One sided strong laws for random variables with infinite mean

DOI 10.1515/math-2017-0070
Received December 12, 2016; accepted May 16, 2017.

Abstract: This paper establishes conditions that secure the almost sure upper and lower bounds for a particular normalized weighted sum of independent nonnegative random variables. These random variables do not possess a finite first moment so these results are not typical. These mild conditions allow us to show that the almost sure upper limit is infinity while the almost sure lower bound is one.

Keywords: Almost sure convergence, Weak law of large numbers, One sided strong laws, Slow variation

MSC: 60F15

This paper extends work done in [1] and [2]. We use those Weak Laws to obtain our almost sure upper and lower bounds. We present conditions that allow us to achieve these bounds for our normalized weighted sums. As in [2] the random variables at the heart of these theorems are independent without a first moment where \( P \{ X_j = 0 \} = 1 - a_j \) and \( P \{ X_j > x \} = \frac{1}{x^{1/a_j}} \), for \( x > 0 \). We set \( A_n = \sum_{j=1}^{n} a_j \), which must go to infinity in order to establish our theorems.

In terms of notation we use \( \lg x = \ln x \), but whenever we have \( \ln 1 \) in a denominator we will just set that equal to one, so we won’t be dividing by zero. Hence \( \lg x = \ln x \), except \( \lg 1 = 1 \). We set \( \lg_1 x = \lg x \), and \( \lg_{k+1} x = \lg(\lg_k x) \), so the \( i \) in \( \lg_i x \) is not the base, it is the iteration of the logarithm. Many times in the proofs and the examples we will pull the logarithm and any other slowly varying function out from an integral and the infinite summations since they are slowly varying, see [3] pages 279-284. Finally, note that the constant \( C \) will be used as a generic bound that is not necessarily the same in each appearance. From [2] we have a Weak Law that helps us produce the almost sure lower bound.

**Theorem 1.** If \( A_n \to \infty \), then

\[
\frac{\sum_{j=1}^{n} a_j X_j}{A_n \lg A_n} \to P.
\]

(1)

We next establish the almost sure upper bound for these same weighted sums. The condition that ensures that the upper bound is infinite is

\[
\sum_{n=1}^{\infty} \frac{a_n}{A_n \lg A_n} = \infty.
\]

(2)

**Theorem 2.** If (2) holds, then

\[
\limsup_{n \to \infty} \frac{\sum_{j=1}^{n} a_j X_j}{A_n \lg A_n} = \infty \quad \text{almost surely.}
\]

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Proof. For all $M > 0$
$$
\sum_{n=1}^{\infty} P\{X_n > M\} = \sum_{n=1}^{\infty} \frac{P\{X_n > MA_n \lg A_n\}}{a_n} = \sum_{n=1}^{\infty} \frac{1}{a_n} \frac{MA_n \lg A_n}{a_n} + \frac{1}{a_n} = \sum_{n=1}^{\infty} \frac{a_n}{MA_n \lg A_n + 1} = \infty.
$$
Thus
$$
\limsup_{n \to \infty} \frac{a_n X_n}{A_n \lg A_n} = \infty \quad \text{almost surely}
$$
hence
$$
\limsup_{n \to \infty} \frac{\sum_{j=1}^{n} a_j X_j}{A_n \lg A_n} \geq \limsup_{n \to \infty} \frac{a_n X_n}{A_n \lg A_n} = \infty \quad \text{almost surely}
$$
which completes this proof.

In order to get the lower limit we assume there is a sequence $\{d_n, n \geq 1\}$ where
$$
\lim_{n \to \infty} \frac{\sum_{j=1}^{n} a_j \lg(a_j d_j)}{A_n \lg A_n} = 1
$$
(3)
and
$$
\sum_{n=1}^{\infty} \frac{a_j^2 d_n}{A_n^2 \lg^2 A_n} < \infty.
$$
(4)
Note that we need both $A_n$ and $a_n d_n$ to go to infinity to achieve our results. But these are mild conditions.

