Research Article

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Semilinear systems with a multi-valued nonlinear term

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Abstract: Introducing a topological degree theory, we first establish some existence results for the inclusion \( h \in Lu - Nu \) in the nonresonance and resonance cases, where \( L \) is a closed densely defined linear operator on a Hilbert space with a compact resolvent and \( N \) is a nonlinear multi-valued operator of monotone type. Using the nonresonance result, we next show that abstract semilinear system has a solution under certain conditions on \( N = (N_1, N_2) \), provided that \( L = (L_1, L_2) \) satisfies \( \dim \ker L_1 = \infty \) and \( \dim \ker L_2 < \infty \). As an application, periodic Dirichlet problems for the system involving the wave operator and a discontinuous nonlinear term are discussed.

Keywords: Semilinear system, Multi-valued operator, Operators of monotone type, Degree theory

MSC: 47H04, 47H05, 47H11, 35A16, 35B10, 35L71

1 Introduction

Semilinear wave equation and abstract semilinear equation have been studied in many ways; see [1–4], for instance. To solve this problem, Mawhin and Willem [5, 6] employed the Leray-Schauder theory combined with monotone type operators in Galerkin arguments; see [1]. Berkovits and Tienari [7] introduced a topological degree theory for multi-valued operators of monotone type with so called elliptic super-regularization method to deal with hyperbolic problems with discontinuous nonlinearity.

Let \( H \) be a real separable Hilbert space. We first observe a semilinear equation

\[
Lu - Nu = h,
\]

where \( L \) is a closed densely defined linear operator on \( H \) with a compact resolvent and \( N \) is a nonlinear operator. In the self-adjoint case, it is known that equation (1) has a solution when

\[
\|Nu - \frac{\lambda_1}{2}u\| \leq \mu \|u\| + \nu \quad \text{for all } u \in H,
\]

where \( \lambda_1 \) is the first positive eigenvalue of \( L \) and \( \mu \in [0, \lambda_1/2), \nu \in [0, \infty) \) are constants. More generally, if \( \ker L = \ker L^* \), \( L^* \) being the adjoint operator of \( L \), then there is a positive number \( \rho \) such that

\[
\|Lu - \frac{\rho}{2}\| \geq \frac{\rho}{2} \|u\| \quad \text{for all } u \in D(L).
\]

In this case, equation (1) admits a solution if there exist \( \mu \in [0, \rho/2) \) and \( \nu \in [0, \infty) \) such that

\[
\|Nu - \frac{\rho}{2}\| \leq \mu \|u\| + \nu \quad \text{for all } u \in H.
\]

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The existence proof for the nonlinear equation was based on the Leray-Schauder theory. See [1, 8, 9].

Next, Berkovits and Fabry [10] considered a system of semilinear equations

\[
\begin{align*}
L_1u_1 - N_1(u_1, u_2) &= h_1, \\
L_2u_2 - N_2(u_1, u_2) &= h_2,
\end{align*}
\]

where \(L_1, L_2\) are closed densely defined linear operators with \(\dim \text{Ker} L_1 = \infty\) and \(\dim \text{Ker} L_2 < \infty\) and \(N_1, N_2\) are nonlinear operators.

In the present paper, our goal is to study the semilinear system including a multi-valued nonlinear term. Especially, we are interested in the following semilinear system

\[
\begin{align*}
L_1u_1 - N_1(u_1, u_2) &\ni h_1, \\
L_2u_2 - N_2(u_1, u_2) &= h_2, \quad (2)
\end{align*}
\]

where \(N_1(u_1, u_2) = N_{1,1}(u_1) + N_{1,2}(u_2)\), \(N_{1,1}\) is a weakly upper semicontinuous bounded multi-valued operator of monotone type, and \(N_{1,2}, N_2\) are continuous bounded operators. The semilinear system (2) can be written as

\[h \in Lu - Nu,\]

where \(L = (L_1, L_2)\) is as above and \(N = (N_1, N_2)\) is a weakly upper semicontinuous bounded multi-valued operator satisfying generalized \((S_+)\) condition with respect to the orthogonal projection to \(\text{Ker} L\).

More generally, to find a solution of the semilinear inclusion

\[h \in Lu - Nu, \quad (3)\]

we introduce a topological degree theory for a wider class including the class \((S_+)\) with elliptic super-regularization method, following the basic lines of the Berkovits-Tienari degree for the class \((S_+)\) given in [7].

Using the fact that some linear injection has nonzero degree, we show that the inclusion (3) has a solution if there are \(\mu \in [0, \rho/2)\) and \(\alpha \in [0, 1]\) such that

\[\|a - \frac{\rho}{2}u\| \leq \mu\|u\| + O(\|u\|^\alpha) \quad \text{for all } u \in H, \quad \|u\| \to \infty \quad \text{and } a \in Nu. \quad (4)\]

Moreover, we are concerned with the solvability of the inclusion (3) under an additional \(h\)-dependent resonance type condition when \(\mu = \rho/2\) in (4) is allowed. For semilinear equations in a more general setting, we refer to [10].

Concerning abstract semilinear systems, it is emphasized that the nonlinear operator \(N = (N_1, N_2)\) is not necessarily of class \((S_+)\). Namely, instead of the Berkovits-Tienari degree for the class \((S_+)\), our degree theory plays an important role in the study of semilinear systems with mixed nonlinear terms like (2). In the nonresonance case, we prove that (2) is solvable under certain conditions on \(N = (N_1, N_2)\). Applying this result, we show the existence of weak solutions of periodic Dirichlet problem for the system involving the wave operator and a discontinuous nonlinear term. Actually, it was inspired by the works [7, 11].

2 Degree theory

Let \(H\) be a real Hilbert space. Given a nonempty subset \(\Omega\) of \(H\), let \(\overline{\Omega}\) and \(\partial \Omega\) denote the closure and the boundary of \(\Omega\) in \(H\), respectively. Let \(B_r(u)\) denote the open ball in \(H\) of radius \(r > 0\) centered at \(u\). The symbol \(\to (\rightharpoonup)\) stands for strong (weak) convergence.

**Definition 2.1.** A multi-valued operator \(F : \Omega \subset H \to 2^H\) is said to be:

1. **upper semicontinuous** if the set \(F^{-1}(A) = \{u \in \Omega \mid Fu \cap A \neq \emptyset\}\) is closed in \(\Omega\) for every closed set \(A\) in \(H\);
2. **weakly upper semicontinuous** if \(F^{-1}(A)\) is closed in \(\Omega\) for every weakly closed set \(A\) in \(H\);
3. **bounded** if it maps bounded sets into bounded sets;
4. **compact** if it is upper semicontinuous and the image of any bounded set is relatively compact;
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Let \((H, \langle \cdot, \cdot \rangle)\) be a real separable Hilbert space and \(E\) a closed subspace of \(H\). Let \(P : H \to E\) and \(Q : H \to E^\perp\) be the orthogonal projections, respectively.

**Definition 2.2.** A multi-valued operator \(F : \Omega \subset H \to 2^H \setminus \emptyset\) is said to be:

1. Of class \((S_+)_P\) if for any sequence \((u_n)\) in \(\Omega\) and for any sequence \((w_n)\) in \(H\) with \(w_n \in Fu_n\) such that \(u_n \to u\), \(Qu_n \to Qu\), and
   \[
   \limsup_{n \to \infty} (w_n, P(u_n - u)) \leq 0,
   \]
   we have \(u_n \to u\);

2. \(P\)-pseudomonotone, written \(F \in (PM)_P\), if for any sequence \((u_n)\) in \(\Omega\) and for any sequence \((w_n)\) in \(H\) with \(w_n \in Fu_n\) such that \(u_n \to u\), \(Qu_n \to Qu\), and
   \[
   \limsup_{n \to \infty} (w_n, P(u_n - u)) \leq 0,
   \]
   we have \(\lim_{n \to \infty} (w_n, P(u_n - u)) = 0\) and if \(u \in \Omega\) and \(w_j \to w\) for some subsequence \((w_j)\) of \((w_n)\) then \(w \in Fu\);

3. \(P\)-quasimonotone, written \(F \in (QM)_P\), if for any sequence \((u_n)\) in \(\Omega\) and for any sequence \((w_n)\) in \(H\) with \(w_n \in Fu_n\) such that \(u_n \to u\) and \(Qu_n \to Qu\), we have
   \[
   \liminf_{n \to \infty} (w_n, P(u_n - u)) \geq 0.
   \]

With \(P = I\), we get the definitions for classes \((S_+)\), \((PM)\), and \((QM)\) in [7].

