Research Article

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Infinitely many solutions for fractional Schrödinger equations with perturbation via variational methods

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Abstract: Using variational methods, we investigate the solutions of a class of fractional Schrödinger equations with perturbation. The existence criteria of infinitely many solutions are established by symmetric mountain pass theorem, which extend the results in the related study. An example is also given to illustrate our results.

Keywords: Fractional Schrödinger equations, Variational methods, Infinitely many solutions

MSC: 26A33, 35A15, 35B20

1 Introduction

In this paper, we are concerned with the existence of solutions of the following fractional Schrödinger equations

\[ (-\Delta)^{\alpha} u + V(x)u = f(x,u) + \lambda h(x)|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1} \]

where \(0 < \alpha < 1, 2\alpha < N, 1 \leq p < 2, \lambda > 0, f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}), h \in L^{\frac{2}{2\alpha}}(\mathbb{R}^N).\) \((-\Delta)^{\alpha}\) is the so-called fractional Laplacian operator of order \(\alpha \in (0, 1)\) and can be defined pointwise for \(x \in \mathbb{R}^N\) by

\[ (-\Delta)^{\alpha} u(x) = -\frac{1}{2} \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2\alpha}} dy \]

along any rapidly decaying function \(u\) of class \(C^\infty(\mathbb{R}^N),\) see Lemma 3.5 of [1]. It can also be characterized as by \((-\Delta)^{\alpha} u = \mathcal{F}^{-1}(|\xi|^{2\alpha} \mathcal{F} u),\) \(\mathcal{F}\) denotes the usual Fourier transform in \(\mathbb{R}^N.\) The potential \(V\) satisfies the following conditions

\[ (V_0) \quad V \in C(\mathbb{R}^N, \mathbb{R}), \inf_{x \in \mathbb{R}^N} V(x) = V_0 > 0 \quad \text{and} \quad \lim_{|x| \to \infty} V(x) = \infty. \]

In [1], the authors proved that \((-\Delta)^{\alpha}\) reduces to the standard Laplacian \(-\Delta\) as \(\alpha \to 1.\) When \(\alpha = 1,\) the problem (1) reduces to the generalized integer order Schrödinger equation. Over the past decades, with the aid of different methods, for various conditions of the potential \(V\) and the nonlinear term \(f,\) the existence and multiplicity of nontrivial solutions for the classical Schrödinger equation have been extensively investigated.

Recently, a great attention has been focused on the study of problems involving the fractional Laplacian. Fractional calculus provide a powerful tool for the description of hereditary properties of various materials and

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memory processes (see [2, 3]). Fractional differential equations play important roles in the modeling of medical, physical, economical and technical sciences. For more details on fractional calculus theory, we refer the readers to
the monographs of Kilbas et al.[4], Lakshmikantham et al. [5], Podlubny [6] and Tarasov [7]. Fractional differential equations involving the Riemann-Liouville fractional derivative or Caputo fractional derivative have received more and more attention.

The fractional Schrödinger equation is a fundamental equation of fractional quantum mechanics. It was discovered by Laskin [8, 9] as a result of extending the Feynman path integral, from the Brownian-like to Lévy-like quantum mechanical paths, where the Feynman path integral leads to the classical Schrödinger equation, and the path integral over Lévy trajectories leads to the fractional Schrödinger equation. Recently, some researchers have investigated the following fractional Schrödinger equations (2) for the different cases of the potential V (for example, V = 1 or V = u) and the nonlinear term f (for example, f(x, u) = |u|^{p-1}u) under the suitable assumptions. We refer the interested readers to[10–20] and the references therein.

\[-\Delta^\alpha u + V(x)u = f(x, u), \ x \in \mathbb{R}^N, \tag{2}\]

In [12], Felmer et al. studied the existence and regularity of solutions for the fractional Schrödinger equations (2) under the famous Ambrosetti-Rabinowitz (A-R) condition, i.e., there exists \(\theta > 2\) such that \(0 < \theta f(x,t) < tf(x,t)\), where \(f(x,u) = \int_0^u f(x,\tau)d\tau\). In [17], by virtue of some nonlinear analysis techniques, the existence of weak solutions for (2) is obtained. In [18], for the case that \(V = 1\) and \(f(x,u) = f(u)\), the authors proved that (2) has at least two nontrivial radial solutions without assuming the A-R condition by variational methods and concentration compactness principle. In [19], using variant Fountain theorems, the author proved the existence of infinitely many nontrivial high or small energy solutions of (2). In [20], the authors obtained the existence of a sequence of radial solutions for \(N \geq 2\), a sequence of non-radial solutions for \(N = 4\) or \(N \geq 6\), and a non-radial solution for \(N = 5\) by the variational methods.

