This paper gives extensions and improvements of Sherman’s inequality for $n$-convex functions obtained by using new identities which involve Green’s functions and Fink’s identity. Moreover, extensions and improvements of Majorization inequality as well as Jensen’s inequality are obtained as direct consequences. New inequalities between geometric, logarithmic and arithmetic means are also established.

**Keywords:** Sherman inequality, Majorization inequality, Jensen inequality, $n$-convex, Green functions, Fink identity, Čebyšev functional, Means

**MSC:** 26D15

1 Introduction

In [1], Sherman proved that the inequality

$$\sum_{i=1}^{m} b_i \phi(y_i) \leq \sum_{j=1}^{l} a_j \phi(x_j) \tag{1}$$

holds for every convex function $\phi : [\alpha, \beta] \to \mathbb{R}$, where vectors $\mathbf{x} = (x_1,...,x_l) \in [\alpha, \beta]^l$, $\mathbf{y} = (y_1,...,y_m) \in [\alpha, \beta]^m$, $\mathbf{a} = (a_1,...,a_l) \in [0,\infty)^l$, $\mathbf{b} = (b_1,...,b_m) \in [0,\infty)^m$ are such that

$$\mathbf{y} = \mathbf{xA}^T \quad \text{and} \quad \mathbf{a} = \mathbf{bA} \tag{2}$$

holds for some row stochastic matrix $\mathbf{A} = a_{ij} \in \mathcal{M}_{m,l}(\mathbb{R})$, i.e. matrix with

$$a_{ij} \geq 0 \quad \text{for all} \quad i = 1,...,m, \quad j = 1,...,l,$$

$$\sum_{j=1}^{l} a_{ij} = 1 \quad \text{for all} \quad i = 1,...,m,$$

while $\mathbf{A}^T$ denotes the transpose of $\mathbf{A}$. If $\phi$ is concave, then the reverse inequality in (1) holds.

This result generalizes classical Majorization inequality, proved by Hardy et al [2], as well as Jensen’s inequality. The purpose of this paper is to extend Sherman’s result to the more general class of $n$-convex functions and to give...
improvements of Sherman’s inequality (1) from which extensions and improvements of Majorization inequality and Jensen’s inequality immediately follow. Some related results can be found in [3-6].

The study of $n$-convex functions on an interval is the subject of a monograph by Popoviciu [7]. Popoviciu defined $n$-convexity of a function $\phi : [\alpha, \beta] \to \mathbb{R}$ in terms of the divided differences (of order $n$) which are defined recursively as follows:

$$[x_0, \ldots, x_n; \phi] = \frac{[x_1, \ldots, x_{n-1}; \phi] - [x_0, \ldots, x_{n-1}; \phi]}{x_n - x_0},$$

where $x_0, x_1, \ldots, x_n \in [\alpha, \beta]$ are mutually different points and the value $[x_0, \ldots, x_n; \phi]$ is independent of their order. This definition may be extended to include the case in which some or all the points coincide. Assuming that $\phi^{(j-1)}(x)$ exists, we define

$$[x_0, \ldots, x_0; \phi] = \frac{\phi^{(j-1)}(x)}{(j-1)!}. \quad (3)$$

A function $\phi : [\alpha, \beta] \to \mathbb{R}$ is $n$-convex ($n \geq 0$) if for all choices of $(n + 1)$ distinct points $x_i \in [\alpha, \beta], i = 0, \ldots, n$, the inequality

$$[x_0, x_1, \ldots, x_n; \phi] \geq 0$$

holds. If this inequality is reversed, then $\phi$ is $n$-concave.

Thus a 1-convex function is nondecreasing and a 2-convex function is convex in the usual sense. An $n$-convex function $\phi$ need not be $n$-times differentiable (e.g. $\phi : [0, 1] \to \mathbb{R}, \phi(x) = x^3$). However iff $\phi^{(n)}$ exists then $\phi$ is $n$-convex iff $\phi^{(n)} \geq 0$ (see [8, p. 16]).

In order to develop some inequalities of type (1) for $n$-convex functions, we use the following Fink’s identity [9]

$$\phi(s) = \frac{n}{\beta - \alpha} \int_{\alpha}^{\beta} \phi(t) dt - \sum_{w=1}^{n-1} \frac{n - w}{w!} \frac{\phi^{(w-1)}(\alpha)(s - \alpha)^w - \phi^{(w-1)}(\beta)(s - \beta)^w}{\beta - \alpha}$$

$$+ \frac{1}{(n-1)!(\beta - \alpha)} \int_{\alpha}^{\beta} (s - t)^{n-1} P(t, s) \phi^{(n)}(t) dt \quad (4)$$

with

$$P(t, s) = \begin{cases} t - \alpha, & \alpha \leq t \leq s \leq \beta \\ t - \beta, & \alpha \leq s < t \leq \beta \end{cases}. \quad (5)$$

which holds for every function $\phi : [\alpha, \beta] \to \mathbb{R}$ such that $\phi^{(n-1)}$ is absolutely continuous on $[\alpha, \beta]$ for some $n \geq 1$.