Throughout this paper, we will always assume that all multi-valued operators considered have nonempty closed convex values.

It is clear from the definitions that \((S_+) = (S_+)_P\), \((PM) = (PM)_P\), and \((QM) = (QM)_P\) if \(\dim E^\perp < \infty\). If all operators are assumed to be bounded and weakly upper semicontinuous, it is easy to see that \((S_+)_P \subset (PM)_P \subset (QM)_P\) and the class \((S_+)\) is invariant under \((QM)_P\)-perturbations.

Let \(H\) be a real separable Hilbert space. Suppose that \(L : D(L) \subset H \to H\) is a closed densely defined linear operator with

\[
\text{Im } L = (\text{Ker } L)^\perp,
\]

and \(K : \text{Im } L \to \text{Im } L \cap D(L)\), the inverse of the restriction of \(L\) to \(\text{Im } L \cap D(L)\), is compact. Let \(P : H \to \text{Ker } L\) and \(Q : H \to \text{Im } L\), be the orthogonal projections, respectively.

To find solutions of a semilinear inclusion, we need the following equivalent formulation; see [7].

**Lemma 2.3.** Let \(L, K, P, Q\) be as above. Suppose that \(N : \overline{G} \to 2^H\) is a multi-valued operator, where \(G\) is an open set in \(H\). Then

\[
h \in Lu - Nu, \quad u \in \overline{G} \cap D(L)
\]

if and only if

\[
\tilde{h} \in Qu - (KQ - P)Nu, \quad u \in \overline{G},
\]

where \(\tilde{h} = (KQ - P)h\).

**Proof.** Suppose that \(\tilde{h} \in Qu - (KQ - P)Nu\) with \(u \in \overline{G}\). Then \(KQh - Ph = Qu - KQa + Pa\) for some \(a \in Nu\). Since \(Qh = Lu - Qa = L - Qa\) and \(-Ph = Pa\), we have

\[
h = Ph + Qh = Lu - Qa - Pa \in Lu - Nu.
\]

Conversely, suppose that \(h \in Lu - Nu\) with \(u \in \overline{G} \cap D(L)\). Then we get

\[
\tilde{h} \in (KQ - P)(L - N)u = Qu - (KQ - P)Nu.
\]

This completes the proof. \(\square\)
Given an open bounded set $G$ in $H$, we consider a class of semilinear operators

$$L - N,$$

where $N : \overline{G} \to 2^H$ is a weakly upper semicontinuous bounded multi-valued operator of class $(S_+)_p$.

We introduce a topological degree theory for the above class, following the basic idea of the Berkovits-Tienari degree for the class $(S_+)$ given in [7]. To do this, we adopt elliptic super-regularization method as in [7, 12].

Let $\Psi : \text{Ker} L \to \text{Ker} L$ be a compact self-adjoint linear injection. To each $F = Q - (KQ - P)N$, we associate a family of Leray-Schauder type operators defined by

$$F_\lambda := I - (KQ - \lambda \Psi^2 P)N \quad \text{for } \lambda > 0.$$  

For the Leray-Schauder degree theory for multi-valued operators, we refer to [13, 14].

We give a fundamental result which is useful for the construction of our degree and its properties.

**Lemma 2.4.** Suppose that $G$ is an open bounded set in $H$ and $N : \overline{G} \to 2^H$ is a weakly upper semicontinuous bounded operator of class $(S_+)_p$. For any closed set $A \subset \overline{G}$ such that $h \notin (L - N)(A \cap D(L))$, there exists a positive number $\lambda_0$ such that

$$h_\lambda \notin F_\lambda(A) \quad \text{for all } \lambda > \lambda_0,$$

where $h_\lambda := (KQ - \lambda \Psi^2 P)h$.

**Proof.** Let $A$ be any closed subset of $\overline{G}$ such that $h \notin (L - N)(A \cap D(L))$. Assume that the assertion is not true. Then we find sequences $(\lambda_n)\in (0, \infty)$ and $(u_n)\in A$ with $\lambda_n \to \infty$ such that $h_{\lambda_n} \in F_{\lambda_n}(u_n)$ for all $n \in \mathbb{N}$, that is,

$$u_n - KQ(a_n + h) + \lambda_n \Psi^2 P(a_n + h) = 0,$$

where $a_n \in Nu_n$. This equation is equivalent to

$$Qu_n - KQ(a_n + h) = 0 \quad \text{and} \quad Pu_n + \lambda_n \Psi^2 P(a_n + h) = 0. \quad (5)$$

Passing to subsequences if necessary, we may suppose that $u_n \to u$ and $a_n \to a$ for some $u, a \in H$. Then we have by (5) and the strong continuity of the operator $K$

$$Qu_n \to KQ(a + h) = Qu \quad \text{and} \quad \Psi^2 Pa_n \to -\Psi^2 Ph = \Psi^2 Pa,$$

which implies $Pa = -Ph$, by the injectivity of $\Psi$. Since $P$ and $\Psi$ are self-adjoint with $P^2 = P$ and $Pa_n \to -Ph$, it follows from (5) that

$$\limsup_{n \to \infty} \langle a_n + h, P(u_n - u) \rangle = \limsup_{n \to \infty} \langle P(a_n + h), P(u_n - u) \rangle$$

$$= \limsup_{n \to \infty} \langle P(a_n + h), -\lambda_n \Psi^2 P(a_n + h) \rangle$$

$$= \limsup_{n \to \infty} \left[ -\lambda_n \| \Psi P(a_n + h) \|^2 \right]$$

$$\leq 0.$$

In view of $N + h \in (S_+)_p$, this implies $u_n \to u \in A$. Note that $Nu$ is closed and convex and so weakly closed. Since $N$ is weakly upper semicontinuous, we have $a \in Nu$. This yields to

$$KQh - Ph = Qu - KQa + Pa \in Qu - (KQ - P)Nu,$$

which contradicts the hypothesis that $h \notin (L - N)(A \cap D(L))$, in view of Lemma 2.3 with $A$ in place of $\overline{G}$. Therefore, the assertion must be true. This completes the proof.

**Corollary 2.5.** Let $G$ be an open bounded set in $H$. If $h \notin (L - N)(\partial G \cap D(L))$, there is a positive number $\lambda_0$ such that $h_\lambda \notin F_\lambda(\partial G)$ for all $\lambda > \lambda_0$ and $d_{LS}(F_\lambda, G, h_\lambda)$ is constant for all $\lambda > \lambda_0$, where $d_{LS}$ denotes the Leray-Schauder degree.
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where

Proof. According to Lemma 2.4 with $A = \partial G$, we can choose a positive number $\lambda_0$ such that $h_\lambda \notin F_\lambda(\partial G)$ for all $\lambda > \lambda_0$. For the second assertion, let $\lambda_1, \lambda_2$ be arbitrary numbers in $(\lambda_0, \infty)$ such that $\lambda_1 < \lambda_2$. Then $F_\lambda, \lambda \in [\lambda_1, \lambda_2]$, defines a Leray-Schauder type homotopy such that $h_\lambda \notin F_\lambda(\partial G)$ for all $\lambda \in [\lambda_1, \lambda_2]$. It follows from the homotopy invariance property of the degree $d_{LS}$ that

$$d_{LS}(F_{\lambda_1}, G, h_{\lambda_1}) = d_{LS}(F_{\lambda_2}, G, h_{\lambda_2}).$$

Since $\lambda_1, \lambda_2$ were arbitrarily chosen in $(\lambda_0, \infty)$, we conclude that $d_{LS}(F_\lambda, G, h_\lambda)$ is constant for all $\lambda > \lambda_0$. This completes the proof.

We are now ready to define a topological degree for the semilinear class involving multi-valued operators of class $(S_+)$.  

**Definition 2.6.** Let $L, K, P, Q$ be as above. Suppose that $N : \overline{G} \to 2^H$ is a weakly upper semicontinuous bounded multi-valued operator of class $(S_+)$. Suppose that $G$ is any open bounded set in $H$. If $h \notin (L - N)(\partial G \cap D(L))$, we define a degree function $d$ by

$$d(L - N, G, h) := \lim_{\lambda \to \infty} d_{LS}(F_\lambda, G, h_\lambda),$$

where $F_\lambda = I - (KQ - \lambda \Psi^2 P)N$ and $h_\lambda = (KQ - \lambda \Psi^2 P)h$.