Some researchers also investigated the related problems, see [21–26].

In [27], by using the mountain pass theorem and Ekeland’s variational principle, the author showed that the generalized fractional Schrödinger equations (1) possess two solutions.

Motivated by the work above, in the present paper, we consider the generalized fractional Schrödinger equations (1). By the variational methods, we obtain the existence of infinitely many solutions without the A-R condition. The form of problem (1) is more general than (2). In (1), the nonlinearity involves a combination of superlinear or asymptotically linear terms and a sublinear perturbation. To our best knowledge, the problem (1) has received considerably less attention. We get the existence of infinitely many solutions for (1), which generalize and improve the recent results in the literature.

To state our results, we make the following assumptions.

\[(H_1)\] Let \(f \in C(\mathbb{R}^N \times R, R)\). There exist constants \(a_1, a_2 \geq 0\), \(q \in [2, \frac{2N+4\alpha}{N}] \subset [2, 2q^*] \) with \(\frac{a_1}{2S^2} + \frac{a_2}{qS^q} < \frac{1}{2}\), such that

\[|f(x,u)| \leq a_1|u| + a_2|u|^{q-1}, \ \forall (x,u) \in \mathbb{R}^N \times R,\]

where \(2q^* = \frac{2N}{N-2\alpha}\) with \(2\alpha < N\), \(S_p\) is the best constant for the embedding of \(X \subset L^q(\mathbb{R}^N)\), the details of \(S_p\) and the definition of \(X\) will be given in the section 2.

\[(H_2)\] \[\lim_{|u| \to \infty} \frac{F(x,u)}{|u|^q} = \infty\] uniformly in \(x \in \mathbb{R}^N\) and there exists \(r_0 > 0\) such that \(F(x,u) \geq 0\), for any \(x \in \mathbb{R}^N\) and \(u \in R\) and \(|u| \geq r_0\), where \(F(x,u) = \int_0^u f(x,\tau)d\tau\).

\[(H_3)\] \(2F(x,u) < uf(x,s), \ \forall (x,u) \in \mathbb{R}^N \times R\).

The main results are as follows.

**Theorem 1.1.** Assume the hypotheses \((V_0), (H_1)-(H_3)\) hold. Suppose that \(F(x, -u) = F(x,u)\) for all \((x,u) \in \mathbb{R}^N \times R\) and \(h \in L^{\frac{N}{N-2\alpha}}(\mathbb{R}^N)\), then there exists a constant \(\lambda_0 > 0\), for \(\lambda \in (0, \lambda_0)\), such that (1) possesses infinitely many solutions.

**Remark 1.2.** A-R condition plays an important role in variational methods, which could guarantee boundedness of the sequence. The problem in bounded domain with A-R condition or weaker conditions was studied in many works.
However, in the unbounded domain $R^N$, there are few papers that considered the problem without A-R condition. Furthermore, there are very few papers which replaced A-R condition by $(H_3)$ in $R^N$. Moreover, the form of (1) is more general. Hence, our results can be viewed as extension to the related results of the fractional Schrödinger equation.

Remark 1.3. We can identify $(-\Delta)^\alpha$ with $-\Delta$ when $\alpha = 1$. Hence, Theorem 1.1 is also valid for $\alpha = 1$.

The rest of this paper is organized as follows. In Section 2, some definitions and lemmas which are essential to prove our main results are stated. In Section 3, we give the main results. Finally, one example is offered to demonstrate the application of our main results.

2 Preliminaries

At first, we present the necessary definitions for the fractional calculus theory and several lemmas which will be used further in this paper.