**Remark 1.1.** For $n = 1$ we take the sum in (4) to be zero.

### 2 Some new identities

It is easy to verify that integration by parts yields that for any function $\phi \in C^2([\alpha, \beta])$ the following identity holds

$$\phi(u) = \phi(\alpha) + (u - \alpha)\phi'(\beta) + \int_{\alpha}^{\beta} G_1(u, s) \phi''(s) ds, \quad (6)$$

where the function $G_1 : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ is Green’s function of the boundary value problem

$$z'' = 0, \quad z(\alpha) = z'(\beta) = 0$$

and is defined by

$$G_1(u, s) = \begin{cases} \alpha - s, & s \leq u, \\ \alpha - u, & u \leq s. \end{cases} \quad (7)$$
Green’s function $G_1$ is continuous and convex in $u$, since it is symmetric, i.e. $G_1(u, s) = G_1(s, u)$, then also in $s$.

Here we introduce three new types of Green’s functions defined on $[\alpha, \beta] \times [\alpha, \beta]$ as follows:

\[
G_2(u, s) = \begin{cases} 
  u - \beta, & s \leq u \\
  s - \beta, & u \leq s 
\end{cases},
\]
(8)

\[
G_3(u, s) = \begin{cases} 
  u - \alpha, & s \leq u \\
  s - \alpha, & u \leq s 
\end{cases},
\]
(9)

\[
G_4(u, s) = \begin{cases} 
  \beta - s, & s \leq u \\
  \beta - u, & u \leq s 
\end{cases}.
\]
(10)

All three functions are continuous, symmetric and convex with respect to both variables $u$ and $s$.

Next we introduce three technical lemmas which give us new identities involving defined Green’s functions.

**Lemma 2.1.** Let $G_k(\cdot, s), s \in [\alpha, \beta], k = 2, 3, 4$, be defined as in (8)-(10). Then for every $\phi \in C^2([\alpha, \beta])$, it holds that

\[
\phi(u) = \phi(\beta) + (u - \beta)\phi'(\alpha) + \int_{\alpha}^{\beta} G_2(u, s)\phi''(s)ds,
\]
(11)

\[
\phi(u) = \phi(\beta) - (\beta - \alpha)\phi'(\beta) + (u - \alpha)\phi'(\alpha) + \int_{\alpha}^{\beta} G_3(u, s)\phi''(s)ds,
\]
(12)

\[
\phi(u) = \phi(\alpha) + (\beta - \alpha)\phi'(\alpha) - (\beta - u)\phi'(\beta) + \int_{\alpha}^{\beta} G_4(u, s)\phi''(s)ds.
\]
(13)

**Proof.** Utilizing integration by parts we have

\[
\int_{\alpha}^{\beta} G_2(u, s)\phi''(s)ds = \int_{\alpha}^{u} G_2(u, s)\phi''(s)ds + \int_{u}^{\beta} G_2(u, s)\phi''(s)ds
\]

\[= \int_{\alpha}^{u} (u - \beta)\phi''(s)ds + \int_{u}^{\beta} (s - \beta)\phi''(s)ds
\]

\[= (u - \beta)\phi'(s)|_{\alpha}^{u} + (s - \beta)\phi'(s)|_{u}^{\beta} - \int_{u}^{\beta} \phi'(s)ds
\]

\[= (u - \beta)\phi'(u) - (u - \beta)\phi'(\alpha)
\]

\[+ (\beta - \beta)\phi'(\beta) - (\beta - u)\phi'(u) - \phi(\beta) + \phi(u)
\]

\[= -(u - \beta)\phi'(\alpha) - \phi(\beta) + \phi(u),
\]
from which (11) follows immediately. Similarly, we can prove other two identities.

**Lemma 2.2.** Let $x \in [\alpha, \beta]^l$, $y \in [\alpha, \beta]^m$, $a \in \mathbb{R}^l$ and $b \in \mathbb{R}^m$ be such that (2) holds for some matrix $A \in \mathcal{M}_{ml}(\mathbb{R})$ whose entries satisfy the condition $\sum_{j=1}^{l} a_{ij} = 1$ for $i = 1, \ldots, m$. Let $G_k(\cdot, s), s \in [\alpha, \beta], k = 1, 2, 3, 4$, be defined as in (7)-(10). Then for every $\phi \in C^2([\alpha, \beta])$, it holds that

\[
\sum_{j=1}^{l} a_{j} \phi(x_j) - \sum_{i=1}^{m} b_{i} \phi(y_i) = \int_{\alpha}^{\beta} \left( \sum_{j=1}^{l} a_{j} G_k(x_j, s) - \sum_{i=1}^{m} b_{i} G_k(y_i, s) \right) \phi''(s)ds.
\]
(14)
Proof. Let us consider Green’s function $G_2$ defined by (8). Applying (11) in the difference \[ \sum_{j=1}^{n} a_j \phi(x_j) - \sum_{i=1}^{m} b_i \phi(y_i), \] we have
\[
\sum_{j=1}^{l} a_j \phi(x_j) - \sum_{i=1}^{m} b_i \phi(y_i) = \sum_{j=1}^{l} a_j \left[ \phi(\beta) + (x_j - \beta)\phi'(\alpha) + \int_{\alpha}^{\beta} G_2(x_j, s)\phi''(s)ds \right] - \sum_{i=1}^{m} b_i \left[ \phi(\beta) + (y_i - \beta)\phi'(\alpha) + \int_{\alpha}^{\beta} G_2(y_i, s)\phi''(s)ds \right].
\] (15)