**Definition 2.7.** A multi-valued operator $N : [0, 1] \times \overline{G} \to 2^H$ is said to be a homotopy of class $(S_+) \rho$ if for any sequence $(t_n, u_n)$ in $[0, 1] \times \overline{G}$ and for any sequence $(a_n)$ in $H$ with $a_n \in N(t_n, u_n)$ such that $t_n \to t$, $u_n \to u$, $Qu_n \to Qu$, and

$$\limsup_{n \to \infty} \langle a_n, P(u_n - u) \rangle \leq 0,$$

we have $u_n \to u$.

**Lemma 2.8.** (Affine homotopy) If $N_1, N_2 : \overline{G} \to 2^H$ are weakly upper semicontinuous bounded operators of class $(S_+) \rho$, then $N : [0, 1] \times \overline{G} \to 2^H$ given by

$$N(t, u) := (1 - t)N_1u + tN_2u \quad \text{for } (t, u) \in [0, 1] \times \overline{G}$$

is a weakly upper semicontinuous bounded homotopy of class $(S_+) \rho$.

Given an open bounded set $G$ in $H$, we consider a class of semilinear homotopies

$L - N,$

where $N : [0, 1] \times \overline{G} \to 2^H$ is a weakly upper semicontinuous bounded multi-valued homotopy of class $(S_+) \rho$.

We make the following observation for estimating the homotopy invariance of the degree $d$.

**Lemma 2.9.** Suppose that $N : [0, 1] \times \overline{G} \to 2^H$ is a weakly upper semicontinuous bounded homotopy of class $(S_+) \rho$, where $G$ is an open bounded set in $H$, and $h : [0, 1] \to H$ is a continuous curve in $H$ such that $h(t) \notin (L - N_t)(\partial G \cap D(L))$ for all $t \in [0, 1]$. Then there is a positive number $\lambda_0$ such that

$$h_\lambda(t) \notin (F_t)_\lambda(\partial G) \quad \text{for all } t \in [0, 1] \text{ and all } \lambda > \lambda_0,$$

where $N_t = N(t, \cdot)$, $(F_t)_\lambda = I - (KQ - \lambda \Psi^2 P)N_t$ and $h_\lambda(t) = (KQ - \lambda \Psi^2 P)h(t)$.

Proof. Assume to the contrary that there exist sequences $(\lambda_n)$ in $(0, \infty)$, $(t_n)$ in $[0, 1]$, and $(u_n)$ in $\partial G$ with $\lambda_n \to \infty$ such that $h_{\lambda_n}(t_n) \in F_{\lambda_n}(t_n, u_n)$ for all $n \in \mathbb{N}$. This can be written as

$$u_n - KQ(a_n + h(t_n)) + \lambda_n \Psi^2 P(a_n + h(t_n)) = 0, \quad (6)$$

where $a_n \in N(t_n, u_n)$. Equation (6) is equivalent to

$$Qu_n - KQ(a_n + h(t_n)) = 0 \quad \text{and} \quad Pu_n + \lambda_n \Psi^2 P(a_n + h(t_n)) = 0. \quad (7)$$
Without loss of generality, we may suppose that $t_n \to t$, $u_n \to u$, and $a_n \to a$ for some $t \in [0, 1]$ and some $u, a \in H$. Then we have $KQ(a + h(t)) = Qu$ and $-\Psi^2 Ph(t) = \Psi^2 Pa$ and so $Pa = Ph(t)$. Since $P$ and $\Psi$ are self-adjoint and $P\alpha_n + Ph(t_n) \to 0$, we obtain from (7) that

$$\lim_{n \to \infty} \langle a_n + h(t_n), P(u_n - u) \rangle \leq 0,$$

which implies, in view of $h(t_n) \to h(t)$, that

$$\lim_{n \to \infty} \langle a_n, P(u_n - u) \rangle \leq 0.$$

Hence it follows from $N \in (\mathcal{S}_+) \eta$ that $u_n \to u \in \partial G$. Since $N$ is weakly upper semicontinuous, we have $a \in N(t, u)$. Therefore, we get

$$KQh(t) - Ph(t) = Qu - KQa + Pa \in Qu - (KQ - P)N(t, u),$$

which contradicts the hypothesis that $h(t) \notin (L - N_t)(\partial G \cap D(L))$. This completes the proof. □

The degree theory plays a decisive role in the study of semilinear inclusions. Especially, the homotopy invariance is a powerful property in the use of the degree, as we will see in Theorem 3.1 below.

**Theorem 2.10.** Let $L$ and $N$ be as in Definition 2.6. Suppose that $G$ is any open bounded subset of $H$ and $h \notin (L - N)(\partial G \cap D(L))$. Then the degree $d$ has the following properties:

(a) (Existence) If $d (L - N, G, h) \neq 0$, then the inclusion $h \in Lu - Nu$ has a solution in $G \cap D(L)$.

(b) (Additivity) If $G_1$ and $G_2$ are disjoint open subsets of $G$ such that $h \notin (L - N)(G_1 \cup G_2) \cap D(L)$, then we have

$$d (L - N, G, h) = d (L - N, G_1, h) + d (L - N, G_2, h).$$

(c) (Homotopy invariance) Suppose that $N : [0, 1] \times \overline{G} \to 2^H$ is a weakly upper semicontinuous bounded homotopy of class $(\mathcal{S}_+) \eta$. If $h : [0, 1] \to H$ is a continuous curve in $H$ such that

$$h(t) \notin Lu - Nu \quad \text{for all } t \in [0, 1] \text{ and all } u \in \partial G \cap D(L),$$

then $d (L - N(t, \cdot), G, h(t))$ is constant for all $t \in [0, 1]$.

**Proof.** (a) Argue by contraposition. If $h \notin Lu - Nu$ for all $u \in \overline{G} \cap D(L)$, Lemma 2.4 implies that there exists a positive number $\lambda_0$ such that $h_\lambda \notin F_\lambda(\overline{G})$ for all $\lambda > \lambda_0$. It follows from the corresponding property of the Leray-Schauder degree that $d_{LS} (F_\lambda, G, h_\lambda) = 0$ for all $\lambda > \lambda_0$. By Definition 2.6, we have $d (L - N, G, h) = 0$.

(b) Applying Lemma 2.4 with $A = \overline{G} \setminus (G_1 \cup G_2)$, we find a positive number $\lambda_0$ such that

$$h_\lambda \notin F_\lambda(\overline{G} \setminus (G_1 \cup G_2)) \quad \text{for all } \lambda > \lambda_0.$$

By the additivity of the Leray-Schauder degree, we have

$$d_{LS} (F_\lambda, G, h_\lambda) = d_{LS} (F_\lambda, G_1, h_\lambda) + d_{LS} (F_\lambda, G_2, h_\lambda) \quad \text{for all } \lambda > \lambda_0.$$

The conclusion follows directly from the definition of the degree $d$.

(c) In view of Lemma 2.9, we can choose a positive number $\lambda_0$ such that

$$h_\lambda(t) \notin (F_t)_\lambda(\partial G) \quad \text{for all } t \in [0, 1] \text{ and all } \lambda > \lambda_0,$$

where $h_\lambda(t) = (KQ - \lambda \Psi^2 P)h(t)$ and $(F_t)_\lambda = I - (KQ - \lambda \Psi^2 P)N_t$. For each fixed $\lambda > \lambda_0$, $(F_t)_\lambda, t \in [0, 1]$ defines a Leray-Schauder type homotopy such that $h_\lambda(t) \notin (F_t)_\lambda(u)$ for all $(t, u) \in [0, 1] \times \partial G$. Hence it follows.
from the homotopy invariance property of the degree $d_{LS}$ that $d_{LS}((F_t)_\lambda, G, h_\lambda(t))$ is constant for all $t \in [0, 1]$. For any $t_1, t_2 \in [0, 1]$, we have by Definition 2.6

$$d(L - N_{t_1}, G, h(t_1)) = \lim_{\lambda \to \infty} d_{LS}((F_{t_1})_\lambda, G, h_\lambda(t_1))$$

$$= \lim_{\lambda \to \infty} d_{LS}((F_{t_2})_\lambda, G, h_\lambda(t_2))$$

$$= d(L - N_{t_2}, G, h(t_2)).$$

We conclude that $d(L - N(t, \cdot), G, h(t))$ is constant for all $t \in [0, 1]$. This completes the proof.

The following result says that some linear injection has nonzero degree. It will be a key tool for proving the existence of a solution for semilinear inclusions in the next section.