In order to establish a variational structure which enables us to reduce the existence of solutions of problem (1) to one of finding critical points of corresponding functional, it is necessary to construct appropriate function spaces. Let us recall that for any fixed $t \in R^N$ and $1 < q < 1$,

$$
|u|_{1} = \max_{t \in R^N} |u(t)|, \quad \|u\|_{L^q(R^N)} = (\int_{R^N} |u(x)|^q \, dx)^{\frac{1}{q}}.
$$

Throughout this paper, we denote by $\|u\|_q$ the $L^q$-norm, for $1 < q \leq \infty$.

Let

$$
H = H^\alpha(R^N) := \{u \in L^2(R^N) : \int \int_{R^N \times R^N} \frac{|u(x) - u(z)|^2}{|x - z|^{N+2\alpha}} \, dx \, dz < +\infty\}
$$

with the inner product and the norm

$$
\langle u, v \rangle_H = \int \int_{R^N \times R^N} \frac{|u(x) - u(z)||v(x) - v(z)|}{|x - z|^{N+2\alpha}} \, dx \, dz + \int_{R^N} u(x)v(x) \, dx, \quad \|u\|_H = (\langle u, u \rangle_H)^{\frac{1}{2}},
$$

while $|u|_{H^\alpha} = (\int \int_{R^N \times R^N} \frac{|u(x) - u(z)|^2}{|x - z|^{N+2\alpha}} \, dx \, dz)^{\frac{1}{2}}$ is the Gagliardo(semi) norm. The space $H^\alpha(R^N)$ can also be described by means of the Fourier transform, which can be denoted by

$$
H^\alpha(R^N) := \{u \in L^2(R^N) : \int_{R^N} (1 + |\xi|^2)^\alpha |\mathcal{F}u(\xi)|^2 \, d\xi < +\infty\},
$$

and the norm is defined as

$$
\|u\|_H = \left( \int_{R^N} (1 + |\xi|^2)^\alpha |\mathcal{F}u(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}.
$$

In the following, we introduce the definition of Schwartz function $\delta$ (is dense in $H^\alpha(R^N)$), that is, the rapidly decreasing $C^\infty$ function on $R^N$. If $u \in \Delta$, the fractional Laplacian $(-\delta)^\alpha$ acts on $u$ as

$$
(-\Delta)^\alpha u(x) = C(N, \alpha) \text{ P.V.} \int_{R^N} \frac{u(x) - u(y)}{|x - y|^{N+2\alpha}} \, dy = C(N, \alpha) \lim_{\varepsilon \to 0^+} \int_{R^N \setminus B(0, \varepsilon)} \frac{u(x) - u(y)}{|x - y|^{N+2\alpha}} \, dy
$$

where the symbol P.V. represents the principal value integrals, the constant $C(N, \alpha)$ depends only on the space dimension $N$ and the order $\alpha$, and it is explicitly given by the formula

$$
\frac{1}{C(N, \alpha)} = \int_{R^N} \frac{1 - \cos \xi}{|\xi|^{N+2\alpha}} \, d\xi.
$$
In [1], the authors show that for all $u \in \delta$,

$$(-\Delta)^\alpha = F^{-1}(\|\cdot\|^{2\alpha} F u)$$

and

$$[u]_{H^\alpha} = \left(\frac{2}{C(N, \alpha)} \int_{\mathbb{R}^N} |\xi|^{2\alpha} |F u|^2 d\xi\right)^{\frac{1}{2}}.$$ 

Furthermore, from the Plancherel formula in Fourier analysis, we can easily see that

$$[u]_{H^\alpha}^2 = \frac{2}{C(N, \alpha)} \|(-\Delta)^\frac{\alpha}{2}\|^2_2.$$ 

Hence, the norms on $H^\alpha(\mathbb{R}^N)$ defined below are all equivalent:

$$\|u\|_H = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(z)|^2}{|x - z|^{N+2\alpha}} dxdz + \int_{\mathbb{R}^N} |u(x)|^2 dx\right)^{\frac{1}{2}},$$

$$\|u\|_H = \left(\int_{\mathbb{R}^N} (1 + |\xi|^{2\alpha})(|F u(\xi)|)^2 d\xi\right)^{\frac{1}{2}}, \quad \|u\|_H = \left(\int_{\mathbb{R}^N} |u(x)|^2 dx + \|(-\Delta)^{\frac{\alpha}{2}}\|^2_2\right)^{\frac{1}{2}},$$

$$\|u\|_H = \left(\int_{\mathbb{R}^N} |u(x)|^2 dx + \int_{\mathbb{R}^N} |\xi|^{2\alpha} |F u(\xi)|^2 d\xi\right)^{\frac{1}{2}}.$$ 