Since (2) holds, then we have
\[
\sum_{i=1}^{m} b_i = \sum_{j=1}^{l} a_j - \sum_{i=1}^{m} \sum_{j=1}^{l} b_i a_{ij} = \sum_{i=1}^{m} b_i - \sum_{i=1}^{m} \sum_{j=1}^{l} a_{ij} = 0
\]
and
\[
\sum_{i=1}^{m} b_i = \sum_{j=1}^{l} a_j - \sum_{i=1}^{m} \sum_{j=1}^{l} b_i a_{ij} = \sum_{i=1}^{m} b_i - \sum_{i=1}^{m} \sum_{j=1}^{l} x_j a_{ij} = 0.
\]
Moreover, after interchanging the order of summation in (15), we easily get
\[
\sum_{j=1}^{l} a_j \phi(x_j) - \sum_{i=1}^{m} b_i \phi(y_i) = \int_{\alpha}^{\beta} \left( \sum_{j=1}^{l} a_j G_2(x_j, s) - \sum_{i=1}^{m} b_i G_2(y_i, s) \right) \phi''(s)ds.
\]

Analogously, we can prove the identities for other three Green’s functions. \qed

Lemma 2.3. Let $x \in [\alpha, \beta]$, $y \in [\alpha, \beta]^m$, $a \in \mathbb{R}^l$ and $b \in \mathbb{R}^m$ be such that (2) holds for some matrix $A \in \mathcal{M}_{ml}(\mathbb{R})$ whose entries satisfy the condition $\sum_{j=1}^{l} a_{ij} = 1$ for $i = 1, \ldots, m$. Let $P(t, s)$ and $G_k(., s)$, $s, t \in [\alpha, \beta]$, $k = 1, 2, 3, 4$, be defined as in (5) and (7)-(10), respectively. If a function $\phi : [\alpha, \beta] \rightarrow \mathbb{R}$ is such that $\phi^{(n-1)}$ is absolutely continuous on $[\alpha, \beta]$ for some $n \geq 3$, then
\[
\sum_{j=1}^{l} a_j \phi(x_j) - \sum_{i=1}^{m} b_i \phi(y_i)
= \sum_{w=0}^{n-3} \frac{n - w - 2}{(\beta - \alpha)w!} \int_{\alpha}^{\beta} \left( \sum_{j=1}^{l} a_j G_k(x_j, s) - \sum_{i=1}^{m} b_i G_k(y_i, s) \right) \left( \phi^{(w+1)}(\beta)(s - \beta)^w - \phi^{(w+1)}(\alpha)(s - \alpha)^w \right)ds
+ \frac{1}{(n-3)!(\beta - \alpha)} \int_{\alpha}^{\beta} \phi^{(n)}(t) \left( \sum_{j=1}^{l} a_j G_k(x_j, s) - \sum_{i=1}^{m} b_i G_k(y_i, s) \right) (s - t)^{n-3} P(t, s)ds dt.
\] (16)

Proof. Applying (4) for $\phi''$, we get
\[
\phi''(s) = \sum_{w=0}^{n-3} \frac{n - w - 2}{w!} \cdot \frac{\phi^{(w+1)}(\beta)(s - \beta)^w - \phi^{(w+1)}(\alpha)(s - \alpha)^w}{\beta - \alpha}
+ \frac{1}{(n-3)!(\beta - \alpha)} \int_{\alpha}^{\beta} (s - t)^{n-3} P(t, s)\phi^{(n)}(t)dt.
\] (17)
By an easy calculation, applying (17) in (14), we get
\[
\sum_{j=1}^{l} a_j \phi(x_j) - \sum_{i=1}^{m} b_i \phi(y_i) = \int_{\alpha}^{\beta} \left( \sum_{j=1}^{l} a_j G_k(x_j, s) - \sum_{i=1}^{m} b_i G_k(y_i, s) \right) \times \\
\left( \sum_{w=0}^{n-3} \frac{n-w-2}{w!} \phi^{(w+1)}(s) - \phi^{(w+1)}(\alpha) - \phi^{(w+1)}(\beta) s - \alpha \right) \\
+ \frac{1}{(n-3)!} \int_{\alpha}^{\beta} (s-t)^{n-3} P(t,s) \phi^{(n)}(t) dt \right) ds.
\]

After interchanging the order of summation and integration and applying Fubini’s theorem we get (16).

3 Sherman’s type inequalities

We begin this section with the following result which concerns Sherman’s type inequalities for real, not necessary nonnegative entries of vectors \(a, b\) and matrix \(A\).

**Theorem 3.1.** Let \(x \in [\alpha, \beta]^l\), \(y \in [\alpha, \beta]^m\), \(a \in \mathbb{R}^l\) and \(b \in \mathbb{R}^m\) be such that (2) holds for some matrix \(A \in \mathcal{M}_{m \times l}(\mathbb{R})\) whose entries satisfy the condition \(\sum_{j=1}^{l} a_{ij} = 1\) for \(i = 1, ..., m\). Let \(G_k(., s), s \in [\alpha, \beta], k = 1, 2, 3, 4\), be defined as in (7)-(10). Then the following statements are equivalent:

(i) For every continuous convex function \(\phi : [\alpha, \beta] \rightarrow \mathbb{R}\), it holds that

\[
\sum_{i=1}^{m} b_i \phi(y_i) \leq \sum_{j=1}^{l} a_j \phi(x_j). 
\]

(ii) For every \(k = 1, 2, 3, 4\) and \(s \in [\alpha, \beta]\), it holds that

\[
\sum_{i=1}^{m} b_i G_k(y_i, s) \leq \sum_{j=1}^{l} a_j G_k(x_j, s). 
\]

Furthermore, the statements (i) and (ii) are also equivalent if one changes the sign of inequality in both (18) and (19).