**Lemma 2.11.** Let $B : H \to H$ be a bounded linear operator of class $(S_+)_P$ such that $L - B$ is injective. Then for any bounded open set $G \subset H$ and $h \in (L - B)(G \cap D(L))$, we have

$$d(L - B, G, h) = \pm 1.$$

**Proof.** Let $G$ be any open bounded set in $H$ and $h \in (L - B)(G \cap D(L))$. By the injectivity of the operator $L - B$, we have $(L - B)v = h$ for some $v \in G \cap D(L)$. In view of part (b) of Theorem 2.10, we can choose a positive number $R$ with $\|v\| < R$ such that

$$d(L - B, G, h) = d(L - B, B_R(0), h).$$

It is clear that $(L - B)u \neq th$ for all $t \in [0, 1]$ and $u \in \partial B_R(0)$. Letting $h : [0, 1] \to H$ be defined by $h(t) := th$ for $t \in [0, 1]$, we obtain from part (c) of Theorem 2.10 that

$$d(L - B, B_R(0), h) = d(L - B, B_R(0), 0).$$

By Definition 2.6, we have

$$d(L - B, B_R(0), 0) = \lim_{\lambda \to \infty} d_{LS}(T_\lambda, B_R(0), 0),$$

where $T_\lambda = I - (KQ - \lambda \Psi^2 P)B$. Since $T_\lambda$ is an injective Leray-Schauder type operator for large $\lambda$, it is known in [15] that $d(L - B, G, h)$ is $+1$ or $-1$. This completes the proof.

**Corollary 2.12.** If $L - \alpha I$ is injective for some positive constant $\alpha$, then we have

$$\deg(L - \alpha I, B_r(0), 0) = \pm 1 \quad \text{for any positive number } r.$$

**Proof.** Note that $\alpha I \in (S_+)_P$. Apply Lemma 2.11 with $B = \alpha I$ and $h = 0$.

### 3 Existence results

This section is devoted to the solvability of semilinear inclusions in the nonresonance and resonance cases, by using the degree theory in the previous section.

Let $(H, \langle \cdot, \cdot \rangle)$ be a real separable Hilbert space. Suppose that $L : D(L) \subset H \to H$ is a closed densely defined linear operator with $\text{Im } L = (\text{Ker } L)^\perp$, and $K : \text{Im } L \to \text{Im } L \cap D(L)$, the inverse of the restriction of $L$ to $\text{Im } L \cap D(L)$, is compact. Let $P : H \to \text{Ker } L$ and $Q : H \to \text{Im } L$ be the orthogonal projections, respectively. Here, Ker $L$ may be infinite dimensional.

Set

$$\mathcal{A} := \{ \rho \in \mathbb{R} \mid \|Lu\| \geq \rho \langle Lu, u \rangle \quad \text{for all } u \in D(L) \}. \quad (8)$$

It is easily checked that

$$\mathcal{A} = \left\{ \rho \in \mathbb{R} \mid \left\| Lu - \frac{\rho}{2} u \right\| \geq \frac{\rho}{2} \left\| u \right\| \quad \text{for all } u \in D(L) \right\}. \quad (9)$$
It is known in [8, 9] that the set \( A \) is a closed interval containing 0 as an interior point of \( A \).

We present a nonresonance theorem on the surjectivity of \( L - N \) when \( N \) is \( P \)-pseudomonotone. The basic idea of proof comes from Theorem 6.1 of [10], where, of course, semilinear equations in a more general setting were dealt with.

**Theorem 3.1.** Let \( L \) be as above. Let \( N : H \rightarrow 2^H \) be a weakly upper semicontinuous bounded operator. Suppose that there exist numbers \( \rho \in (0, \sup A] \), \( \mu \in [0, \rho/2) \), and \( \alpha \in [0, 1) \) such that

\[
\|a - \frac{\rho}{2} u\| \leq \mu \| u \| + O(\| u \|^\alpha) \quad \text{for } u \in H, \| u \| \rightarrow \infty, \text{and } a \in Nu.
\]

(a) If \( N \) is \( P \)-quasimonotone, then the range of the operator \( L - N \) is dense in \( H \).

(b) If \( N \) is \( P \)-pseudomonotone, then the inclusion

\[ h \in Lu - Nu \]

has a solution in \( D(L) \) for every \( h \in H \).

**Proof.** Let \( h \) be an arbitrary element of \( H \). We consider the homotopy equation

\[
th \in Lu - (1 - t) \frac{\rho}{2} u - tNu \quad \text{for } t \in [0, 1] \text{ and } u \in D(L).
\]

Since \( \rho \in A \) implies, in view of (9), that the linear operator \( L - (\rho/2)I \) is injective, equation (11) with \( t = 0 \) has only the trivial solution. From Corollary 2.12, we know that

\[
deg \left( L - \frac{\rho}{2} I, B_r(0), 0 \right) \neq 0 \quad \text{for any positive number } r.
\]

We first claim that the set of solutions of (11) is bounded in \( H \). In fact, we assume that there are sequences \( (u_n) \) in \( D(L) \) and \( (t_n) \) in \( [0, 1] \) with \( \| u_n \| \rightarrow \infty \) such that

\[ t_n h = Lu_n - (1 - t_n) \frac{\rho}{2} u_n - t_n a_n \quad \text{for all } n \in \mathbb{N}, \]

where \( a_n \in Nu_n \). For all \( n \in \mathbb{N} \), we have by (9) and (10)

\[
\frac{\rho}{2} \| u_n \| \leq \| Lu_n - \frac{\rho}{2} u_n \| = \| t_n a_n - \frac{\rho}{2} u_n + h \| \leq \mu \| u_n \| + \| h \| + O(\| u_n \|^\alpha)
\]

and hence

\[
\left( \frac{\rho}{2} - \mu \right) \| u_n \| \leq \| h \| + O(\| u_n \|^\alpha),
\]

which contradicts the unboundedness of the sequence \( (u_n) \) chosen. Thus, the solution set is bounded in \( H \). Now we can choose a positive constant \( R \) such that

\[
th \notin Lu - (1 - t) \frac{\rho}{2} u - tNu \quad \text{for all } t \in [0, 1] \text{ and all } u \in D(L) \text{ with } \| u \| \geq R.
\]

There are three cases to consider. Firstly, we suppose that \( N \in (S_+)_P \). Notice by Lemma 2.8 that \( N_1 : [0, 1] \times \overline{B_R(0)} \rightarrow 2^H \) defined by

\[
N_1(t, u) := (1 - t) \frac{\rho}{2} I u + tNu \quad \text{for } (t, u) \in [0, 1] \times \overline{B_R(0)}
\]

is a weakly upper semicontinuous bounded homotopy of class \( (S_+)_P \). Part (c) of Theorem 2.10 implies, in view of (13) and (12), that

\[
deg \left( L - N, B_R(0), h \right) = deg \left( L - \frac{\rho}{2} I, B_R(0), 0 \right) \neq 0.
\]
We consider the homotopy equation

\[
\text{deg} \left( L - (1 - t) \frac{\rho}{2} I - tN, B_R(0), th \right) \neq 0.
\]

By part (a) of Theorem 2.10, there exists a \( u_t \in B_R(0) \cap D(L) \) such that

\[
th = Lu_t - (1 - t) \frac{\rho}{2} u_t - ta_t,
\]

where \( a_t \in Nu_t \). According to the assertion (15), for a sequence \((t_n)\) in \((0, 1)\) with \( t_n \to 1 \), there is a corresponding sequence \((u_n)\) in \( B_R(0) \cap D(L) \) such that

\[
t_n h = Lu_n - (1 - t_n) \frac{\rho}{2} u_n - t_n a_n,
\]

where \( a_n \in Nu_n \). Hence it follows that \( Lu_n - a_n \to h \) and so \( h \in \text{Im}(L - N) \). Since \( h \in H \) was arbitrary, we conclude that the range of \( L - N \) is dense in \( H \). Thus, statement (a) holds.

Thirdly, we suppose that \( N \) is \( P \)-quasimonotone. In virtue of the second case, we take a sequence \((u_n)\) in \( B_R(0) \cap D(L) \) such that \( Lu_n - a_n \to h \), where \( a_n \in Nu_n \). Without loss of generality, we may suppose that \( u_n \to u \) and \( a_n \to a \) for some \( u, a \in H \). Since \( \text{Im} L = (\text{Ker} L)^\perp \) and \( Pu_n \to Pu \), we have

\[
\lim_{n \to \infty} \langle a_n, Pu(u_n - u) \rangle = \lim_{n \to \infty} \langle Lu_n, Pu(u_n - u) \rangle = \lim_{n \to \infty} \langle h, Pu(u_n - u) \rangle = 0.
\]

Since \( Qu_n = KQLu_n \) and \( K \) is compact, it is obvious that \( Qu_n \to Qu \). The \( P \)-quasimonotonicity of the operator \( N \) implies that \( a \in Nu \). Since the graph of \( L \) is weakly closed and \( Lu_n \to a + h \), we obtain that

\[
u \in D(L) \quad \text{and} \quad h \in Lu - Nu.
\]

We have just proved that statement (b) is valid. This completes the proof.

