) = 0,
\]

and next, the following recurrence relations hold true

\[
a_{n+2}(\delta) = (2 - \delta)a_{n+1}(\delta) + (\delta^2 + \delta - 1)a_n(\delta), \quad n \in \mathbb{N}_0
\]

\[
b_{n+2}(\delta) = (2 - \delta)b_{n+1}(\delta) + (\delta^2 + \delta - 1)b_n(\delta), \quad n \in \mathbb{N}_0
\]
for every $n \in \mathbb{N}_0$. We note that $a_1(\delta) = 1$ and $b_1(\delta) = \delta$.

We note also that $\delta$-Fibonacci numbers are the simplest members of the family of the quasi-Fibonacci numbers of $(k$-th, $\delta$-as) order (see [1–3] and the references therein). So the following natural question arises: does there exist some algebraic relation connecting $a_n(\delta)$ and $b_n(\delta)$, $n \in \mathbb{N}_0$, $\delta \in \mathbb{C}$, with the Fibonacci numbers? The answer is positive. There exist two basic relations of such type resulting from two following fundamental properties of $a_n(\delta)$ and $b_n(\delta)$ (see formulae (1.1), (3.14), (3.15), (5.6), (5.7) in [1]):

$$a_n(1) = F_{n+1}, \quad b_n(1) = F_n,$$

and

$$a_n(\delta) = \sum_{k=0}^{n} \binom{n}{k} F_{k-1} (-\delta)^{k-1} = \sum_{k=0}^{n} \binom{n}{k} F_{k+1} (1-\delta)^{n-k} \delta^{k},$$

and

$$b_n(\delta) = \sum_{k=1}^{n} \binom{n}{k} (-1)^{k} F_k \delta^{k} = \sum_{k=1}^{n} \binom{n}{k} F_k (1-\delta)^{n-k} \delta^{k},$$

for every $n \in \mathbb{N}_0$. We note that one can easily verify all relations (6)-(8) by induction (on the basis of recurrence relations (4) and (5)).

Moreover, it follows from (7) and (8) that the sequences $\{a_n(\delta)\}_{n=0}^{\infty}$ and $\{b_n(\delta)\}_{n=0}^{\infty}$ are the binomial transformation (denoted for shortness by $Binom(\cdot)$) of sequences $\{F_{k-1} (-\delta)^{k}\}_{k=0}^{\infty}$ and $\{-F_k (-\delta)^{k}\}_{k=0}^{\infty}$, respectively. Similarly, we have

$$\left\{ \frac{a_n(\delta)}{(1-\delta)^n} \right\}_{n=0}^{\infty} = Binom \left( \left\{ F_{k+1} \left( \frac{\delta}{1-\delta} \right)^{k} \right\}_{k=0}^{\infty} \right)$$

and

$$\left\{ \frac{b_n(\delta)}{(1-\delta)^n} \right\}_{n=0}^{\infty} = Binom \left( \left\{ F_k \left( \frac{\delta}{1-\delta} \right)^{k} \right\}_{k=0}^{\infty} \right).$$

We note also that, after [4], from (7) and (8) one can derive the generating functions $A(x; \delta)$ and $B(x; \delta)$ of $\{a_n(\delta)\}_{n=0}^{\infty}$ and $\{b_n(\delta)\}_{n=0}^{\infty}$, respectively. For example, we obtain

$$B(x; \delta) = \frac{1}{1-(1-\delta)x} F \left( \frac{x}{1-(1-\delta)x} \right) = \frac{\delta x}{(1-\delta-x^2)^2(x^2 + (\delta-2)x + 1)},$$

where $F(x) = \frac{x}{1-x-x^2}$ is the generating function of the Fibonacci numbers. However, with regard to the length of our paper, we will not make use of the functions $A(x; \delta)$ and $B(x; \delta)$ as the alternative sources for deriving the presented here formulæ.