**Proof.** (i)\(\Rightarrow\) (ii) Let (i) hold. Since \(G_k(., s), s \in [\alpha, \beta], k = 1, 2, 3, 4\), is continuous and convex on \([\alpha, \beta]\), then also

\[
\sum_{i=1}^{m} b_i G_k(y_i, s) \leq \sum_{j=1}^{l} a_j G_k(x_j, s).
\]

(ii)\(\Rightarrow\) (i) Let (ii) hold. Let us consider the function \(G_2(., s), s \in [\alpha, \beta]\), defined by (8). For every function \(\phi \in C^2([\alpha, \beta])\) from (14) we have

\[
\sum_{j=1}^{l} a_j \phi(x_j) - \sum_{i=1}^{m} b_i \phi(y_i) = \int_{\alpha}^{\beta} \left( \sum_{j=1}^{l} a_j G_2(x_j, s) - \sum_{i=1}^{m} b_i G_2(y_i, s) \right) \phi''(s) ds.
\]

If \(\phi\) is convex, then \(\phi'' \geq 0\) on \([\alpha, \beta]\). Furthermore, if

\[
\sum_{j=1}^{l} a_j G_2(x_j, s) - \sum_{i=1}^{m} b_i G_2(y_i, s) \geq 0,
\]

then also (18) holds. Note that it is not necessary to demand the existence of the second derivative of the function \(\phi\). The differentiability condition can be directly eliminated by using the fact that a continuous convex function is possible to approximate uniformly by convex polynomials (see [8, p. 172]). The same conclusion we have for other three Green’s functions.

The last part statement of theorem can be proved analogously.
Next, we develop Sherman’s type inequalities for $n$-convex functions.

**Theorem 3.2.** Let $x \in [\alpha, \beta]^l$, $y \in [\alpha, \beta]^m$, $a \in \mathbb{R}^l$ and $b \in \mathbb{R}^m$ be such that (2) holds for some matrix $A \in \mathcal{M}_{nl}(\mathbb{R})$ whose entries satisfy the condition $\sum_{i=1}^{l} a_{ij} = 1$ for $i = 1, \ldots, m$. Let $P(t, s)$ and $G_k(\cdot, s)$, $s, t \in [\alpha, \beta]$, $k = 1, 2, 3, 4$, be defined as in (5) and (7)-(10), respectively. Let a function $\phi : [\alpha, \beta] \to \mathbb{R}$ be $n$-convex function such that $\phi^{(n-1)}$ is absolutely continuous on $[\alpha, \beta]$ for some $n \geq 3$.

If

$$
\int_{\alpha}^{\beta} \left( \sum_{j=1}^{l} a_j G_k(x_j, s) - \sum_{i=1}^{m} b_i G_k(y_i, s) \right) (s-t)^{n-3} P(t, s) ds \geq 0,
$$

(20)

then

$$
\sum_{j=1}^{l} a_j \phi(x_j) - \sum_{i=1}^{m} b_i \phi(y_i)
$$

$$
\geq \sum_{w=0}^{n-3} \frac{n-w-2}{(\beta-\alpha)w!} \int_{\alpha}^{\beta} \left( \sum_{j=1}^{l} a_j G_k(x_j, s) - \sum_{i=1}^{m} b_i G_k(y_i, s) \right) \phi^{(w+1)}(\beta)(s-\beta)^w - \phi^{(w+1)}(\alpha)(s-\alpha)^w ds.
$$

If the reverse of (20) holds, then the reverse of (21) holds.

**Proof.** Under the assumptions of theorem, (16) holds. Since $\phi^{(n-1)}$ is absolutely continuous on $[\alpha, \beta]$, then $\phi^{(n)}$ exists almost everywhere (see [10]). By assumption, $\phi$ is $n$-convex on $[\alpha, \beta]$, therefore $\phi^{(n)} \geq 0$ on $[\alpha, \beta]$. Using this fact and the assumption (20) in combination with (16), we obtain (21).

Let us denote

$$
F_k(\cdot) = \sum_{w=0}^{n-3} \frac{n-w-2}{(\beta-\alpha)w!} \int_{\alpha}^{\beta} G_k(\cdot, s) \phi^{(w+1)}(\beta)(s-\beta)^w - \phi^{(w+1)}(\alpha)(s-\alpha)^w ds.
$$

(22)

When we take in account Sherman’s condition of nonnegativity of vectors $a$, $b$ and matrix $A$, we obtain the following extensions.

**Theorem 3.3.** Let $x \in [\alpha, \beta]^l$, $y \in [\alpha, \beta]^m$, $a \in [0, \infty)^l$ and $b \in [0, \infty)^m$ be such that (2) holds for some row stochastic matrix $A \in \mathcal{M}_{nl}(\mathbb{R})$. Let $P(t, s)$ and $G_k(\cdot, s)$, $s, t \in [\alpha, \beta]$, $k = 1, 2, 3, 4$, be defined as in (5) and (7)-(10), respectively. Let a function $\phi : [\alpha, \beta] \to \mathbb{R}$ be $n$-convex such that $\phi^{(n-1)}$ is absolutely continuous on $[\alpha, \beta]$ for some $n \geq 3$.

(i) If $n$ is even, then (21) holds.

(ii) If $n$ is odd, then for $t \leq s$, the inequalities (20) and (21) hold, while for $s \leq t$, the reverse inequalities in (20) and (21) hold.