Next, we show the existence of a solution of the semilinear inclusion under an additional \( h \)-dependent resonance type condition when \( \mu = \rho/2 \) in condition (10) is allowed.

**Theorem 3.2.** Let \( L \) be as above. Let \( N : H \to 2^H \) be a weakly upper semicontinuous bounded operator. Suppose that there are \( \rho \in (0, \sup \mathcal{A}) \) and \( \alpha \in (0, 1) \) such that

\[
\left\| a - \frac{\rho}{2} u \right\| \leq \frac{\rho}{2} \| u \| + O(\| u \|^\alpha) \quad \text{for } u \in H, \| u \| \to \infty, \text{ and } a \in Nu.
\]

Let \( h \in H \) be given and suppose that for any sequence \((u_n)\) in \( D(L) \) such that \( \| u_n \| \to \infty \) and \( \| Lu_n \| = o(\| u_n \|) \) for \( n \to \infty \), there exists an integer \( n_0 \) such that

\[
\langle a_n + h, Pu_n \rangle > 0 \quad \text{for all } n \geq n_0 \text{ and all } a_n \in Nu_n.
\]

(a) If \( N \) is \( P \)-quasimonotone, then \( h \in \text{Im}(L - N) \).

(b) If \( N \) is \( P \)-quasimonotone, then the inclusion \( h \in Lu - Nu \)

has a solution in \( D(L) \).

**Proof.** We consider the homotopy equation

\[
th \in Lu - (1 - t) \frac{\rho}{2} u - tNu \quad \text{for } (t, u) \in [0, 1] \times D(L).
\]

We have to show that the solution set

\[
S = \{ u \in D(L) | th \in Lu - (1 - t) \frac{\rho}{2} u - tNu \text{ for some } t \in [0, 1] \}
\]

is of class \( (S_+) \). Applying the assertion (14) in the first case with \( N(t, \cdot) \) in place of \( N \), we see that

\[
\deg \left( L - (1 - t) \frac{\rho}{2} I - tN, B_R(0), th \right) \neq 0.
\]
is bounded in $H$. Assume to the contrary that there exist sequences $(u_n)$ in $D(L)$ and $(t_n)$ in $(0,1]$ with $\|u_n\| \to \infty$ such that
\[
t_n h = Lu_n - (1 - t_n) \frac{\rho}{2} u_n - t_n a_n \quad \text{for all } n \in \mathbb{N},
\]
where $a_n \in Nu_n$. Let $\bar{p} \in \mathcal{A}$ be any positive number with $\bar{p} > \rho$. For all $u \in D(L)$, we have by (8)
\[
\|Lu - \frac{\rho}{2} u\|^2 = \|Lu\|^2 - \rho(Lu,u) + \left(1 - \frac{\rho}{\bar{p}}\right) \|Lu\|^2
\geq \left(1 - \frac{\rho}{\bar{p}}\right) \|Lu\|^2 + \left(\frac{\rho}{2}\right)^2.
\]
Hence it follows from (19) and (16) that
\[
\left(1 - \frac{\rho}{\bar{p}}\right) \|Lu_n\|^2 + \left(\frac{\rho}{2}\right)^2 \leq \|Lu_n - \frac{\rho}{2} u_n\|^2
\leq \left(\|a_n - \frac{\rho}{2} u_n\| + \|h\|\right)^2
\leq \left(\frac{\rho}{2}\right)^2 \|u_n\|^2 + \|h\| + O(\|u_n\|^2),
\]
which implies
\[
\|Lu_n\| = o(\|u_n\|).
\]
Set $z_n := u_n/\|u_n\|$ and $w_n := Lz_n$. Then $\|u_n\| = \|Lu_n\|/\|u_n\| \to 0$ implies $Qz_n = Kw_n \to 0$. Since $\langle Lu_n, Pu_n \rangle = 0$, we have by (19)
\[
\langle a_n + h, Pu_n \rangle = -(1 - t_n)t_n^{-1} \frac{\rho}{2} \langle u_n, Pu_n \rangle.
\]
It follows from $\langle u_n, Pu_n \rangle = \|u_n\|^2 - \langle u_n, Qu_n \rangle$ that
\[
\langle a_n + h, Pu_n \rangle = -(1 - t_n)t_n^{-1} \frac{\rho}{2} \|u_n\|^2 [1 - (z_n, Qz_n)].
\]
Hence we obtain from $Qz_n \to 0$ that
\[
\langle a_n + h, Pu_n \rangle \leq 0 \quad \text{for some large } n,
\]
which contradicts hypothesis (17). Thus, we have shown that the solution set $S$ is bounded in $H$. The rest of proof proceeds in a similar way to that of Theorem 3.1. \hfill \square

4 Semilinear systems

In this section, we first examine under what conditions the operators are of class $(S_+)_P$ or $P$-quasimonotone and then establish some existence results for semilinear systems in the nonresonance case.

Let $H_1, H_2$ be two real separable Hilbert spaces and let $H = H_1 \times H_2$ be the Hilbert space with inner product defined by
\[
\langle u, v \rangle = \langle u_1, v_1 \rangle + \langle u_2, v_2 \rangle \quad \text{for } u = (u_1, u_2), v = (v_1, v_2) \in H_1 \times H_2.
\]
For $k = 1,2$, let $L_k : D(L_k) \subset H_k \to H_k$ be a closed densely defined linear operator with $\text{Im } L_k = (\text{Ker } L_k)^\perp$. Suppose that $K_k : \text{Im } L_k \to \text{Im } L_k$, the inverse of the restriction of $L_k$ to $\text{Im } L_k \cap D(L_k)$, is compact. For $k = 1,2$, let $P_k : H_k \to \text{Ker } L_k$ and $Q_k : H_k \to \text{Im } L_k$ be the orthogonal projections, respectively.

Define the diagonal operator $L : D(L) \subset H \to H$ by setting
\[
Lu = (L_1 u_1, L_2 u_2) \quad \text{for } u = (u_1, u_2) \in D(L) = D(L_1) \times D(L_2).
\]
Then $K : \text{Im } L \to \text{Im } L$, the inverse of $L$ to $\text{Im } L \cap D(L)$, is compact, where
\[
Ku = (K_1 u_1, K_2 u_2) \quad \text{for } u = (u_1, u_2) \in \text{Im } L.
\]
Let $P : H \to \ker L$ and $Q : H \to \im L$ be the orthogonal projections, respectively. We write

$$Pu = (P_1u_1, P_2u_2) \quad \text{and} \quad Qu = (Q_1u_1, Q_2u_2) \quad \text{for} \ u = (u_1, u_2) \in H.$$  

In what follows, we suppose that $\dim \ker L_1 = \infty$ and $\dim \ker L_2 < \infty$.

We show that $N = (N_1, N_2)$ is of class $(S_+)_p$ under strong monotonicity on the first component $N_1$. For the single-valued case, we refer to [12, Lemma 6.2].

**Proposition 4.1.** Suppose that $N = (N_1, N_2) : H \to 2^H$ is a bounded multi-valued operator such that

(a) $N_1(v, \cdot) : H_2 \to 2^{H_1}$ is upper semicontinuous with compact values for each $v \in H_1$;

(b) for each $z \in H_2$ there exist a positive number $\delta = \delta(z)$ and a positive constant $c = c(z)$ such that

$$\langle b - d, v' - v \rangle \geq c \|v' - v\|^2$$

for all $v', v \in H_1$, $z' \in B_\delta(z)$, $b \in N_1(v', z')$, and $d \in N_1(v, z')$.

Then the operator $N$ is of class $(S_+)_p$.