**Reflections**

The $\delta$-Fibonacci numbers [1] and the $\delta$-Lucas numbers [5] represent the simplest "quasi-Fibonacci" numbers of any order defined by R. Witula and D. Slota (see [1, 2, 5, 6] and the references therein), which in fact are the recursively defined polynomials of the respective order. The above numbers have been introduced so that their definitions indicate an easy way for generating many standard identities for these numbers (including the sums of powers, the sums of the respective scalar products). Relations defining the quasi-Fibonacci numbers constitute the "platform" for discussing the recursively defined sequences, alternative for Binet’s formulæ or generating functions — very effective platform, according to us. It is worth to emphasize that all the quasi-Fibonacci numbers seem to exist independently of the background of the recurrence sequences discussed in literature.

Since the $\delta$-Fibonacci numbers and the $\delta$-Lucas numbers are in fact the binomial transformations of the scaled sequences of Fibonacci and Lucas numbers, thus in some moment we realized that it would be welcome, for emphasizing the meaning of these numbers, to have at our disposal the general formulæ for $\delta$-Fibonacci numbers and $\delta$-Lucas numbers for "the most generally expressed" parameters $\delta$ from the set of complex numbers. So, these are the roots of this work.
2 Main results

First we will show relations between the special cases of $\delta$-Fibonacci numbers for complex values of $\delta$ and the Fibonacci and Lucas numbers.

**Theorem 2.1.** Let $\delta = -\frac{1}{2}(1 + i \sqrt{5} \tan \varphi)$ and $\varphi \neq \frac{\pi}{2}(2\mathbb{Z} + 1)$. Then

$$b_n(\delta) = -\left(\frac{\sqrt{5}}{2 \cos \varphi}\right)^n \left[\begin{array}{c} \frac{F_n T_n(\cos \varphi) + i \frac{\sqrt{5}}{2} L_n \sin \varphi U_{n-1}(\cos \varphi)}{\frac{\sqrt{5}}{2} L_n T_n(\cos \varphi) + i F_n \sin \varphi U_{n-1}(\cos \varphi)} \end{array}\right],$$

for even $n \in \mathbb{N}_0$, and

$$b_n(\delta) = -\left(\frac{\sqrt{5}}{2 \cos \varphi}\right)^n \left[\begin{array}{c} \frac{F_n T_n(\cos \varphi) + i \frac{\sqrt{5}}{2} L_n \sin \varphi U_{n-1}(\cos \varphi)}{\frac{\sqrt{5}}{2} L_n T_n(\cos \varphi) + i F_n \sin \varphi U_{n-1}(\cos \varphi)} \end{array}\right],$$

for odd $n \in \mathbb{N}$,

where $T_n$ and $U_n$ are the $n$-th Chebyshev polynomials of the first and second kind, respectively.

Indeed, we get

$$\sqrt{5}b_n\left(-\frac{1}{2}(1 + i \sqrt{5} \tan \varphi)\right) = \left(1 - \frac{\sqrt{5} - 1 \frac{1 + i \sqrt{5} \tan \varphi}{2}\right)^n - \left(1 + \frac{\sqrt{5} + 1 \frac{1 + i \sqrt{5} \tan \varphi}{2}\right)^n$$

$$= \left(\frac{1}{4}(5 - \sqrt{5} - i(\sqrt{5} - 1) \sqrt{5} \tan \varphi)\right)^n - \left(\frac{1}{4}(5 + \sqrt{5} + i(\sqrt{5} + 1) \sqrt{5} \tan \varphi)\right)^n$$

$$= \left(\frac{\sqrt{5} \beta e^{-i \varphi}}{2 \cos \varphi}\right)^n - \left(\frac{\sqrt{5} \alpha e^{i \varphi}}{2 \cos \varphi}\right)^n = \left(\frac{\sqrt{5}}{2 \cos \varphi}\right)^n \left[(-\beta)^n e^{-in \varphi} - \alpha^n e^{in \varphi}\right]$$

$$= \left(\frac{\sqrt{5}}{2 \cos \varphi}\right)^n \left[\begin{array}{c} -\sqrt{5} F_n \cos n \varphi - i L_n \sin n \varphi \end{array}\right],$$

for even $n \in \mathbb{N}_0$, and

$$= \left(\frac{\sqrt{5}}{2 \cos \varphi}\right)^n \left[\begin{array}{c} -\sqrt{5} F_n \cos n \varphi - i L_n \sin n \varphi \end{array}\right],$$

for odd $n \in \mathbb{N}$,

where we set (and these notations hold in the entire paper):

$$\alpha := \frac{1 + \sqrt{5}}{2}, \quad \beta := \frac{1 - \sqrt{5}}{2}$$

(\alpha denotes the golden ratio) and where we used the Binet’s formulae for the Fibonacci and Lucas numbers, respectively

$$F_n = \alpha^n - \beta^n, \quad L_n = \alpha^n + \beta^n, \quad n = 0, 1, 2, \ldots$$

Similarly we obtain the following theorem.