In order to investigate the problem (1), we define the following space

$$X = \{u \in L^2(\mathbb{R}^N); \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(z)|^2}{|x - z|^{N+2\alpha}} dxdz + \int_{\mathbb{R}^N} V(x)u(x)^2 dx < +\infty\}$$

with the inner product and the norm

$$\langle u, v \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(z)|^2}{|x - z|^{N+2\alpha}} dxdz + \int_{\mathbb{R}^N} V(x)u(x)v(x) dx, \quad \|u\|^2 = \langle u, u \rangle.$$ 

It is easy to see that $X$ is a Hilbert space with the inner product $\langle u, v \rangle$, $X \subset H$ and $X \subset L^q(\mathbb{R}^N)$ for all $q \in [2, 2\alpha^*_N)$ with the embeddings being continuous.

**Lemma 2.1** ([27]). Assume that the condition $(V_0)$ holds. Then there exists a constant $c_0 > 0$ such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(z)|^2}{|x - z|^{N+2\alpha}} dxdz + \int_{\mathbb{R}^N} V(x)u(x)^2 dx \geq c_0 \|u\|^2_H, \quad \forall u \in H. \quad (3)$$

**Lemma 2.2** ([27]). Assume that the condition $(V_0)$ holds. Then $X$ is compactly embedded in $X \subset L^q(\mathbb{R}^N)$ for all $q \in [2, 2\alpha^*_N)$.

**Remark 2.3.** Lemma 2.2 implies that $S_r \|u\|_r \leq \|u\|$, where $S_r > 0$ is the best constant for the embedding of $X \subset L^q(\mathbb{R}^N)$.

We consider the functional $\varphi: X \to \mathbb{R}$, defined by

$$\varphi(u) = \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^N} F(x, u)dx - \frac{\lambda}{p} \int_{\mathbb{R}^N} h(x)|u|^p dx. \quad (4)$$
Then \( \varphi \) is continuously differentiable under the assumption \((H_1)\), and
\[
\langle \varphi'(u), v \rangle = \int_{R^N} \int_{R^N} \frac{[u(x) - u(z)][v(x) - v(z)]}{|x - z|^{N+2\alpha}} \, dx \, dz + \int_{R^N} V(x)u(x)v(x) \, dx \\
- \int_{R^N} f(x, u(x))v(x) \, dx - \lambda \int_{R^N} h(x)|u(x)|^{p-2}u(x)v(x) \, dx,
\]
for all \( u, v \in X \). Hence, the critical point of \( \varphi \) is the solution of problem (1). Next, we only need to consider the critical point of \( \varphi \).

**Definition 2.4.** Let \((E, \| \cdot \|)\) be a Banach space and \( \varphi \in C^1(E, R) \). We say \( \varphi \) satisfies \((Cerami)_c\) condition, if any sequence \( \{u_m\} \subset E \) for which \( \varphi(u_m) \to c \) and \( \|\varphi'(u_m)\|((1 + \|u_m\|) \to 0 \) as \( m \to 0 \) posses a convergent subsequence, and \( \{u_m\} \) is called a \((Cerami)_c\) sequence.

From now on, we will denote \((Cerami)_c\) condition by the abbreviation \((C)_c\) condition.

**Lemma 2.5 ([28] Theorem9.12).** Let \((E, \| \cdot \|)\) be an infinite dimensional Banach space, \( X = Y \oplus Z \) with \( \dim Y < \infty \). Let \( \varphi \in C^1(X, R) \) be an even functional, which satisfies the \((C)_c\) condition and \( \varphi(0) = 0 \). In addition, if \( \varphi \) satisfies:

1. There exist \( a, \rho > 0 \) such that \( \varphi|_{B_{\rho} \cap Z} \geq a \) where \( B_{\rho} = \{u \in X: \|u\| < \rho\} \),
2. For any finite dimensional subspace \( W \subset X \), there exists \( R(W) \) such that \( \varphi(u) \leq 0 \) on \( W \setminus B_{R(W)} \).