(iii) If (21) holds and $F_k$, defined by (22), is convex on $[\alpha, \beta]$, then

$$
\sum_{j=1}^{l} a_j \phi(x_j) - \sum_{i=1}^{m} b_i \phi(y_i)
$$

$$
\geq \sum_{w=0}^{n-3} \frac{n-w-2}{(\beta-\alpha)w!} \int_{\alpha}^{\beta} \left( \sum_{j=1}^{l} a_j G_k(x_j, s) - \sum_{i=1}^{m} b_i G_k(y_i, s) \right) \phi^{(w+1)}(\beta)(s-\beta)^w - \phi^{(w+1)}(\alpha)(s-\alpha)^w ds
$$

(23)
If \( (21) \) holds and \( \phi^{(w+1)}(\alpha) \leq 0, \phi^{(w+1)}(\beta) \geq 0 \) for even \( w \) and \( \phi^{(w+1)}(\alpha) \leq 0, \phi^{(w+1)}(\beta) \leq 0 \) for odd \( w \), then \( (23) \) holds.

Proof. (i)-(ii) Since \( G_k(\cdot, s), s \in [\alpha, \beta] \), is convex on \( [\alpha, \beta] \), by Sherman’s theorem

\[
\sum_{j=1}^{l} a_j G_k(x_j, s) - \sum_{i=1}^{m} b_i G_k(y_i, s) \geq 0.
\]

If \( n \) is even, then

\[
\int_{\alpha}^{\beta} \left( \sum_{j=1}^{l} a_j G_k(x_j, s) - \sum_{i=1}^{m} b_i G_k(y_i, s) \right) (s - t)^{n-3} P(t, s) ds \geq 0, \quad \alpha \leq s \leq t \leq \beta,
\]

(24)

while for odd \( n \), the reversed inequality in (24) holds while the inequality (25) remains the same. Now, applying Theorem 3.2, we conclude (i) and (ii).

(iii) If \( (21) \) holds, the right hand side can be written in the form

\[
\sum_{j=1}^{l} a_j F_k(x_j) - \sum_{i=1}^{m} b_i F_k(y_i).
\]

where \( F_k \) is defined as in (22). If \( F_k \) is convex, then by Sherman’s theorem we have

\[
\sum_{j=1}^{l} a_j F_k(x_j) - \sum_{i=1}^{m} b_i F_k(y_i) \geq 0,
\]

i.e. the right hand side of (21) is nonnegative and \( (23) \) holds.

(iv) For even \( w \) we have

\[
(s - \alpha)^w \geq 0 \quad \text{and} \quad (s - \beta)^w \geq 0, \quad \alpha \leq s \leq \beta
\]

while for odd \( w \) the first inequality remains the same while the second is reversed. So if for even \( w \) we have \( \phi^{(w+1)}(\alpha) \leq 0, \phi^{(w+1)}(\beta) \geq 0 \) and for odd \( w \) we have \( \phi^{(w+1)}(\alpha) \leq 0, \phi^{(w+1)}(\beta) \leq 0 \), then

\[
\sum_{w=0}^{n-3} \left( \phi^{(w+1)}(\beta)(s - \beta)^w - \phi^{(w+1)}(\alpha)(s - \alpha)^w \right) \geq 0,
\]

i.e. the right hand side of (21) is nonnegative what we need to prove. \( \square \)

Remark 3.4. Note that from (23), Sherman’s inequality (1) immediately follows. Moreover, (23) improves Sherman’s inequality for a different choice of Green’s functions \( G_k, k = 1, 2, 3, 4 \).

4 Related results

Motivated by (21), we define

\[
T_k(\phi) = \sum_{j=1}^{l} a_j \phi(x_j) - \sum_{i=1}^{m} b_i \phi(y_i) - \sum_{w=0}^{n-3} \frac{n - w - 2}{(\beta - \alpha)^w} \times
\]

\[
\int_{\alpha}^{\beta} \left( \sum_{j=1}^{l} a_j G_k(x_j, s) - \sum_{i=1}^{m} b_i G_k(y_i, s) \right) \left( \phi^{(w+1)}(\beta)(s - \beta)^w - \phi^{(w+1)}(\alpha)(s - \alpha)^w \right) ds.
\]
Remark 4.1. Note that under assumptions of Theorem 3.2, for every n-convex function \( \phi : [\alpha, \beta] \to \mathbb{R} \), we have \( T_k(\phi) \geq 0 \).

In this section, we present upper bounds for \( T_k(\phi) \). We start with upper bounds when \( \phi \) is such that its \( n \)-th derivative \( \phi^{(n)} \) belongs to \( L_p[\alpha, \beta] \).

Theorem 4.2. Assume that \( (p, q) \) is a pair of conjugate exponents, i.e. \( 1 \leq p, q \leq \infty, 1/p + 1/q = 1 \). Let \( x, y, a, b, A, B, k, G_k, \) and \( P(t, s) \) be as in Theorem 3.2. Let \( \phi : [\alpha, \beta] \to \mathbb{R} \) be such that \( \phi^{(n-1)} \) is absolutely continuous and \( \phi^{(n)} \in L_p[\alpha, \beta] \) for some \( n \geq 3 \). Then

\[
|T_k(\phi)| \leq \frac{1}{(n-3)!(\beta - \alpha)^{1/n}} \left( \int_{[\alpha, \beta]} (s-t)^{n-3} P(t, s) ds \right)^{1/n} \|\phi^{(n)}\|_p.
\]

The constant on the right hand side is sharp for \( 1 < p \leq \infty \) and the best possible for \( p = 1 \).