**Theorem 2.2.** Let $\delta = -\frac{1}{2}(1 + i \sqrt{5} \tan \varphi)$ and $\varphi \neq \frac{\pi}{2}(2\mathbb{Z} + 1)$. Then

$$a_n(\delta) = \left(\frac{\sqrt{5}}{2 \cos \varphi}\right)^n \left[\begin{array}{c} \frac{\sqrt{5} S_{n-1} T_n(\cos \varphi) + i F_{n-1} \sin \varphi U_{n-1}(\cos \varphi)}{\frac{\sqrt{5}}{2} S_{n-1} T_n(\cos \varphi) + i F_{n-1} \sin \varphi U_{n-1}(\cos \varphi)} \end{array}\right],$$

for odd $n \in \mathbb{N}$, and

$$a_n(\delta) = \left(\frac{\sqrt{5}}{2 \cos \varphi}\right)^n \left[\begin{array}{c} \frac{\sqrt{5} S_{n-1} T_n(\cos \varphi) + i F_{n-1} \sin \varphi U_{n-1}(\cos \varphi)}{\frac{\sqrt{5}}{2} S_{n-1} T_n(\cos \varphi) + i F_{n-1} \sin \varphi U_{n-1}(\cos \varphi)} \end{array}\right],$$

for even $n \in \mathbb{N}$.

where again $T_n$ and $U_n$ denote the Chebyshev polynomials.

Indeed, one can deduce that

$$\sqrt{5}a_n\left(-\frac{1}{2}(1 + i \sqrt{5} \tan \varphi)\right) = \left(\frac{\sqrt{5}}{2 \cos \varphi}\right)^n \left[\alpha(-\beta)^n e^{-in \varphi} - \beta \alpha^n e^{in \varphi}\right]$$

$$= \left(\frac{\sqrt{5}}{2 \cos \varphi}\right)^n \left[(-\beta)^n e^{-in \varphi} + \alpha^n e^{in \varphi}\right]$$
Now we focus on the special cases of $\varphi$. One may note that if $\varphi := \arctan \left( \frac{\sqrt{5}}{2} \tan \varphi_0 \right)$, $\varphi_0 \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right)$, then

$$-\frac{1}{2} (1 + i \sqrt{5} \tan \varphi) = -\frac{1}{2} (1 + i \tan \varphi_0) = -\frac{1}{2 \cos \varphi_0} e^{i \varphi_0},$$

$$\cos \varphi = \frac{1}{\sqrt{1 + \frac{1}{5} \tan^2 \varphi_0}} = \frac{\sqrt{5} \cos \varphi_0}{\sqrt{1 + 4 \cos^2 \varphi_0}},$$

$$\sin \varphi = \frac{\tan \varphi_0}{\sqrt{5 + \tan^2 \varphi_0}} = \frac{\sin \varphi_0}{\sqrt{1 + 4 \cos^2 \varphi_0}}.$$

Thus, as a consequence we obtain

- for $\varphi_0 = \frac{\pi}{3}$:

$$b_n(-e^{i\pi/3}) = -2^{n/2} \times \left[ \frac{F_n T_n(\cos \varphi) + \frac{i}{2} \sqrt{\frac{5}{10}} L_n U_{n-1}(\cos \varphi)}{L_n T_n(\cos \varphi) + \frac{i}{2} \sqrt{\frac{5}{10}} L_n U_{n-1}(\cos \varphi)} \right],$$

for odd $n \in \mathbb{N}$,

$$a_n(-e^{i\pi/3}) = 2^{n/2} \times \left[ \frac{F_{n-1} T_n(\cos \varphi) + \frac{i}{2} \sqrt{\frac{5}{10}} L_n U_{n-1}(\cos \varphi)}{F_{n-1} T_n(\cos \varphi) + \frac{i}{2} \sqrt{\frac{5}{10}} L_n U_{n-1}(\cos \varphi)} \right],$$

for even $n \in \mathbb{N}$,

where $\varphi := \arctan \left( \frac{\sqrt{5}}{2} \right)$ which yields $\cos \varphi = \frac{5}{8} \cdot \sin \varphi = \frac{3}{8}$.