Then, \( \varphi \) possesses an unbounded sequence of critical values.

### 3 Main results

Without loss of generality, if we take a subsequence \( \{u_m\} \), we also use the same notation \( \{u_m\} \).

**Lemma 3.1.** Assume that \((V_0)\), \((H_1)-(H_3)\) hold. Then there exists a constant \( \lambda_0 > 0 \), for any \( \lambda \in (0, \lambda_0) \), \( \varphi \) satisfies the \((C)_c\) condition.

**Proof.** Let \( \{u_m\} \subset X \) be a \((C)_c\) sequence, that is
\[
\varphi(u_n) \to c, \quad \|\varphi'(u_n)\|((1 + \|u_n\|) \to 0, \text{ as } n \to \infty,
\]
which also implies
\[
\langle \varphi'(u_n), u_n \rangle \to 0, \text{ as } n \to \infty.
\]
First, we prove that \( \{u_m\} \) is bounded.

We argue it by contradiction. If \( \{u_m\} \) is unbounded in \( X \), then there exists a subsequence \( \{u_m\} \) with \( \|u_n\| \to \infty \), as \( n \to \infty \). From (4)-(5) and the Hölder inequality, one has
\[
\varphi(u_n) - \frac{\langle \varphi'(u_n), u_n \rangle}{2} = -\int_{R^N} F(x, u_n) \, dx - \left(\frac{1}{p} - \frac{1}{2}\right) \lambda \int_{R^N} h(x)|u_n|^p \, dx + \frac{1}{2} \int_{R^N} f(x, u_n)u_n \, dx \\
\geq -\left(\frac{1}{p} - \frac{1}{2}\right) \lambda \int_{R^N} h(x)|u_n|^p \, dx \\
\geq -\left(\frac{1}{p} - \frac{1}{2}\right) \lambda_0 \|h\|_{2,p} \|u_n\|_2^p.
\]
By (6)-(7), we can easily get that \( \|u_n\|_2 \) is bounded.

Also from (4)-(5), we have
\[
\langle \varphi'(u_n), u_n \rangle = \|u_n\|_2 - \int_{R^N} f(x, u_n)u_n \, dx - \lambda \int_{R^N} h(x)|u_n|^p \, dx.
\]
Then, it follows from \((H_1)\) that

\[
\|u_n\|_2 \leq \langle \varphi'(u_n), u_n \rangle + \int_{\mathbb{R}^N} f(x, u_n) u_n \, dx + \lambda_0 \|h\|_{L^{\frac{2}{2-\alpha}}} \|u_n\|_2^2
\]

\[
\leq \langle \varphi'(u_n), u_n \rangle + a_1 \|u_n\|_2^2 + a_2 \|u_n\|_q^q + \lambda_0 \|h\|_{L^{\frac{2}{2-\alpha}}} \|u_n\|_2^2. \tag{8}\]

By the Fractional Gagliardo-Nirenberg inequality ([29], corollary 2.3) and the definition of the norm in \(X\), we know that

\[
\|u\|_{L^q(R^N)} \leq \xi^\frac{1}{q} \|u\|_{L^2(R^N)}^{1-\frac{1}{q}}, \tag{9}\]

where \(\xi = 2^{-\alpha} \pi^{-\frac{N}{2}} \left( \frac{\Gamma(N)}{\Gamma(N/2)} \right)^{-\frac{1}{2}} \Gamma \left( \frac{N-\alpha}{2} \right) \Gamma \left( \frac{N+\alpha}{2} \right)\) and \(s \left( \frac{1}{2} - \frac{\alpha}{N} \right) + \frac{\alpha}{2} - s = 1\).