Proof. Applying the well-known Hölder inequality to (16), we have

\[
|T_k(\phi)| = \frac{1}{(n-3)!(\beta - \alpha)^{1/n}} \left( \int_{[\alpha, \beta]} (s-t)^{n-3} P(t, s) ds \right)^{1/n} \|\phi^{(n)}\|_p.
\]

The proof of the sharpness is analogous to the one in proof of [5, Theorem 13].

Let us consider the Čebyšev functional \( \Delta(f, g) \), of two Lebesgue integrable functions \( f, g : [\alpha, \beta] \to \mathbb{R} \), defined by

\[
\Delta(f, g) := \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} f(t)g(t) dt - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} f(t) dt \cdot \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} g(t) dt.
\]

Let \( x, y, a, b, A, B, k, G_k, \) and \( P(t, s) \) be as in Theorem 3.2. We define

\[
B_k(t) = \int_{\alpha}^{\beta} \left( \sum_{j=1}^{m} a_j G_k(x_j, s) - \sum_{i=1}^{m} b_i G_k(y_i, s) \right) (s-t)^{n-3} P(t, s) ds.
\]

Considering the functions \( B_k \) we obtain the following estimations for the given remainders \( R_k \).

Theorem 4.3. Let \( x, y, a, b, A, B, k, G_k, \) and \( P(t, s) \) be as in Theorem 3.2. Let \( \phi : [\alpha, \beta] \to \mathbb{R} \) be such that \( \phi^{(n)} \) is absolutely continuous with \( (-\alpha)(\beta - \cdot)(\phi^{(n+1)})^2 \in L[\alpha, \beta] \) for some \( n \geq 3 \). Then

\[
|R_k(\phi)| \leq \frac{1}{2(\beta - \alpha)(n-3)!} \left| \Delta(B_k, B_k) \right|^2 \left( \int_{\alpha}^{\beta} (t-\alpha)(\beta - t)[\phi^{(n+1)}(t)]^2 dt \right)^{1/n}.
\]

where the reminder \( R_k(\phi) \) is given in formula

\[
T_k(\phi) = \frac{\phi^{(n-1)}(\beta) - \phi^{(n-1)}(\alpha)}{(n-3)!(\beta - \alpha)^2} \int_{\alpha}^{\beta} B_k(t) dt + R_k(\phi)
\]

and \( T_k(\phi) \) and \( B_k \) are defined by (26) and (29), respectively.
Proof. Comparing identities (31) and (16) we have
\[
R_k(\phi) = \frac{1}{(n-3)!} \int_\alpha^\beta B_k(t)\phi^{(n)}(t)dt - \frac{\phi^{(n-1)}(\beta) - \phi^{(n-1)}(\alpha)}{(n-3)!} \int_\alpha^\beta B_k(t)dt
\]
\[
= \frac{1}{(n-3)!} \int_\alpha^\beta B_k(t)\phi^{(n)}(t)dt
\]
\[
- \frac{1}{(n-3)!} \int_\alpha^\beta B_k(t)dt \cdot \frac{1}{\beta - \alpha} \int_\alpha^\beta \phi^{(n)}(t)dt
\]
\[
= \frac{1}{(n-3)!} \Delta(B_k, \phi^{(n)}).
\]

Applying [11, Theorem 1] on the functions $B_k$ and $\phi^{(n)}$ we obtain (30).

\[
\text{Theorem 4.4.} \text{ Let } \phi : [\alpha, \beta] \to \mathbb{R} \text{ be such that } \phi^{(n)} \text{ is absolutely continuous with } \phi^{(n+1)} \geq 0 \text{ on } [\alpha, \beta] \text{ for some } n \geq 3. \text{ Then for } R_k \text{ given by } (31)
\]
\[
|R_k(\phi)| \leq \frac{1}{(n-3)!} \|B_k\|_\infty \left[ \frac{\phi^{(n-1)}(\beta) + \phi^{(n-1)}(\alpha)}{2} \right]
\]
\[
\Delta(B_k, \phi^{(n)}).
\]

Proof. We have $R_k(\phi) = \frac{1}{(n-3)!} \Delta(B_k, \phi^{(n)}).

Applying [11, Theorem 2] on the functions $B_k$ and $\phi^{(n)}$ and using the identity
\[
\int_\alpha^\beta (\beta - t)(\beta - t)\phi^{(n+1)}(t)dt = \int_\alpha^\beta 2\phi^{(n)}(t)dt
\]
\[
= (\beta - \alpha) \left[ \phi^{(n-1)}(\beta) + \phi^{(n-1)}(\alpha) \right] - 2 \left[ \phi^{(n-2)}(\beta) - \phi^{(n-2)}(\alpha) \right],
\]
we obtain (32).

\section{5 Applications}

In this section, considering the particular cases of the previous results, we show some consequences. As applications, we obtain extensions and improvements of Majorization and discrete Jensen’s inequality as well as inequalities between geometric, logarithmic and arithmetic means.

As a direct consequence of Theorem 3.3 we get the following corollary.

\textbf{Corollary 5.1.} Let \( x, y, a, b, A \), \( P(t, s) \) and \( G_k(\cdot, s) \), \( s, t \in [\alpha, \beta] \), \( k = 1, 2, 3, 4 \), be as in Theorem 3.3. Let a function \( \phi : [\alpha, \beta] \to \mathbb{R} \) be 3-convex such that \( \phi'' \) is absolutely continuous on \([\alpha, \beta]\).