Furthermore, we have (see [7, 8]):

$$T_{2N} \left( \frac{\sqrt{5}}{8} \right) = N \sum_{k=0}^{N} \frac{(-1)^k}{2N - k} \left( \frac{2N - k}{N} \right)^{N-k},$$

$$T_{2N-1} \left( \frac{\sqrt{5}}{8} \right) = \frac{2N - 1}{\sqrt{10}} \sum_{k=0}^{N-1} \frac{(-1)^k}{2N - k} \left( \frac{2N - k - 1}{N} \right)^{N-k},$$

since

$$T_n(x) = \frac{n}{2} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{n-k} \binom{n-k}{k} (2x)^{n-2k}.$$

Similar formulae hold for the values of Chebyshev polynomials of the second kind

$$U_{2N} \left( \frac{\sqrt{5}}{8} \right) = N \sum_{k=0}^{N} \frac{(-1)^k}{2N - k} \left( \frac{5}{2} \right)^{N-k},$$

$$U_{2N-1} \left( \frac{\sqrt{5}}{8} \right) = \sqrt{\frac{2}{5}} \sum_{k=0}^{N-1} \frac{(-1)^k}{2N - k} \left( \frac{5}{2} \right)^{N-k},$$

because of identity

$$U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{n-k} \binom{n-k}{k} (2x)^{n-2k}.$$

Moreover, we observe that all four numbers

$$2b_{2N-1}(-e^{i\pi/3}), 2a_{2N-1}(-e^{i\pi/3}),$$

$$2b_{2N}(-e^{i\pi/3}), 2a_{2N}(-e^{i\pi/3})$$
belong to the number class $\mathbb{Z} + i\sqrt{3}\mathbb{Z}$ since in definition of the coefficients of $T_n(x)$ we have $\frac{n}{n-k} \binom{n-k}{k} \in \mathbb{Z}$, $k \leq \left[\frac{n}{2}\right]$.

- for $\varphi_0 = \frac{\pi}{6}$:

$$b_n\left(-\frac{\sqrt{3}}{3}e^{i\pi/6}\right) = -\left(\frac{4}{3}\right)^{n/2} \times \left[ F_n T_n (\cos \varphi) + \frac{i\sqrt{3}}{\sqrt{3}} L_n U_{n-1} (\cos \varphi) \right], \quad \text{for even } n \in \mathbb{N}_0,$$

$$a_n\left(-\frac{\sqrt{3}}{3}e^{i\pi/6}\right) = \left(\frac{4}{3}\right)^{n/2} \times \left[ \frac{\sqrt{3}}{\sqrt{3}} L_n T_n (\cos \varphi) + \frac{i}{\sqrt{6}} F_n U_{n-1} (\cos \varphi) \right], \quad \text{for odd } n \in \mathbb{N},$$

where $\varphi := \arctan\left(\frac{\sqrt{3}}{15}\right)$ which gives $\cos \varphi = \frac{\sqrt{3}}{4}$, $\sin \varphi = \frac{1}{4}$.

- for $\varphi_0 = \frac{\pi}{4}$:

$$b_n\left(-\frac{\sqrt{2}}{2}e^{i\pi/4}\right) = \left(\frac{3}{2}\right)^{n/2} \times \left[ F_n T_n (\cos \varphi) + \frac{i}{\sqrt{6}} L_n U_{n-1} (\cos \varphi) \right], \quad \text{for even } n \in \mathbb{N}_0,$$

$$a_n\left(-\frac{\sqrt{2}}{2}e^{i\pi/4}\right) = \left(\frac{3}{2}\right)^{n/2} \times \left[ \frac{1}{\sqrt{3}} L_n T_n (\cos \varphi) + \frac{i}{\sqrt{6}} F_n U_{n-1} (\cos \varphi) \right], \quad \text{for odd } n \in \mathbb{N},$$

where $\varphi := \arctan\left(\frac{\sqrt{3}}{2}\right)$ which gives $\cos \varphi = \sqrt{\frac{3}{8}}$, $\sin \varphi = \sqrt{\frac{1}{8}}$.