**Proof.** Let $(u_n)$ be any sequence in $H$ and $(a_n)$ any sequence in $H$ with $a_n \in Nu_n$ such that

$$u_n \to u, \quad Qu_n \to Qu, \quad \text{and} \quad \limsup_{n \to \infty} \langle a_n, P(u_n - u) \rangle \leq 0. \quad (20)$$

Let $u_n = (v_n, z_n)$ and $u = (v, z)$. Since $\dim \ker L_2 < \infty$ and $Qu_n \to Qu$, we have

$$z_n \to z \quad \text{in} \ H_2. \quad (21)$$

Since $Qu_n \to Qu$ and $N$ is bounded, we get by (21)

$$\limsup_{n \to \infty} \langle a_n, P(u_n - u) \rangle = \limsup_{n \to \infty} \langle a_n, u_n - u \rangle = \limsup_{n \to \infty} \langle b_n, v_n - v \rangle, \quad (22)$$

where $a_n = (b_n, c_n) \in Nu_n$, that is, $b_n \in N_1(v_n, z_n)$. By hypothesis (a), we see that a sequence $(d_n)$ has a strongly convergent subsequence in $H_1$, denoted again by $(d_n)$, where $d_n \in N_1(v, z_n)$, and hence

$$\lim_{n \to \infty} \langle d_n, v_n - v \rangle = 0,$$

which implies

$$\limsup_{n \to \infty} \langle b_n - d_n, v_n - v \rangle = \limsup_{n \to \infty} \langle b_n, v_n - v \rangle. \quad (23)$$

It follows from (20), (22), and (23) that

$$\limsup_{n \to \infty} \langle b_n - d_n, v_n - v \rangle \leq 0.$$

Noting that $b_n \in N_1(v_n, z_n)$ and $d_n \in N_1(v, z_n)$, we get by hypothesis (b)

$$0 \leq \limsup_{n \to \infty} c \|v_n - v\|^2 \leq \limsup_{n \to \infty} \langle b_n - d_n, v_n - v \rangle \leq 0.$$

Since $v_n \to v$ in $H_1$, we have by (21)

$$u_n = (v_n, z_n) \to (v, z) = u.$$

We conclude that $N \in (S_+)_p$. \hfill $\square$

**Proposition 4.2.** Suppose that $N = (N_1, N_2) : H \to 2^H$ is a bounded multi-valued operator such that

(a) $N_1(v, \cdot) : H_2 \to 2^{H_1}$ is upper semicontinuous with compact values for each $v \in H_1$;

(b) for each $z \in H_2$ there exists a positive number $\delta = \delta(z)$ such that

$$\langle b - d, v' - v \rangle \geq 0$$

for all $v', v \in H_1$, $z' \in B_\delta(z)$, $b \in N_1(v', z')$, and $d \in N_1(v, z')$. 


Then $N$ is $P$-quasimonotone.

Proof. Let $(u_n)$ be a sequence in $H$ and $(a_n)$ a sequence in $H$ with $a_n \in Nu_n$ such that

$$u_n \rightharpoonup u \quad \text{and} \quad Qu_n \rightarrow Qu.$$ 

As in the proof of Proposition 4.1, an analogous argument shows that

$$\liminf_{n \to \infty} \langle a_n, P(u_n - u) \rangle = \liminf_{n \to \infty} \langle b_n - d_n, v_n - v \rangle \geq 0,$$

where $a_n = (b_n, c_n) \in Nu_n$, $u_n = (v_n, z_n)$, $u = (v, z)$, $b_n \in N_1(v_n, z_n)$, and $d_n \in N_1(v, z_n)$. The last inequality follows from hypothesis (b), which proves that $N$ is $P$-quasimonotone. □

In Propositions 4.3 and 4.4 below, the main point is that monotone type hypothesis on the second component $N_2$ is not required and so $N = (N_1, N_2)$ is not necessarily of class $(S_+)$ or pseudomonotone.

**Proposition 4.3.** Suppose that $N = (N_1, N_2) : H \to 2^H$ is bounded, where $N_1(v, z) = N_1,1(v) + N_1,2(z)$, such that

(a) $N_1,1 : H_1 \to 2^{H_1}$ is weakly upper semicontinuous and of class $(S_+)$;

(b) $N_1,2 : H_2 \to H_1$ is continuous;

(c) $N_2 : H \to H_2$ is demicontinuous.

Then $N$ is of class $(S_+)P$.

Proof. Let $(u_n)$ be any sequence in $H$ and $(a_n)$ any sequence in $H$ with $a_n \in Nu_n$ such that

$$u_n \rightharpoonup u, \quad Qu_n \to Qu, \quad \text{and} \quad \limsup_{n \to \infty} \langle a_n, P(u_n - u) \rangle \leq 0.$$ 

(24)

Let $u_n = (v_n, z_n)$ and $u = (v, z)$. As in the proof of Proposition 4.1, we have $z_n \rightharpoonup z$ in $H_2$ and

$$\limsup_{n \to \infty} \langle a_n, P(u_n - u) \rangle = \limsup_{n \to \infty} \langle b_n, v_n - v \rangle,$$

(25)

where $a_n = (b_n, c_n) \in Nu_n$, that is, $b_n \in N_1,1(v_n) + N_1,2(z_n)$. From hypothesis (b) we obtain that

$$\limsup_{n \to \infty} \langle b_n - N_1,2(z_n), v_n - v \rangle = \limsup_{n \to \infty} \langle b_n, v_n - v \rangle.$$ 

(26)

It follows from (24), (25), and (26) that

$$\limsup_{n \to \infty} \langle b_n - N_1,2(z_n), v_n - v \rangle = \limsup_{n \to \infty} \langle a_n, P(u_n - u) \rangle \leq 0.$$

Since $N_1,1$ is of class $(S_+)$ and $b_n - N_1,2(z_n) \in N_1,1(v_n)$, this implies $v_n \rightharpoonup v$ in $H_1$. Therefore, we have $u_n = (v_n, z_n) \rightharpoonup (v, z) = u$. This means that $N$ is of class $(S_+)P$. □

**Proposition 4.4.** Suppose that $N = (N_1, N_2) : H \to 2^H$ is bounded, where $N_1(v, z) = N_1,1(v) + N_1,2(z)$, such that

(a) $N_1,1 : H_1 \to 2^{H_1}$ is weakly upper semicontinuous and pseudomonotone;

(b) $N_1,2 : H_2 \to H_1$ is continuous;

(c) $N_2 : H \to H_2$ is weakly continuous.

Then $N$ is $P$-pseudomonotone.

Proof. Let $(u_n)$ be any sequence in $H$ and $(a_n)$ any sequence in $H$ with $a_n \in Nu_n$ such that

$$u_n \rightharpoonup u, \quad Qu_n \to Qu, \quad \text{and} \quad \limsup_{n \to \infty} \langle a_n, P(u_n - u) \rangle \leq 0.$$ 

Let $u_n = (v_n, z_n)$ and $u = (v, z)$. As in the proof of Proposition 4.3, we have $z_n \rightharpoonup z$ in $H_2$ and

$$\limsup_{n \to \infty} \langle b_n - N_1,2(z_n), v_n - v \rangle = \limsup_{n \to \infty} \langle a_n, P(u_n - u) \rangle \leq 0.$$
where \( a_n = (b_n, c_n) \in Nu_n \), that is, \( b_n \in N_{1,1}(v_n) + N_{1,2}(z_n) \) and \( c_n = N_2(u_n) \). Since \( N_{1,1} \) is pseudomonotone, we have
\[
\lim_{n \to \infty} \langle b_n - N_{1,2}(z_n), v_n - v \rangle = 0
\]
and hence
\[
\lim_{n \to \infty} (a_n, P(u_n - u)) = 0.
\]
Suppose that \( a_j = (b_j, c_j) \to a = (b, c) \) for some subsequence \( (a_n) \) of \( (a_n) \). By hypotheses (b) and (c), we get
\[
N_{1,2}(z_j) \to N_{1,2}(z) \quad \text{and} \quad c_j = N_2(u_j) \to N_2(u) = c.
\]
The pseudomonotonicity of \( N_{1,1} \) implies that \( b - N_{1,2}(z) \in N_{1,1}(v) \). Therefore, we have
\[
a = (b, c) \in (N_{1,1}(v) + N_{1,2}(z), N_2(u)) = Nu.
\]
Consequently, \( N \) is \( P \)-pseudomonotone.

**Proposition 4.5.** Suppose that \( N = (N_1, N_2) : H \to 2^H \) is bounded, where \( N_1(v, z) = N_{1,1}(v) + N_{1,2}(z) \), such that

(a) \( N_{1,1} : H \to 2^{H_1} \) is weakly upper semicontinuous and quasimonotone;

(b) \( N_{1,2} : H_2 \to H_1 \) is continuous;

(c) \( N_2 : H \to H_2 \) is demicontinuous.

Then \( N \) is \( P \)-quasimonotone.