For \(0 < s = \frac{(q-2)N}{2\alpha} < q \in [2, \frac{2N+4\alpha}{N}] \subset [2, 2^*_N)\), it is easy to see that

\[0 < s < 2. \tag{10}\]

Then it follows (8)-(10) that

\[
1 = \frac{\|u_n\|_2^2}{\|u_n\|_2^2} \leq \frac{\langle \varphi'(u_n), u_n \rangle + a_1 \|u_n\|_2^2 + a_2 \|u_n\|_q^q}{\|u_n\|_2^2} + \frac{\lambda_0 \|h\|_{L^{\frac{2}{2-\alpha}}} \|u_n\|_2^2}{\|u_n\|_2^2}
\]

\[
\leq \frac{\langle \varphi'(u_n), u_n \rangle + a_1 \|u_n\|_2^2 + a_2 \|u_n\|_q^q}{\|u_n\|_2^2} + \frac{\lambda_0 \|h\|_{L^{\frac{2}{2-\alpha}}} \|u_n\|_2^2}{\|u_n\|_2^2}
\]

\[\to 0, \text{ as } n \to \infty. \]

This is a contradiction. Hence, we know that \(\{u_n\}\) is bounded in \(X\).

Next we prove \(\varphi\) satisfies the \((C)\_c\) condition.

For any \(\{u_n\} \subset X\) being a \((C)_c\) sequence, from the boundedness of \(\{u_n\}\), we know there exists a weakly convergent subsequence \(\{u_n\}\) such that \(u_n \rightharpoonup u\) weakly in \(X\). From Lemma 2.2, we can obtain that \(u_n \to u\) strongly in \(L^q(R^N)\) for \(q \in [2, \frac{2N+4\alpha}{N}]\).

Then we prove that \(u_n \to u\) in \(X\).

From (4)-(5), we have

\[
\|u_n - u\|_2^2 = \langle \varphi'(u_n) - \varphi'(u), u_n - u \rangle + \int_{\mathbb{R}^N} [f(x, u_n(x)) - f(x, u(x))](u_n - u) \, dx
\]

\[+ \lambda \int_{\mathbb{R}^N} h(x)|u_n - u|^p \, dx. \tag{11}\]

It is easy to see that

\[
\langle \varphi'(u_n) - \varphi'(u), u_n - u \rangle \to 0, \text{ as } n \to \infty. \tag{12}\]

Based on the fact that \(h \in L^{\frac{2}{2-\alpha}}(R^N), u_n \to u\) in \(L^q(R^N)\) and Hölder inequality, one has

\[
\lambda \int_{\mathbb{R}^N} h(x)|u_n - u|^p \, dx \leq \lambda_0 \|h\|_{L^{\frac{2}{2-\alpha}}} \|u_n\|_2^2 \to 0, \text{ as } n \to \infty. \tag{13}\]

From \((H_1)\) and the Hölder inequality, we can obtain

\[
\int_{\mathbb{R}^N} [f(x, u_n(x)) - f(x, u(x))](u_n - u) \, dx \leq \int_{\mathbb{R}^N} [f(x, u_n(x)) - f(x, u(x))]|u_n - u| \, dx
\]

\[
\leq \int_{\mathbb{R}^N} a_1(u + u_n) + a_2(|u|^q - 1 + |u_n|^q - 1)|u_n - u| \, dx
\]

\[
\leq a_1 \|u\|_2 \|u_n - u\|_2 + a_2 (\|u_n\|_q^q - 1 + \|u\|_q^q - 1) \|u_n - u\|_q
\]

\[\to 0, \text{ as } n \to \infty. \tag{14}\]
It follows from (11)-(14) that \( \|u_n - u\|^2 \to 0 \), which shows that \( u_n \to u \) in \( X \). Hence, \( \varphi \) satisfies the \((C)_c\) condition. We complete the proof of Lemma 3.1.

Let \( \{e_j\} \) be a total orthonormal basis of \( X \). We define
\[
X_j := \text{span}\{e_j\}, \quad Y_k := \oplus_{j=1}^{k} X_j \quad \text{and} \quad Z_k = \overline{\oplus_{j=k+1}^{\infty} X_j}, \quad k \in \mathbb{N}.
\]
Clearly, \( X = Y_k \oplus Z_k \) with \( \text{dim} \ Y_k < \infty \).

**Lemma 3.2.** Assume that \((V_0), (H_1)\)-(\(H_3\)) hold. Then there exist constants \( a, \rho > 0 \) such that \( \varphi|_{\partial B_\rho \cap Z_k} \geq a \), where \( B_\rho = \{ u \in X : \|u\| < \rho \} \).