Now, from (4) and (5) one may also obtain the result given below (in fact, one of the authors deduced these formulae by "observing" the numerical values of $a_n(-i)$ and $b_n(-i)$ for $n = 0, 1, ..., 20$, and then he verified them easily by induction).

**Lemma 2.3.** The following recurrent identities hold

$$
\begin{align*}
\begin{cases}
a_0(-i) = a_1(-i) = 1, \\
a_{n+2}(-i) = (2 + i)(a_{n+1}(-i) - a_n(-i)), \\
a_{2n+2}(-i) = -(2 + i)^n n^3 F_n = 2i - 1)^{n+1} F_n, \\
a_{2n+3}(-i) = -(2 + i)^n n^3 (F_n + i F_{n+1}) \\
= (2i - 1)^n (F_n + i F_{n+1}), \\
b_0(-i) = 0, \\
b_1(-i) = -i, \\
b_{n+2}(-i) = (2 + i)(b_{n+1}(-i) - b_n(-i)), \\
b_{2n}(-i) = (2 + i)^n n^2 F_n = -(2i - 1)^n F_n, \\
b_{2n+1}(-i) = (2 + i)^n n^2 (F_n + i F_{n+1}) \\
= -(2i - 1)^n (F_n + i F_{n+1}),
\end{cases}
\end{align*}
$$

for every $n \in \mathbb{N}_0$.

Now one may deduce from (7) and (8) the corollary.

**Corollary 2.4.** For the Fibonacci numbers the following identities hold

$$
(2i - 1)^{n+1} F_n = \sum_{k=0}^{2n+2} \binom{2n+2}{k} i^k F_{k-1}
= \sum_{k=0}^{2n+2} \binom{2n+2}{k} (1 + i)^{2n+2-k} (-i)^k F_{k+1},
$$

(12)
\[(2i - 1)^n F_n = \sum_{k=1}^{2n} \binom{2n}{k} i^k F_k. \quad (13)\]

\[(2i - 1)^{n+1} (F_n + iF_{n+1}) = \sum_{k=0}^{2n+3} \binom{2n + 3}{k} i^k F_{k-1}. \quad (14)\]

\[(2i - 1)^n (F_n + iF_{n+1}) = \sum_{k=1}^{2n+1} \binom{2n + 1}{k} i^k F_k\]

\[= -\sum_{k=1}^{2n+1} \binom{2n + 1}{k} (1 + i)^{2n+1-k} (-i)^k F_k. \quad (15)\]

for every \(n = 0, 1, \ldots\).

### 3 Some general recurrence relations

The above formulae can be used to generate the interesting general relations for Fibonacci numbers, but to this aim we need some technical result which holds true for both the Fibonacci and Lucas numbers (and even for the Gibonacci numbers).

**Lemma 3.1.** Let \(\alpha_{k,n}, \beta_{k,n} \in \mathbb{C}, k = 0, 1, \ldots, b + c, b, c, n \in \mathbb{N}_0, a, r_0 \in \mathbb{Z}\). Let \(X, Y, Z \in \{F, L\}\) (in fact, one can assume that \(X, Y, Z\) are the Gibonacci numbers - see the respective definition at the end of this section). If there exists an \(r_0 \in \mathbb{Z}\) such that the following equalities

\[X_{a,n+r} = \sum_{k=0}^{b+c} \alpha_{k,n} Y_{k+r} = \sum_{k=0}^{b+c} \beta_{k,n} (-1)^{k-r} Z_{k-r}\]

(16) hold for \(r = r_0, r_0 + 1\) and for every \(n \in \mathbb{N}_0\), then these equalities hold for all \(r \in \mathbb{Z}\) and \(n \in \mathbb{N}_0\).