(i) If \( t \leq s \), then
\[
\int_\alpha^\beta \left( \sum_{j=1}^l a_j G_k(x_j, s) - \sum_{i=1}^m b_i G_k(y_i, s) \right) P(t, s)ds \geq 0 \quad (33)
\]
and
\[
\sum_{j=1}^l a_j \phi(x_j) - \sum_{i=1}^m b_i \phi(y_i) \geq \frac{\phi'(\beta) - \phi'(\alpha)}{\beta - \alpha} \int_\alpha^\beta \left( \sum_{j=1}^l a_j G_k(x_j, s) - \sum_{i=1}^m b_i G_k(y_i, s) \right)ds. \quad (34)
\]

Moreover, if in addition \( \phi'(\beta) \geq \phi'(\alpha) \), then the left hand side of (34) is nonnegative.

(ii) If \( s \leq t \), then the reverse inequalities in (33) and (34) hold. Moreover, if in addition \( \phi'(\beta) \leq \phi'(\alpha) \), then the left hand side of the reversed (34) is nonpositive.

\textbf{Remark 5.2.} Note that (34) improves Sherman’s inequality (1) when \( \phi'(\beta) \geq \phi'(\alpha) \). Moreover, if \( l = m \) and all weights \( a_j \) and \( b_i \) are equal, we get the following extension of weighted majorization inequality
\[
\sum_{i=1}^m a_i \phi(x_i) - \sum_{i=1}^m a_i \phi(y_i) \geq \frac{\phi'(\beta) - \phi'(\alpha)}{\beta - \alpha} \int_\alpha^\beta \left( \sum_{i=1}^m a_i G_k(x_i, s) - \sum_{i=1}^m a_i G_k(y_i, s) \right)ds,
\]
i.e. we get improvements when $\phi'(\beta) \geq \phi'(\alpha)$.

If we denote $A_m = \sum_{i=1}^{m} a_i$ and put $y_1 = y_2 = ... = y_m = \frac{1}{A_m} \sum_{i=1}^{m} a_i x_i$, we get the following Jensen’s type inequality

$$\frac{1}{A_m} \sum_{i=1}^{m} \phi(x_i) - \phi \left( \frac{1}{A_m} \sum_{i=1}^{m} \phi(x_i) \right) \geq \frac{\phi'(\beta) - \phi'(\alpha)}{\beta - \alpha} \int_{\alpha}^{\beta} \left( \frac{1}{A_m} \sum_{i=1}^{m} G_k(x_i, s) - G_k \left( \frac{1}{A_m} \sum_{i=1}^{m} \phi(x_i), s \right) \right) ds,$$

i.e. if in addition $\phi'(\beta) \geq \phi'(\alpha)$, then we get double inequality

$$\phi \left( \frac{1}{A_m} \sum_{i=1}^{m} \phi(x_i) \right) \leq \frac{\phi'(\beta) - \phi'(\alpha)}{\beta - \alpha} \int_{\alpha}^{\beta} \left( \frac{1}{A_m} \sum_{i=1}^{m} G_k(x_i, s) - G_k \left( \frac{1}{A_m} \sum_{i=1}^{m} \phi(x_i), s \right) \right) ds + \phi \left( \frac{1}{A_m} \sum_{i=1}^{m} \phi(x_i) \right),$$

which improves Jensen’s inequality.

Especially, choosing Green’s function $G_2$, defined by (8) and setting $m = 2$, $A_2 = 2$, $x_1 = \alpha$, $x_2 = \beta$, $y_1 = y_2 = \frac{\alpha + \beta}{2}$, we get

$$\phi \left( \frac{\alpha + \beta}{2} \right) \leq \frac{\phi'(\beta) - \phi'(\alpha)}{4} (\beta - \alpha) + \phi \left( \frac{\alpha + \beta}{2} \right) \leq \frac{\phi(\alpha) + \phi(\beta)}{2}. \tag{35}$$

In the sequel, applying the double inequality (35) to some concrete functions, we derive some new inequalities.

Example 5.3. Let $\phi : [\alpha, \beta] \to \mathbb{R}$ be defined by $\phi(x) = e^x$. Since $\phi$ is 3-convex by definition, $\phi''$ is absolutely continuous on $[\alpha, \beta]$ and $\phi'(\beta) \geq \phi'(\alpha)$ is satisfied, then by (35) we have

$$e^{\frac{\alpha + \beta}{2}} \leq e^{\frac{\beta}{2}} - e^{\frac{\alpha}{2}} (\beta - \alpha) + e^{\frac{\alpha + \beta}{2}} \leq e^{\frac{\alpha}{2}} + e^{\frac{\beta}{2}}.$$ 

By substituting $x = e^\alpha$, $y = e^\beta$, we get

$$\sqrt{xy} \leq \frac{y - x}{\ln y - \ln x} \frac{(\ln y - \ln x)^2}{4} + \sqrt{xy} \leq \frac{x + y}{2},$$

i.e. we get the inequalities between the geometric mean $G = \sqrt{xy}$, the logarithmic mean $L = \frac{y - x}{\ln y - \ln x}$ and the arithmetic mean $A = \frac{x + y}{2}$, in form

$$G \leq \frac{(\ln y - \ln x)^2}{4} + G \leq A.$$

Example 5.4. Let $0 < \alpha < \beta$, $\phi : [\alpha, \beta] \to \mathbb{R}$ be defined by $\phi(x) = x^{r+1}$, $r \geq 1$. Since $\phi$ is a 3-convex by definition, $\phi''$ is absolutely continuous on $[\alpha, \beta]$ and $\phi'(\beta) \geq \phi'(\alpha)$ is satisfied, then by (35) we have

$$\left( \frac{\alpha + \beta}{2} \right)^{r+1} \leq \frac{r + 1}{4} (\beta - \alpha) (\beta^r - \alpha^r) + \left( \frac{\alpha + \beta}{2} \right)^{r+1} \leq \frac{\alpha^r + \beta^r}{2}.$$ 

As a direct consequence of Theorem 4.2 we get the following corollary.