What is fascinating is that even though we can split (3) into two pieces, we need both parts. When $a_j = j^\alpha$, where $\alpha > -1$, we select $d_j = j$ which gives us
$$
\sum_{j=1}^{n} \frac{a_j \lg a_j}{A_n \lg A_n} \to \frac{\alpha}{\alpha + 1}
$$
and
$$
\sum_{j=1}^{n} \frac{a_j \lg d_j}{A_n \lg A_n} \to \frac{1}{\alpha + 1}.
$$
This shows that both terms are necessary. Combining that with $A_n \sim n^{\alpha+1}/(\alpha + 1)$, we see that (3) and (4) hold, see Example 4.

**Theorem 3.** If there exist a sequence $\{d_n, n \geq 1\}$ satisfying (3) and (4) then
$$
\liminf_{n \to \infty} \frac{\sum_{j=1}^{n} a_j X_j}{A_n \lg A_n} = 1 \quad \text{almost surely.}
$$

Proof. From Theorem 1 we have
$$
\liminf_{n \to \infty} \frac{\sum_{j=1}^{n} a_j X_j}{A_n \lg A_n} \leq 1 \quad \text{almost surely.}
$$
We will show that conditions (3) and (4) establishes
$$
\liminf_{n \to \infty} \frac{\sum_{j=1}^{n} a_j X_j}{A_n \lg A_n} \geq 1 \quad \text{almost surely.}
$$
(5)
Using our sequence $\{d_n, n \geq 1\}$ we have
$$
\frac{\sum_{j=1}^{n} a_j X_j}{A_n \lg A_n} \geq \frac{\sum_{j=1}^{n} a_j X_j I(X_j \leq d_j)}{A_n \lg A_n} = \frac{\sum_{j=1}^{n} a_j [X_j I(X_j \leq d_j) - EX_j I(X_j \leq d_j)]}{A_n \lg A_n} + \frac{\sum_{j=1}^{n} a_j EX_j I(X_j \leq d_j)}{A_n \lg A_n}.
$$
By the Khintchine-Kolmogorov Convergence Theorem, see [4] page 113, and the Kronecker lemma, the first term converges to zero almost surely since

$$
\sum_{n=1}^{\infty} \frac{a_n^2 \mathbb{E}X_n^2 I(X_n \leq d_n)}{A_n^2 \lg^2 A_n} = \sum_{n=1}^{\infty} \frac{a_n^2}{A_n^2 \lg^2 A_n} \int_{0}^{d_n} \frac{x^2 \, dx}{(x + 1/a_n)^2} \lesssim \sum_{n=1}^{\infty} \frac{a_n^2 d_n}{A_n^2 \lg^2 A_n} < \infty
$$

by (4). We now examine our expectation

$$
\sum_{j=1}^{n} \frac{a_j \mathbb{E}X_j I(X_j \leq d_j)}{A_n \lg A_n} = \frac{1}{A_n \lg A_n} \sum_{j=1}^{n} a_j \int_{1/a_j}^{d_j+1/a_j} \frac{du}{u^2} - \frac{1}{A_n \lg A_n} \sum_{j=1}^{n} \frac{d_j + 1/a_j}{1/a_j} \int_{1/a_j}^{d_j+1/a_j} \frac{du}{u^2}.
$$

The first term converges to one by (3) since

$$
\frac{1}{A_n \lg A_n} \sum_{j=1}^{n} a_j \int_{1/a_j}^{d_j+1/a_j} \frac{du}{u} = \frac{1}{A_n \lg A_n} \sum_{j=1}^{n} a_j \left[ \lg(d_j + 1/a_j) - \lg(1/a_j) \right] = \frac{1}{A_n \lg A_n} \sum_{j=1}^{n} a_j \lg(a_j d_j + 1) \to 1
$$

while the second term goes to zero since

$$
\left| - \frac{1}{A_n \lg A_n} \sum_{j=1}^{n} \int_{1/a_j}^{d_j+1/a_j} \frac{du}{u^2} \right| \leq \frac{1}{A_n \lg A_n} \sum_{j=1}^{n} \int_{1/a_j}^{\infty} \frac{du}{u^2} = \frac{1}{A_n \lg A_n} \sum_{j=1}^{n} a_j = \frac{A_n}{A_n \lg A_n} = \frac{1}{\lg A_n} \to 0.
$$