**Proof.** We discuss here only the case \(r_0 = 0\).

We will prove by induction that for every \(m \in \mathbb{N}\) and \(r \in \mathbb{Z}, -m + 1 \leq r \leq m\), equality (16) is an identity with respect to \(n \in \mathbb{N}_0\). For \(m = 1\) the above thesis follows from the assumption in Lemma. Thus, let us suppose that the statement is true for some \(m \in \mathbb{N}\) and for all \(r \in \mathbb{Z}, -m + 1 \leq r \leq m\). Then we find

\[X_{a,n-m} = X_{a,n-m+2} - X_{a,n-m+1} = \sum_{k=0}^{b+c} \alpha_{k,n} (Y_{k-m+2} - Y_{k-m+1})\]

\[= \sum_{k=0}^{b+c} \beta_{k,n} ((-1)^{k+m} Z_{k+m-2} + (-1)^{k+m} Z_{k+m-1})\]

\[\sum_{k=0}^{b+c} \alpha_{k,n} Y_{k-m} = \sum_{k=0}^{b+c} \beta_{k,n} (-1)^{k+m} Z_{k+m}.\]

\[X_{a,n+m+1} = X_{a,n+m} + X_{a,n+m-1} = \sum_{k=0}^{b+c} \alpha_{k,n} (Y_{k+m} + Y_{k+m-1})\]

\[= \sum_{k=0}^{b+c} \beta_{k,n} ((-1)^{k-m} Z_{k-m} + (-1)^{k-m+1} Z_{k-m+1})\]

\[= \sum_{k=0}^{b+c} \alpha_{k,n} Y_{k+m+1} = \sum_{k=0}^{b+c} \beta_{k,n} (-1)^{k-m-1} Z_{k-m-1}.\]
Hence, in view of the inductive assumption, relation (16) holds for every \( n \in \mathbb{N}_0 \) and \( r \in \mathbb{Z}, -m \leq r \leq m + 1 \). Therefore, by virtue of the mathematical induction rule, the Lemma is true.

From the above Lemma and Corollary 2.4 we get the next result.

**Corollary 3.2.** The following identities hold

\[
(2i - 1)^n F_{n+r} = \sum_{k=0}^{2n} \binom{2n}{k} i^k F_{k+r},
\]

\[
(2i - 1)^n (F_{n+r} + iF_{n+r+1}) = \sum_{k=0}^{2n+1} \binom{2n+1}{k} i^k F_{k+r},
\]

for every \( n \in \mathbb{N}_0 \) and \( r \in \mathbb{Z} \).

We note also that formulae (7) and (8) are not the only technical tools for generating the binomials transform formulae of scaled Fibonacci numbers. One can also apply the following formulae (see formulae (5.1) and (5.2) in [1]).

**Lemma 3.3.** Let \( \delta, \gamma \in \mathbb{C} \) and \( \gamma \neq 0 \). Then

\[
\begin{aligned}
\gamma^n a_n \left( \frac{\delta}{\gamma} \right) &= \sum_{k=0}^{n} \binom{n}{k} (\gamma - 1)^{n-k} a_k(\delta), \\
\gamma^n b_n \left( \frac{\delta}{\gamma} \right) &= \sum_{k=0}^{n} \binom{n}{k} (\gamma - 1)^{n-k} b_k(\delta).
\end{aligned}
\]

As examples of the previous lemma we may also deduce the additional relations between Fibonacci numbers. Let us note

\[
a_n(-1) = F_{2n-1}, \quad b_n(-1) = -F_{2n}.
\]

So, if we set \( \delta = -1, \gamma = -i \), then by using (19), (10) and (11) we obtain

\[
(1 - 2i)^n F_{n-1} = \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k (1 + i)^{2n-k} F_{2k-1}.
\]

\[
(1 - 2i)^n (F_n - iF_{n-1}) = \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^{k+1} (1 + i)^{2n+1-k} F_{2k-1}.
\]

\[
(1 - 2i)^n F_n = \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k (1 + i)^{2n-k} F_{2k}.
\]

\[
(1 - 2i)^n (F_{n+1} - iF_n) = \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^{k+1} (1 + i)^{2n+1-k} F_{2k}.
\]