**Proof.** From \((H_1)\), (4) and Hölder inequality, for any \( u \in Z_k, q \in [2, \frac{2N+4\alpha}{N}) \), we have
\[
\varphi(u) = \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^N} F(x, u) dx - \frac{\lambda}{p} \int_{\mathbb{R}^N} h(x)|u|^p dx \\
\geq \frac{1}{2} \|u\|^2 - \frac{a_1}{2} \|u\|^2 - \frac{a_2}{q} \|u\|^q - \lambda_0 \|h\|_{\frac{2}{2-q}} \|u\|^2 \\
= \frac{1}{2} \|u\|^2 - \frac{a_1}{2} \|u\|^2 - \frac{a_2}{2} \|u\|^q - \lambda_0 \|h\|_{\frac{2}{2-q}} \|u\|^2.
\]
We denote \( c = \frac{1}{2} - (\frac{a_1}{2S^q_2} + \frac{a_2}{qS^q_2}) \), then \((H_1)\) implies \( c > 0 \). Let \( \alpha_k = \sup_{u \in Z_k, \|u\|=1} \|u\|, \) for \( r \in [1, 2^*] \).

By a similar proof to the Lemma 3.2 of [19], we can deduce that \( \alpha_k \to 0 \) as \( k \to \infty \). Assume \( 0 < \rho \leq 1 \), then we have
\[
\varphi|_{\partial B_\rho \cap Z_k} \geq c \|u\|^2 - \lambda_0 \|h\|_{\frac{2}{2-q}} \alpha_k \|u\|^q = \|u\|^p[c \|u\|^{2-p} - \lambda_0 \|h\|_{\frac{2}{2-q}} \alpha_k^p].
\]
Then it is easy to see that we can choose constant \( \rho \in (0, 1] \) such that \( \varphi|_{\partial B_\rho \cap Z_k} > 0 \), as \( k \to \infty \), which implies the conclusion of Lemma 3.2.

**Lemma 3.3.** Assume that \((V_0), (H_1)-(H_3)\) hold. Then for any finite dimensional subspace \( W \subset X \), there exists \( R = R(W) \) such that \( \varphi(u) \leq 0 \) on \( W \setminus B_R(W) \).

**Proof.** First, we prove that for any finite dimensional subspace \( W \subset X \) and \( \|u\| \to \infty, \) \( u \in W \), there holds \( \varphi(u) \to -\infty \).

On the contrary, we assume that for some sequence \( \{u_n\} \subset W \) with \( \|u_n\| \to \infty \), there exists a positive constant \( M > 0 \) such that
\[
\varphi(u_n) \geq -M.
\] (15)
Let \( v_n = \frac{u_n}{\|u_n\|} \), then \( \|v_n\| = 1 \). From the boundedness of \( v_n \), there exists a weakly convergent subsequence \( v_n \) such that \( v_n \to v \) weakly in \( X \). Then for the finite dimensional subspace \( W \subset X \), we know that \( v_n \to v \) strongly in \( W \). By the equivalence of finite dimensional spaces, we can get that \( v_n \to v \) a.e. in \( \mathbb{R}^N \), which also implies \( \|v\| = 1 \).

