Research Article

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Multiplicity solutions of a class fractional Schrödinger equations

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Abstract: In this paper, we study the existence of nontrivial solutions to a class fractional Schrödinger equations

\[ (-\Delta)^s u + V(x)u = \lambda f(x, u) \quad \text{in} \quad \mathbb{R}^N, \]

where \((-\Delta)^s u(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N\) is a fractional operator and \(s \in (0, 1)\). By using variational methods, we prove this problem has at least two nontrivial solutions in a suitable weighted fractional Sobolev space.

Keywords: Fractional Laplacian, Variational methods, Nontrivial solution

MSC: 35P15, 35P30, 35R11

1 Introduction

There are a lot of interesting problems in the standard framework of the Laplacian (and, more generally, of uniformly elliptic operators), widely studied in the literature. A natural question is whether or not the existence results got in this classical context can be extended to the non-local framework of the fractional Laplacian type operators.

First, we focus on the so-called fractional Schrödinger equation

\[ i \frac{\partial \psi}{\partial t} = (-\Delta)^s \psi + V(x)\psi - |\psi|^{p-1}\psi, \]

where \((x,t) \in \mathbb{R} \times (0, +\infty)\), and \(V : \mathbb{R}^N \to \mathbb{R}\) an external potential function. The fractional Laplacian operator \((-\Delta)^s u\) with \(0 < s < 1\) of a function \(\phi \in \ell\) is defined by

\[ \mathcal{F}((-\Delta)^s \phi)(\xi) = |\xi|^{2s} \mathcal{F}(\phi)(\xi), \]

where \(\ell\) denotes the Schwartz space of rapidly decreasing \(C^\infty\) functions in \(\mathbb{R}^N\). \(\mathcal{F}\) is the Fourier transform, i.e.,

\[ \mathcal{F}((-\Delta)^s \phi)(\xi) = |\xi|^{2s} \mathcal{F}(\phi)(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{2\pi i \xi \cdot x} \phi(x) dx. \]
This equation was introduced by Laskin ([21, 22]), and comes from an expansion of the Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths. When \( s = 1 \), the Lévy dynamics becomes the Brownian dynamics, and (1) reduces to the classical Schrödinger equation

\[
\frac{i \partial \psi}{\partial t} = -\Delta \psi + V(x)\psi - |\psi|^{p-1}\psi.
\]

Standing wave solutions to this equation are solutions of the form

\[
\psi(x, t) = e^{-i\omega t} u(x),
\]

where \( u \) solves the elliptic equation

\[-\Delta \psi + V(x)\psi - |\psi|^{p-1}\psi = 0.
\]

In this paper we study the following fractional Schrödinger equation

\[(-\Delta)^s u + V(x) u = \lambda f(x, u) \text{ in } \mathbb{R}^N\]

where \( \lambda \) is a parameter.

The fractional Schrödinger equations are an important model in the study of the fractional quantum mechanics. Recently, this has been widely investigated by many authors in the last decades, see [3–15, 18–20, 22–26] and references therein. In most of the papers mentioned above the existence of positive solutions has been considered under different assumptions on \( V \) and \( f \). We refer the reader to [16, 17] and to the references included for a self-contained overview of the basic properties of fractional Sobolev spaces.

In [3], the author used the Ekeland variational principle and the mountain pass theorem to obtain a nontrivial solution for (2) with the Ambrosetti-Rabinowitz condition:

there is a constant \( \mu > 2 \) such that

\[
0 \leq \mu \int_0^s f(x, t) dt \leq sf(x, s), \text{ for all } x \in \mathbb{R}^N, s \in \mathbb{R} \setminus \{0\}.
\]

(AR)

In [4–6], the authors used variant fountain theorems and the \( \mathbb{Z}_2 \) version of mountain pass theorem to establish the existence of infinitely many nontrivial high-energy or small-energy solutions for (2). In [7], the authors used the concentration compactness principle to show that (2) \((V(x) = 1)\) has at least two nontrivial radial solutions without the (AR) condition.

In [25], Bisci and Rădulescu studied the following equation

\[(-\Delta)^s u + V(x) u = \lambda (f(x, u) + kg(x, u)) \text{ in } \mathbb{R}^N\]

when the potential \( V \in C(\mathbb{R}^N) \) satisfies

\[(V) \inf_{\mathbb{R}^N} V(x) > 0\]

and

\[\lim_{|y| \to +\infty} \mu \{x \in B(y, r) : V(x) \leq M\} < +\infty, \forall M > 0,\]

where \( \mu \) denotes the Lebesgue measure in \( \mathbb{R}^N \), \( B(y, r) \) denotes the open ball in \( \mathbb{R}^N \) with center \( y \) and radius \( r > 0 \), and established two existence theorems for two nontrivial solutions when the nonlinearity \( f \) and \( g \) satisfy

\[(h) \max \{|f(x, t)|, |g(x, t)|\} \leq W(x)|t|^q,\]

where \( W \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) and \( q \in (0, 1) \).