Hence (5) does hold, concluding the proof. \(\square\)

We conclude with three examples, each showing how to select \(\{d_n, n \geq 1\}\) so that (3) and (4) hold. What is quite remarkable is that if we select \(\{d_n, n \geq 1\}\) correctly we see that all three conditions hold even though (2) and (4) seem contradictory.

**Example 4.** Let \(L(x)\) be any slowly varying function and \(\alpha > -1\), then

$$
\liminf_{n \to \infty} \frac{\sum_{j=1}^{n} j^\alpha L(j) X_j}{n^{\alpha+1} L(n) \lg n} = 1 \quad \text{almost surely}
$$

and

$$
\limsup_{n \to \infty} \frac{\sum_{j=1}^{n} j^\alpha L(j) X_j}{n^{\alpha+1} L(n) \lg n} = \infty \quad \text{almost surely}.
$$

**Proof.** Since \(a_j = j^\alpha L(j)\) we have \(A_n \sim n^{\alpha+1} L(n)/(\alpha + 1)\) and \(\lg A_n \sim (\alpha + 1) \lg n\), thus

$$
\sum_{n=1}^{\infty} \frac{a_n}{A_n \lg A_n} \geq C \sum_{n=1}^{\infty} \frac{n^{\alpha} L(n)}{n^{\alpha+1} L(n) \lg n} = C \sum_{n=1}^{\infty} \frac{1}{n \lg n} = \infty.
$$

In this case we set \(d_j = j\). Thus \(a_j d_j = j^{\alpha+1} L(j)\) and \(\lg(a_j d_j) \sim (\alpha + 1) \lg j\), hence

$$
\frac{\sum_{j=1}^{n} a_j \lg(a_j d_j)}{A_n \lg A_n} \sim \frac{(\alpha + 1) \sum_{j=1}^{n} j^\alpha L(j) \lg j}{n^{\alpha+1} L(n) \lg n} \to 1.
$$
and
\[ \sum_{n=1}^{\infty} \frac{a_n^2 d_n}{A_n^2 \log^2 A_n} \leq C \sum_{n=1}^{\infty} \frac{n^{2\alpha+1} L^2(n)}{n^{\alpha+2} L^2(n) \log^2 n} = C \sum_{n=1}^{\infty} \frac{1}{n \log^2 n} < \infty. \]

Next we look at a borderline case, where once again \( A_n \to \infty \) is necessary.

**Example 5.** If \( \alpha > -1 \), then for all \( \delta \)
\[ \liminf_{n \to \infty} \frac{\sum_{j=1}^{n} \frac{(\log j)^{\alpha} (\log j)^{\delta}}{(\log n)^{\alpha+1} (\log n)^{\delta+1}} X_j}{\log n} = 1 \text{ almost surely} \]
and
\[ \limsup_{n \to \infty} \frac{\sum_{j=1}^{n} \frac{(\log j)^{\alpha} (\log j)^{\delta}}{(\log n)^{\alpha+1} (\log n)^{\delta+1}} X_j}{\log n} = \infty \text{ almost surely}. \]

**Proof.** Since \( a_j = (\log j)^{\alpha} (\log j)^{\delta} \), we have \( A_n \sim (\log n)^{\alpha+1} (\log n)^{\delta}/(\alpha + 1) \) and \( \log A_n \sim (\alpha + 1) \log n \), thus
\[ \sum_{n=1}^{\infty} \frac{a_n^2 d_n}{A_n^2 \log^2 A_n} \leq C \sum_{n=1}^{\infty} \frac{1}{n \log n \log^2 n} \leq C \sum_{n=1}^{\infty} \frac{1}{n \log n \log^2 n} < \infty. \]