Let \( V := \{ x \in \mathbb{R}^N : \varphi(x) \neq 0 \} \), then we know the measure of the set \( V \) is positive, i.e., \( \text{meas}(V) > 0 \). Hence, for \( x \in V \), from \( u_n = \|u_n\|v_n \) we can deduce
\[
|u_n| \to \infty, \quad \text{as} \quad n \to \infty.
\] (16)
For any \( 0 < x_1 < x_2 \), we denote \( \Omega_n(x_1, x_2) = \{ x \in \mathbb{R}^N : x_1 \leq |u_n| < x_2 \} \), which implies \( \Omega_n(x_1, x_2) \subset V \). Then for \( n \) large enough, it is easy to see that \( \chi_{\Omega_n(r_0, \infty)}(x) = 1 \), where \( \chi_{\Omega_n(r_0, \infty)} \) is the characteristic function on \( \Omega_n(r_0, \infty) \), \( r_0 > 0 \) is given in \((H_2)\). Hence, we have
\[
\chi_{\Omega_n(r_0, \infty)}(x)v_n \to v, \quad \text{as} \quad n \to \infty, \quad \text{for a.e.} \quad x \in \Omega_n(r_0, \infty).
\] (17)
On one hand, from (4), (15) and Remark 2.3, for any \( \lambda \in (0, \lambda_0) \), we have
\[
\lim_{n \to +\infty} \frac{\int_{R^N} F(x, u_n) \, dx}{\| u_n \|^2} = \lim_{n \to +\infty} \frac{\frac{1}{2} \| u_n \|^2 - \frac{\lambda}{p} \int_{R^N} h(x) |u_n|^p \, dx - \varphi(u_n)}{\| u_n \|^2} 
\leq \lim_{n \to +\infty} \left( \frac{\frac{1}{2} \| u_n \|^2 + \frac{\lambda_0}{2} \| h \|_{\frac{2 - q}{q}} \| u \|^p + M}{\| u_n \|^2} \right) = \frac{1}{2}.
\]
which is a contradiction with (18). Therefore \( \varphi(u) \to -\infty \) for \( \| u \| \to \infty, u \in W \). Hence, we can easily choose \( R = R(W) \) such that \( \varphi(u) \leq 0 \) on \( W \setminus B_{R(W)} \). Then we complete the proof.

**Proof of Theorem 1.1.** Let \( Y = Y_k, Z = Z_k \), then \( X = Y \oplus Z \) with \( \dim Y < \infty \). From the condition that \( F(x, -u) = F(x, u) \) and \( h \in L^{\frac{2 - q}{q}}(R^N) \), we know \( \varphi \) is even and \( \varphi(0) = 0 \). Lemma 3.1-3.3 imply that \( \varphi \) satisfies other conditions of Theorem 2.5. Consequently, we can deduce that \( \varphi \) possesses an unbounded sequence of critical values, which are the solutions of the fractional Schrödinger equation (1).

Finally, we give one example to illustrate the usefulness of our main result. Consider the following fractional Schrödinger equations.

**Example 3.4.**
\[
(-\Delta)^{\frac{1}{2}} u + (1 + x^2) u = \frac{8S^{S/3}}{8^{S/3}} (\sin^2 x) u^{S/3} - \frac{\ln (1 + |\sin x|^2)}{e^{3x}(1 + x^2)} |u|^{-1} u, \quad x \in R^2.
\]

Obviously, \( \alpha = 1/2, N = 2, p = 1, \lambda = 1, f(x, u) = \frac{8S^{S/3}}{8^{S/3}} (\sin^2 x) u^{S/3} \) is continuous, \( V(x) = 1 + x^2 \) and \( h(x) = \frac{\ln (1 + |\sin x|^2)}{e^{3x}(1 + x^2)} \) is a \( L^2 \) integrable function.

First, we can see that \( 2\alpha = 1 < N = 2, F(x, u) = \frac{8S^{S/3}}{8^{S/3}} (\sin^2 x) u^{S/3} \), then we have
\[
|f(x, u)| = \left| \frac{8S^{S/3}}{8^{S/3}} (\sin^2 x) u^{S/3} \right| \leq \frac{8S^{S/3}}{8^{S/3}} |u|^{S/3}.
\]
with \( \frac{S}{3} = q \in [2, 3) \subset [2, 2^*_q = 4) \), \( a_1 = 0, a_2 = \frac{S^{S/3}}{8^{S/3}} \), and \( \frac{a_1}{2^*_q} + \frac{a_2}{q^*_S} = \frac{3}{8} < \frac{1}{2} \), which shows that \( (H_1) \) of Theorem 1.1 holds.

From the fact that \( 2F(x, u) = \frac{8S^{S/3}}{8^{S/3}} (\sin^2 x) u^{S/3} \leq \frac{8S^{S/3}}{8^{S/3}} (\sin^2 x) u^{S/3} \), we can verify \( (H_3) \) of Theorem 1.1 also is satisfied.

It is also easy to check that the hypotheses \( (V_0), (H_2) \) and other conditions of Theorem 1.1 hold. Then all the conditions in Theorem 1.1 are satisfied. In virtue of Theorem 1.1, we conclude that (20) possesses infinitely many solutions.
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