Motivated by the above papers, we shall assume that \( f(x, t) \) satisfies the following conditions:

\[(f_1) f \in C([\mathbb{R}^N, 1] \cap \mathbb{R}), \text{ and}\]

\[
\lim_{t \to 0} \frac{f(x, t)}{t} = 0.
\]
\[(f_2)\]
\[
\lim_{|t| \to \infty} \frac{|f(x, t)|}{t} = 0.
\]
\[(f_3)\]
\[
\sup_{t \in \mathbb{R}} F(t) > 0.
\]
where \(F(x, u) = \int_0^u f(x, s) \, ds\).

And on the potential function \(V\) we assume
\[(V_1)\]
\(V \in C(\mathbb{R}^N, \mathbb{R})\) is a positive weight and there exists a constant \(V_0 > 0\) such that \(V(x) \geq V_0\) for all \(x \in \mathbb{R}^N\).
\[(V_2)\]
\[
\frac{1}{V(x)} \in L^1(\mathbb{R}^N).
\]
The main purpose of this paper is to generalize the main results of [24, 25]. Now we state our main results:

**Theorem 1.1.** Assume that \(f\) and \(V\) satisfy \((f_1)\) \((f_2)\), \((f_3)\), \((V_1)\), and \((V_2)\). Then problem \((1.2)\) admits at least two nontrivial nonnegative solutions.

This paper is organized as follows. In Section 2, we will give some notation and introduce our main idea. In Section 3, we prove Theorem 1.1.

**Remark 1.2.** If \(g(x, u) = 0\), then \((h) \Rightarrow \ (f_2)\). However, there are many functions which satisfy \((f_2)\), but do not satisfy the condition \((h)\). For example, the function
\[
f(x, t) = \begin{cases} 
  t^2, & \text{if } 0 \leq t \leq M_0, \ M_0 > 1 \\
  \frac{M_1^2 - M_0^2}{M_1 - M_0} (t - M_0) + M_0^2, & \text{if } M_0 \leq t \leq M_1, \\
  t^\frac{1}{2}, & \text{if } t \geq M_1, \\
  0, & \text{if } x < 0.
\end{cases}
\]
does not satisfy \((h)\), but it satisfies our conditions \((f_2)\).

## 2 Preliminary

In this section, for the reader’s convenience, we collect some basic results that will be used in the forthcoming sections. In the following, we denote the \(N\)-dimensional Lebesgue measure of a set \(A \subseteq \mathbb{R}^N\) by \(\text{meas}(A)\). We use "\(\rightharpoonup\)" and "\(\rightarrow\)" to denote weak and strong convergence in the related function space. For any Euclidean space \((\mathbb{R}^N, | \cdot |)\) we will denote
\[
B_\rho = \{ u \in \mathbb{R}^N : |u - u_0| \leq \rho \} (u_0 \in \mathbb{R}^N, \rho > 0).
\]

### 2.1 Variational formulation of the problem

First we introduce a variational setting for problem \((2)\). The Gagliardo seminorm is defined for all measurable function \(u : \mathbb{R}^N \to \mathbb{R}\) by
\[
[u]_{2,s,N} = \left( \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2s}} \, dx \, dy \right)^{\frac{1}{2}},
\]
we can define the fractional Sobolev space
\[
H^s(\mathbb{R}^N) = \{ u \in L^2(\mathbb{R}^N) : u \text{ measurable, } [u]_{2,s,N} < \infty \}.
\]
with

\[ \|u\|_{H^s} = (\|u\|_2^2 + [u]_{2,s}^2)^{\frac{1}{2}}, \]

where \( \|u\|_2 = \left( \int_{\mathbb{R}^N} |v|^{2s} \, ds \right)^{\frac{1}{s}} \), which is the norm of \( L^2(\mathbb{R}^N) \). The space \( H^s(\mathbb{R}^N) \) is a Hilbert space with the inner product

\[ \langle u, v \rangle = \int_{\mathbb{R}^N} \frac{u(x) - u(y)(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy + \int_{\mathbb{R}^N} uv \, dx. \]

The space \( H^s(\mathbb{R}^N) \) can be described by means of the Fourier transform. Indeed, it is defined by

\[ H^s(\mathbb{R}^N) = \{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1 + |\xi|^2)^s |\mathcal{F}u(\xi)|^2 \, d\xi < +\infty \}. \]

In this case, the inner product and the norm are defined as

\[ \langle u, v \rangle = \int_{\mathbb{R}^N} (1 + |\xi|^2)^s \mathcal{F}u(\xi) \mathcal{F}v(\xi) \, d\xi, \]

\[ \|u\|_{H^s} = \left( \int_{\mathbb{R}^N} (1 + |\xi|^2)^s |\mathcal{F}u(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}. \]

In order to give the relationship of the above two norms, we introduce the definition of Schwartz function \( \ell \), that is, the rapidly decreasing \( C^\infty \) function on \( \mathbb{R}^N \). If \( u \in \ell \), the fractional Laplacian \( (-\Delta)^s \) acts on \( u \) as

\[ (-\Delta)^s u(x) = C(N, s) \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B(0, \varepsilon)} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, dy, \]

where \( C(N, s) \) is the following constant

\[ C(N, s) = \left( \int_{\mathbb{R}^N} \frac{1 - \cos(\zeta_1)}{|\xi|^{n+2s}} \right)^{-1}. \]