**Proof.** Let \((u_n)\) be a sequence in \( H \) and \((a_n)\) a sequence in \( H \) with \( a_n \in Nu_n \) such that
\[
u_n \rightharpoonup u \quad \text{and} \quad Qu_n \rightharpoonup Qu.
\]
If \( u_n = (v_n, z_n) \) and \( u = (v, z) \), we have by the quasimonotonicity of \( N_{1,1} \)
\[
\liminf_{n \to \infty} (a_n, P(u_n - u)) = \liminf_{n \to \infty} (b_n - N_{1,2}(z_n), v_n - v) \geq 0,
\]
where \( a_n = (b_n, c_n) \in Nu_n \), that is, \( b_n \in N_{1,1}(v_n) + N_{1,2}(z_n) \). Thus, \( N \) is \( P \)-quasimonotone.

We are now in position to prove the existence of a solution for semilinear systems, by using the nonresonance theorem in the previous section.

**Theorem 4.6.** Let \( L_1, L_2 \) be as above such that \( \dim Ker L_1 = \infty \) and \( \dim Ker L_2 < \infty \). Suppose that \( N = (N_1, N_2) : H \to 2^H \) is a weakly upper semicontinuous bounded operator such that \( N_1 : H \to 2^{H_1}, N_2 : H \to H_2 \) satisfy the conditions of Proposition 4.1. Further, suppose that there exist numbers \( \rho \in (0, \sup A), \mu \in [0, \rho/2) \), and \( \alpha \in [0, 1) \) such that
\[
\left\| a_1 - \frac{\rho}{2} u_1 \right\| \leq \mu \| u_1 \| + O(\| u \|^\alpha),
\]
\[
\left\| N_2(u_1, u_2) - \frac{\rho}{2} u_2 \right\| \leq \mu \| u_2 \| + O(\| u \|^\alpha),
\]
for \( u = (u_1, u_2) \in H, \| u \| \to \infty \), and \( a_1 \in N_1(u_1, u_2) \). Then for every \( (h_1, h_2) \in H_1 \times H_2 \), the system
\[
\begin{align*}
L_1 u_1 - N_1(u_1, u_2) &\ni h_1 \\
L_2 u_2 - N_2(u_1, u_2) &= h_2
\end{align*}
\]
has a solution in \( D(L_1) \times D(L_2) \).

**Proof.** Note that the norm induced by the inner product on the space \( H = H_1 \times H_2 \) is equivalent to the norm given by
\[
\| (u_1, u_2) \|_1 := \| u_1 \| + \| u_2 \| \quad \text{for} \ (u_1, u_2) \in H_1 \times H_2.
\]
Apply Theorem 3.1, based on Proposition 4.1 and \( (S_\perp) \rho \subset (PM) \rho \).
We close this section with somewhat more concrete semilinear system in a viewpoint of applications.

**Theorem 4.7.** Let $L_1, L_2$ be as above such that $\dim \ker L_1 = \infty$ and $\dim \ker L_2 < \infty$. Suppose that $N = (N_1, N_2) : H \to 2^H$ is a bounded operator such that $N_1 : H \to 2^{H_1}, N_2 : H \to H_2$ satisfy the conditions of Proposition 4.3 or Proposition 4.4. Moreover, suppose that there are $u$ for

\[
\begin{align*}
&L_1 u - N_1.1(u_1) - N_1.2(u_2) \ni h_1, \\
&L_2 u - N_2(u_1, u_2) = h_2
\end{align*}
\]

for $u = (u_1, u_2) \in H, \|u\| \to \infty$, and $a_1 \in N_1.1(u_1)$. Then for every $(h_1, h_2) \in H_1 \times H_2$, the system

\[
\begin{align*}
&L_1 u - N_1.1(u_1) - N_1.2(u_2) \ni h_1, \\
&L_2 u - N_2(u_1, u_2) = h_2
\end{align*}
\]

has a solution in $D(L_1) \times D(L_2)$.

**Proof.** Apply Theorem 3.1 with Proposition 4.3 or Proposition 4.4.

\[\square\]

## 5 Application

In this section, we study the existence of periodic solutions for the system involving the wave operator and a discontinuous nonlinear term, based on a nonresonance theorem for semilinear systems in the previous section.

Motivated by the works [7, 11], we consider the following periodic problem

\[
\begin{align*}
v_{tt} - v_{xx} - h_1(x, t) - g_1(x, t, v, z) &\in [g(x, t, v, z), \overline{g}(x, t, v)] & &\text{in } (0, \pi) \times \mathbb{R}, \\
z_{tt} - z_{xx} - 4z - g_2(x, t, v, z) &\in h_2(x, t) & &\text{in } (0, \pi) \times \mathbb{R}, \\
v(0, \cdot) = v(\pi, \cdot) = 0, &\quad z(0, \cdot) = z(\pi, \cdot).
\end{align*}
\]

(27)

where $g : [0, \pi] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a possibly discontinuous function in the third variable and $g_1, g_2$ satisfy the Carathéodory condition. Here,

\[
\begin{align*}
\underline{g}(x, t, s) &= \liminf_{n \to s} g(x, t, \eta) & &\text{and} & &\overline{g}(x, t, s) &= \limsup_{n \to s} g(x, t, \eta).
\end{align*}
\]

To seek a weak solution of the problem (27), we will consider the corresponding semilinear system; see Definition 5.1 and (30) below.

Let $\Omega = (0, \pi) \times (0, 2\pi)$ and let $H = L^2(\Omega)$ be the real Hilbert space with usual inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $\varphi_{nm}(x, t) = \pi^{-1} \sin(nx) \exp(\im t)$ for $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. Each $u \in H$ has a representation

\[
u = \sum_{(n, m) \in \mathbb{N} \times \mathbb{Z}} u_{nm} \varphi_{nm},
\]

where $u_{nm} = \langle u, \varphi_{nm} \rangle$.

We define a linear operator $L_1 : D(L_1) \subset H \to H$ by

\[
L_1 u := \sum_{(n, m) \in \mathbb{N} \times \mathbb{Z}} (n^2 - m^2) u_{nm} \varphi_{nm},
\]

where

\[
D(L_1) = \{ u \in H \mid \sum_{(n, m) \in \mathbb{N} \times \mathbb{Z}} |n^2 - m^2|^2 |u_{nm}|^2 < \infty \}.
\]

Then $L_1$ is a self-adjoint densely defined operator and

\[
\ker L_1 = \text{span} \{ \varphi_{n, m} \varphi_{n, -m} \mid n \in \mathbb{N} \} \quad \text{and} \quad \text{im} L_1 = (\ker L_1)^\perp.
\]
Note that $\lambda = 1$ is the first positive eigenvalue of $L_1$ which corresponds to $(n, m) = (1, 0)$ and

$$\|L_1 u\|^2 \geq (L_1 u, u) \quad \text{for all } u \in D(L_1).$$

(28)

The partial inverse $L_1^{-1} : \text{Im} \ L_1 \to \text{Im} \ L_1 \cap D(L_1)$ is given by

$$L_1^{-1} u := \sum_{(n,m) \in \Gamma_1} (n^2 - m^2)^{-1} u_{nm} \psi_{nm},$$

where $\Gamma_1 = \{(n, m) \in \mathbb{N} \times \mathbb{Z} \mid n^2 \neq m^2\}$. Note that the spectrum of the operator $L_1^{-1}$, denoted by $\sigma(L_1^{-1})$, has no limit point except 0 and $\dim \ker(L_1^{-1} - \lambda I)$ is finite for every nonzero $\lambda \in \sigma(L_1^{-1})$. This implies that $L_1^{-1}$ is compact. See e.g., [2, 4, 6].

Next, we define another linear operator $L_2 : D(L_2) \subset H \to H$ by

$$L_2 u := \sum_{(n,m) \in \mathbb{N} \times \mathbb{Z}} (n^2 - m^2 - 4) u_{nm} \psi_{nm},$$

where

$$D(L_2) = \{ u \in H \mid \sum_{(n,m) \in \mathbb{N} \times \mathbb{Z}} |n^2 - m^2 - 4| |u_{nm}|^2 < \infty \}.$$

Then $L_2$ is a closed densely defined operator and

$$\ker L_2 = \text{span} \{ \varphi_{20} \} \quad \text{and} \quad \text{Im} \ L_2 = (\ker L_2)^\perp.$$

Note that $\lambda = 1$ is the first positive eigenvalue of $L_2$ corresponding to $(n, m) = (3, 2)$ and

$$\|L_2 u\|^2 \geq (L_2 u, u) \quad \text{for all } u \in D(L_2).$$

(29)

The partial inverse $L_2^{-1} : \text{Im} \ L_2 \to \text{Im} \ L_2 \cap D(L_2)$ is given by

$$L_2^{-1} u := \sum_{(n,m) \in \Gamma_2} (n^2 - m^2 - 4)^{-1} u_{nm} \psi_{nm},$$

where $\Gamma_2 = \{(n, m) \in \mathbb{N} \times \mathbb{Z} \mid n^2 - m^2 \neq 4\}$, is compact.