In this case we set \( d_j = j \log j \). Thus \( a_j d_j = (\log j)^{\alpha+1} (\log j)^{\delta} \) and \( \log(a_j d_j) \sim (\alpha + 1) \log j \), hence
\[ \sum_{j=1}^{n} \frac{a_j d_j}{A_n \log A_n} \sim \frac{(\alpha + 1) \sum_{j=1}^{n} (\log j)^{\alpha} (\log j)^{\delta+1}}{(\log n)^{\alpha+1} (\log n)^{\delta+1}} \to 1 \]
and
\[ \sum_{n=1}^{\infty} \frac{a_n^2 d_n}{A_n^2 \log^2 A_n} \leq C \sum_{n=1}^{\infty} \frac{1}{n \log n \log^2 n} \leq C \sum_{n=1}^{\infty} \frac{1}{n \log n \log^2 n} < \infty. \]

**Example 6.** If \( \alpha > -1 \), then for all \( \delta \)
\[ \liminf_{n \to \infty} \frac{\sum_{j=1}^{n} \frac{(\log j)^{\alpha} (\log j)^{\delta}}{(\log n)^{\alpha+1} (\log n)^{\delta+1}} X_j}{\log n} = 1 \text{ almost surely} \]
and
\[ \limsup_{n \to \infty} \frac{\sum_{j=1}^{n} \frac{(\log j)^{\alpha} (\log j)^{\delta}}{(\log n)^{\alpha+1} (\log n)^{\delta+1}} X_j}{\log n} = \infty \text{ almost surely}. \]

**Proof.** Since \( a_j = (\log j)^{\alpha} (\log j)^{\delta} \), we have \( A_n \sim (\log n)^{\alpha+1} (\log n)^{\delta}/(\alpha + 1) \) and \( \log A_n \sim (\alpha + 1) \log j \), thus
\[ \sum_{n=1}^{\infty} \frac{a_n^2 d_n}{A_n^2 \log^2 A_n} \leq C \sum_{n=1}^{\infty} \frac{1}{n \log n \log^2 n} \leq C \sum_{n=1}^{\infty} \frac{1}{n \log n \log^2 n} < \infty. \]

In this case we set \( d_j = j \log j \). Thus \( a_j d_j = (\log j)^{\alpha+1} (\log j)^{\delta} \) and \( \log(a_j d_j) \sim (\alpha + 1) \log j \), hence
\[ \sum_{j=1}^{n} \frac{a_j d_j}{A_n \log A_n} \sim \frac{(\alpha + 1) \sum_{j=1}^{n} (\log j)^{\alpha} (\log j)^{\delta+1}}{(\log n)^{\alpha+1} (\log n)^{\delta+1}} \to 1 \]
and
\[ \sum_{n=1}^{\infty} \frac{a_n^2 d_n}{A_n^2 \log^2 A_n} \leq C \sum_{n=1}^{\infty} \frac{1}{n \log n \log^2 n} \leq C \sum_{n=1}^{\infty} \frac{1}{n \log n \log^2 n} < \infty. \]
Remark 7. It would be nice to replace (2) with the milder condition $A_n \to \infty$, but they aren’t equivalent. If we have rapidly growing sequences we can see the difference. For example, let $A_n = e^{n^2}$, then clearly $A_n \to \infty$. However

$$a_n = A_n - A_{n-1} = e^{n^2} - e^{(n-1)^2} < e^{n^2}$$

and

$$\sum_{n=1}^{\infty} \frac{a_n}{A_n \lg A_n} < \sum_{n=1}^{\infty} \frac{e^{n^2}}{n \lg(e^{n^2})} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

References