Corollary 5.5. Assume that $(p, q)$ is a pair of conjugate exponents, i.e. $1 \leq p, q \leq \infty, 1/p + 1/q = 1$. Let $x, y, a, b, A, P(i, s)$ and $G_k(x, s), s, t \in [\alpha, \beta], k = 1, 2, 3, 4$, be as in Theorem 4.2. Let a function $\phi : [\alpha, \beta] \to \mathbb{R}$ be such that $\phi''$ is absolutely continuous and $\phi''' \in L_p[\alpha, \beta]$. Then

$$\left| \sum_{j=1}^{l} a_j \phi(x_j) - \sum_{i=1}^{m} b_i \phi(y_i) - \frac{\phi'(\beta) - \phi'(\alpha)}{\beta - \alpha} \int_{\alpha}^{\beta} \left( \sum_{j=1}^{l} a_j G_k(x_j, s) - \sum_{i=1}^{m} b_i G_k(y_i, s) \right) ds \right| \tag{36}$$
\[
\frac{1}{\beta - \alpha} \left( \int_\alpha^\beta \left( \int_\alpha^\beta \left( \sum_{j=1}^l a_j G_k(x_j, s) - \sum_{i=1}^m b_i G_k(y_i, s) \right) P(t, s) \, ds \right) \, dt \right)^\frac{1}{q} \| \phi'''' \|_p.
\]

The constant on the right hand side is sharp for \(1 < p \leq \infty\) and the best possible for \(p = 1\).

**Remark 5.6.** Choosing Green’s function \(G_2\), defined by (8), and setting \(l = m = 2, a_i = b_i = 1, i = 1, 2,\) \(x_1 = \alpha, x_2 = \beta, y_1 = y_2 = \frac{\alpha + \beta}{2}\) with \(t \leq s\), from (36) we get

\[
\left| \phi(\alpha) + \phi(\beta) - 2\phi \left( \frac{\alpha + \beta}{2} \right) - \frac{\phi(t) - \phi(\alpha)}{2} (\beta - \alpha) \right| 
\leq \frac{\beta - \alpha}{2} \left( \int_\alpha^\beta (t - \alpha)^q \, dt \right)^\frac{1}{q} \| \phi'''' \|_p.
\]

If we choose \(q = 1, p = \infty\), we obtain

\[
\left| \phi(\alpha) + \phi(\beta) - 2\phi \left( \frac{\alpha + \beta}{2} \right) - \frac{\phi(t) - \phi(\alpha)}{2} (\beta - \alpha) \right| 
\leq \frac{(\beta - \alpha)^3}{4} \| \phi'''' \|_\infty.
\]

Choosing \(q = \infty, p = 1\), we have

\[
\left| \phi(\alpha) + \phi(\beta) - 2\phi \left( \frac{\alpha + \beta}{2} \right) - \frac{\phi(t) - \phi(\alpha)}{2} (\beta - \alpha) \right| 
\leq \frac{(\beta - \alpha)^2}{2} (\phi''(\beta) - \phi''(\alpha)).
\]

If additionally \(\phi\) is 3-convex, then

\[
0 \leq \phi(\alpha) + \phi(\beta) - 2\phi \left( \frac{\alpha + \beta}{2} \right) - \frac{\phi(t) - \phi(\alpha)}{2} (\beta - \alpha) \leq \frac{(\beta - \alpha)^2}{2} (\phi''(\beta) - \phi''(\alpha)),
\]

**Example 5.7.** Let \(\phi : [\alpha, \beta] \to \mathbb{R}\) be defined by \(\phi(x) = e^x\). Then

\[
e^x + e^\beta \leq e^{\alpha + \beta} \leq e^\alpha + e^\beta - e^\alpha (\beta - \alpha) \leq e^\alpha + e^\beta - e^\alpha (\beta - \alpha)^2.
\]

By substituting \(x = e^\alpha, y = e^\beta\), we get the inequalities between means \(G = \sqrt[\alpha \beta]{\frac{e^x - e^\beta}{\ln y - \ln x}}\) and \(A = \frac{x + y}{2}\):

\[L \leq A + L \left( \frac{\ln y - \ln x)^2}{4} \right) \leq G + L \left( \frac{(\ln y - \ln x)^3}{4} \right).
\]

**Example 5.8.** Let \(0 < \alpha < \beta, \phi : [\alpha, \beta] \to \mathbb{R}\) be defined by \(\phi(x) = x^{r+1}, r \geq 1\). Then

\[
\left( \frac{\alpha + \beta}{2} \right)^{r+1} \leq \frac{\alpha^{r+1} + \beta^{r+1}}{2} - \frac{r + 1}{4} (\beta - \alpha) (\beta^r - \alpha^r)
\]

\[
\leq \left( \frac{\alpha + \beta}{2} \right)^{r+1} + \frac{r(r + 1)}{4} (\beta - \alpha)^2 (\beta^{r-1} - \alpha^{r-1}).
\]

**Remark 5.9.** Choosing one of the other three Green’s functions, we can estimate new similar results.
References