Firstly, we suppose that $g : [0, \pi] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is $2\pi$-periodic in the second variable such that

(g1) $g$ and $\overline{g}$ are superpositionally measurable, that is, $\overline{g}(\cdot, u(\cdot, \cdot))$ and $\overline{g}(\cdot, \cdot, u(\cdot, \cdot))$ are measurable on $\Omega$ for any measurable function $u : \Omega \to \mathbb{R}$;

(g2) $g$ satisfies the growth condition:

$$|g(x, t, s)| \leq k_0(x, t) + c_0|s| \quad \text{for almost all } (x, t) \in \Omega \text{ and all } s \in \mathbb{R},$$

where $k_0 \in H$ is nonnegative and $c_0$ is a positive constant;

(g3) there is a positive constant $\alpha$ such that

$$(g(x, t, s) - g(x, t, \eta))(s - \eta) \geq \alpha |s - \eta|^2 \quad \text{for almost all } (x, t) \in \Omega \text{ and all } s, \eta \in \mathbb{R}.$$

We define a multi-valued operator $N_{1,1} : H \to 2^H$ by setting

$$N_{1,1}(u) := \{ w \in H \mid g(x, t, u(x, t)) \leq w(x, t) \leq \overline{g}(x, t, u(x, t)) \} \quad \text{for almost all } (x, t) \in \Omega.$$ 

(32)

Under conditions (g1) and (g2), the multi-valued operator $N_{1,1}$ is bounded, upper semicontinuous, and $Nu$ is nonempty, closed, and convex for every $u \in H$; see [16, Theorem 1.1]. Under additional condition (g3), the operator $N_{1,1}$ is of class $(S_+)$. 

Secondly, we suppose that $g_1 : [0, \pi] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is $2\pi$-periodic in the second variable such that

(g4) $g_1$ satisfies the Carathéodory condition, that is, $g_1(\cdot, \cdot, s)$ is measurable on $\Omega$ for all $s \in \mathbb{R}$ and $g_1(x, t, \cdot)$ is continuous on $\mathbb{R}$ for almost all $(x, t) \in \Omega$;
Definition 5.1. A point \((x, t, s)\) is clearly bounded and continuous under conditions (g6) and (g7).

We define the Nemytskii operator \(N_{1,2} : H \to H\) by

\[
N_{1,2}(u)(x, t) := g_1(x, t, u(x, t)) \quad \text{for } u \in H \text{ and } (x, t) \in \Omega.
\]

Under conditions (g4) and (g5), it is obvious that the operator \(N_{1,2}\) is bounded and continuous; see e.g., [17].

Thirdly, we suppose that \(g_2 : [0, \pi] \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}\) is \(2\pi\)-periodic in the second variable such that (g6) \(g_2\) satisfies the Carathéodory condition;
(g7) there exist a nonnegative function \(k_2 \in H\) and a positive constant \(c_2\) such that

\[
|g_2(x, t, s, p)| \leq k_2(x, t) + c_2(|s| + |p|) \quad \text{for almost all } (x, t) \in \Omega \text{ and all } (s, p) \in \mathbb{R}^2.
\]

The Nemytskii operator \(N_2 : H \times H \to H\) given by

\[
N_2(u, v)(x, t) := g_2(x, t, u(x, t), v(x, t)) \quad \text{for } (u, v) \in H \times H \text{ and } (x, t) \in \Omega
\]
is clearly bounded and continuous under conditions (g6) and (g7).

Definition 5.1. A point \((v, z) \in H \times H\) is said to be a weak solution of the problem (27) if there exists a point \(w \in N_{1,1}(v)\) such that

\[
\langle v, y_{tt} - y_{xx} \rangle - \langle w, y \rangle - \langle N_{1,2}(z), y \rangle = \langle h_1, y \rangle \quad \text{and} \quad \langle z, y_{tt} - y_{xx} - 4y \rangle - \langle N_2(v, z), y \rangle = \langle h_2, y \rangle
\]
for all \(y \in C^2\), where \(C^2\) denotes the space of twice continuously differentiable functions \(y : \mathbb{R} \to \mathbb{R}\) such that

\[
y(0, \cdot) = y(\pi, \cdot) = 0 \quad \text{and} \quad y(\cdot, 0) - y(\cdot, 2\pi) = y(\cdot, 0) - y(\cdot, 2\pi) = 0.
\]

In view of the above definitions, (27) has a weak solution \((v, z)\) in \(H \times H\) if and only if \((v, z) \in D(L_1) \times D(L_2)\) is a solution of the semilinear system

\[
\begin{align*}
L_1v - N_{1,1}(v) - N_{1,2}(z) & \ni h_1 \\
L_2z - N_2(v, z) & = h_2.
\end{align*}
\tag{30}
\]

Theorem 5.2. Let \(g, g_1 : [0, \pi] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) and \(g_2 : [0, \pi] \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}\) satisfy the conditions (g1)–(g7).
Suppose that there are \(\mu \in [0, 1/2)\) and \(\beta_1, \beta_2 \in [0, \infty)\) such that

\[
\left|w(x, t) + g_1(x, t, p) - \frac{1}{2} s \right| \leq \mu |s| + \beta_1,
\]

\[
\left|g_2(x, t, s, p) - \frac{1}{2} p \right| \leq \mu |p| + \beta_2,
\]
for almost all \((x, t) \in \Omega\) and all \((s, p) \in \mathbb{R}^2\) with \(|(s, p)| \to \infty\), where \(g(x, t, s) \leq w(x, t) \leq \Gamma(x, t, s)\). Then for every \((h_1, h_2) \in H \times H\), the given problem (27) has a weak solution.

Proof. Let \(L = (L_1, L_2)\) and \(N = (N_1, N_2)\) be defined as above. Notice that \(L_1, L_2\) are closed densely defined linear operators with \(\text{dim Ker } L_1 = \infty\) and \(\text{dim Ker } L_2 < \infty\). Moreover, \(N_{1,1}\) is a bounded upper semi-continuous multi-valued operator of class \((S_+)\), and the Nemytskii operators \(N_{1,2}\) and \(N_2\) are bounded and continuous. It follows from (28) and (29) that

\[
\|Lu\|^2 \geq \langle LU, u \rangle \quad \text{for all } u \in D(L),
\]
which means that \(1 \in \mathcal{A}\), in the sense of (8). By hypotheses, we have

\[
\left\|w + N_{1,2}(z) - \frac{1}{2} v \right\| \leq \mu \|v\| + \xi_1,
\]

\[
\left\|N_2(v, z) - \frac{1}{2} z \right\| \leq \mu \|z\| + \xi_2,
\]
for all \(v, z \in H, \| (v, z) \| \to \infty\), and all \(w \in N_{1,1}(v)\), where \(\xi_1, \xi_2\) are some constants. Applying Theorem 4.7 with \(\rho = 1\) and \(\alpha = 0\), the system
\[
\begin{align*}
L_1 v - N_{1,1}(v) - N_{1,2}(z) &\geq h_1 \\
L_2 z - N_{2}(v, z) &\geq h_2
\end{align*}
\]  
(31)

has a solution for every \((h_1, h_2) \in H \times H\). Therefore, (27) has a weak solution. This completes the proof. 

Remark 5.3. Berkovits-Tienari [7] studied the periodic Dirichlet problem of the form
\[ u_{tt} - u_{xx} \in [g(x, t, u), P(x, t, u)] \quad \text{in} \quad (0, \pi) \times \mathbb{R}, \]
where \(g(x, t, \cdot)\) is nondecreasing on \(\mathbb{R}\), based on the degree for the class \((S_+)\) introduced in [7]. In this note, the main point is that when treating semilinear system (27) with mixed nonlinear terms, certain monotonicity assumptions on \(g_1\) and \(g_2\) are not required. As was seen in Section 4, the class \((S_+)P\) is more general than the class \((S_+)\) and our degree theory for the class \((S_+)P\) plays a decisive role in the study of semilinear systems.

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References