which implies, by Lemma 3.1, that

\[
(1 - 2i)^n F_{n+r} = \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k (1 + i)^{2n-k} F_{2k+r},
\]

\[
(1 - 2i)^n (F_{n+r} - iF_{n+r-1}) = \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-1)^{k+1} (1 + i)^{2n+1-k} F_{2k+r-1}.
\]

for every \( n \in \mathbb{N}_0 \) and \( r \in \mathbb{Z} \). Similarly, if we set \( \delta = 2, \gamma = 2i \), then we can generate the identities (we note that especially for the left hand side the subtle calculations are needed - more precisely, one should use the auxiliary fact
given in the footnote\textsuperscript{1}):

\begin{equation}
-(1 - 2i)^n (F_{r-1}F_n + F_r F_{n-2}) = 2^{-2n} \sum_{k=0}^{2n} \binom{2n}{k} (2i - 1)^{2n-k} (F_{r+2} - F_{r+1}(-1)^k) 5^{k/2},
\end{equation}

\begin{equation}
(1 - 2i)^n (F_{r-1}F_{n+1} + F_r F_{n-1} - i(F_{r-1}F_n + F_r F_{n-2}))
= 2^{-2n-1} \sum_{k=0}^{2n+1} \binom{2n+1}{k} (2i - 1)^{2n+1-k} (F_{r+2} - F_{r+1}(-1)^k) 5^{k/2},
\end{equation}

for every $n \in \mathbb{N}_0$ and $r \in \mathbb{Z}$ since $a_n(2) = 5^{\lfloor n/2 \rfloor}$ and $b_n(2) = (1 - (-1)^n)5^{\lfloor n/2 \rfloor}$ (see [1]).

Finally, let us note that most of the presented results can be reformulated for the general Gibonacci numbers, i.e., the recurrent sequences $\{G_n\}_{n=-\infty}^{\infty}$ satisfying the recurrent relation

\[ G_{n+2} = G_{n+1} + G_n, \quad n \in \mathbb{Z} \]

(only the recurrence relation is important, the initial conditions are not). It follows easily from the result given below and being the counterpart of Lemma 3.1 for more general case of Gibonacci numbers.

**Theorem 3.4.** Let $a_{k,n} \in \mathbb{C}$, $k = 0, 1, ..., b, n + c, b, c, n \in \mathbb{N}_0$, $a \in \mathbb{Z}$. If the following identity holds with respect to $n \in \mathbb{N}_0$ and $r \in \mathbb{Z}$

\[ F_{an+r} = \sum_{k=0}^{bn+c} a_{k,n} F_{k+r}, \]

then for every Gibonacci sequence $\{G_n\}_{n=-\infty}^{\infty}$ the identity

\[ G_{an+r} = \sum_{k=0}^{bn+c} a_{k,n} G_{k+r} \]

is satisfied for $n \in \mathbb{N}_0$ and $r \in \mathbb{Z}$.

**Proof.** Since $G_n = G_0 F_{n-1} + G_1 F_n$, $n \in \mathbb{Z}$, thus we obtain

\[ \sum_{k=0}^{bn+c} a_{k,n} G_{k+r} = G_0 \sum_{k=0}^{bn+c} a_{k,n} F_{k+r-1} + G_1 \sum_{k=0}^{bn+c} a_{k,n} F_{k+r} = G_0 F_{an+r-1} + G_1 F_{an+r} = G_{an+r}. \]

\[ \Box \]

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**References**


\textsuperscript{1} Let $a_n, G_n \in \mathbb{C}$, $n \in \mathbb{Z}$ and $r_0 \in \mathbb{Z}$. Suppose that $a_{n+2} = a_{n+1} + a_n$, $G_{n+2} = G_{n+1} + G_n$, $n \in \mathbb{Z}$.

If $a_0 = G_{r_0}$, $a_1 = -G_{r_0+1}$, then $a_k = -F_{k-1}G_{r_0-1} - F_{k-2}G_{r_0+1}$, for every $k = 0, 1, 2, ...$

If $a_0 = G_{r_0+1}$, $a_1 = G_{r_0}$, then $a_k = F_k G_{r_0} + F_{k-1} G_{r_0+1}$, for every $k = 0, 1, 2, ...$