In [17], it is proved that

\[ (-\Delta)^s u = \mathcal{F}^{-1} (|\xi|^{2s} \mathcal{F}u), \]

\[ [u]_{2,s} = \frac{2}{C(N, s)} \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}u|^2 \, d\xi, \]

and that

\[ [u]_{2,s} = \frac{2}{C(N, s)} \|(-\Delta)^{\frac{s}{2}} u\|_2^2. \]

As a consequence, the norms on \( H^s(\mathbb{R}^N) \) defined above

\[ u \to \left( \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} \, dx \, dy + \int_{\mathbb{R}^N} u^2 \, dx \right)^{\frac{1}{2}}, \]

\[ u \to \left( \int_{\mathbb{R}^N} (1 + |\xi|^2)^s |\mathcal{F}u(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}, \]

\[ u \to \left( \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 \, d\xi + \int_{\mathbb{R}^N} u^2 \, dx \right)^{\frac{1}{2}}, \]

\[ u \to \left( \|(-\Delta)^{\frac{s}{2}} u\|_2^2 + \int_{\mathbb{R}^N} u^2 \, dx \right)^{\frac{1}{2}}. \]

are all equivalent.
Moreover, it is easy to see that \( H^s(\mathbb{R}^N) \) is a uniformly convex Banach space and the embedding \( H^s(\mathbb{R}^N) \hookrightarrow L^\theta(\mathbb{R}^N) \) is continuous for any \( \theta \in [2, 2_s^*] \) by Theorem 6.7 of [17], that is, there exists a positive constant \( C_\ast \) such that
\[
\|u\|_{L^\theta(\mathbb{R}^N)} \leq C_\ast \|u\|_{2,s} \quad \text{for all } u \in H^s(\mathbb{R}^N).
\]

We will work in the following linear subspace
\[
E = \{u \in H^s(\mathbb{R}^N), \int_{\mathbb{R}^N} V(x)|u|^2dx < +\infty\},
\]
which denotes the completion of \( C^\infty_0(\mathbb{R}^N) \) with respect to the norm
\[
\|u\|_E = \left(\|u\|_{2,s}^2 + \|u\|_{2,V}^2\right)^{\frac{1}{2}}, \quad \|u\|_{2,V}^2 = \int_{\mathbb{R}^N} V(x)|u|^2dx.
\]

We also know that \((E, \|\cdot\|_E)\) is a uniformly convex Banach space, see ([27] Lemma 10). The dual space of \((E, \|\cdot\|_E)\) is denoted by \((E^*, \|\cdot\|_{E^*})\). \((\cdot, \cdot)\) denotes the pairing between \(E\) and its dual space \(E^*\). We define the nonlinear operator \(\Phi : E \to E^*\) as
\[
(\Phi(u), v) = \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}}dxdy + \int_{\mathbb{R}^N} V(x)u(x)v(x)dx.
\]

It can be seen that a weak solution of problem (2) is a function \(u \in E\) such that
\[
(\Phi(u), v) = \int_{\mathbb{R}^N} f(x, u)vdx.
\]
for all \(v \in E\). Clearly, for all \(u \in E\), \((\Phi(u), u) = \|u\|_E^2\).

Now we introduce the minimal hypotheses on the reaction term of (2):
\[
(H) \quad f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \quad \text{is a Carathéodory mapping, } F(x, t) = \int_0^t f(x, s)ds \quad \text{for all } (x, t) \in \mathbb{R}^N \times \mathbb{R} \quad \text{and} \quad q \in (2, 2_s^*) \quad \text{such that} \quad |f(x, u)| \leq a(1 + |u|^q-1),
\]
a.e. in \(\mathbb{R}^N\) and for all \(t \in \mathbb{R}\) \((a > 0)\).

We set for all \(u \in E\)
\[
I(u) = \frac{\|u\|_E^2}{2} - \int_{\mathbb{R}^N} F(x, u)dx.
\]
defined on \(E\), and for any \(u \in E\), it holds that:
\[
(I'(u), v) = (\Phi(u), v) - \int_{\mathbb{R}^N} f(x, u)vdx.
\]

By hypotheses \((H)\), we have \(I \in C^1(E)\). We denote by \(K(I)\) the set of all critical points of \(I\). If \(u \in K(I)\), then (4) hold for all \(v \in E\), i.e., \(u\) is a weak solution to (2).

### 2.2 Some preliminary lemmas

We first recall some embedding results related to the fractional Sobolev space \(E\), for more details, see [27].

**Lemma 2.1** ([27, Lemma 1]). Let \((V_1)\) hold. Then the embeddings \(E \hookrightarrow H^s(\mathbb{R}^N) \hookrightarrow L^\theta(\mathbb{R}^N)\) are continuous with
\[
\min\{1, V_0\} \|u\|_{2,s}^2 \leq \|u\|_E^2,
\]
for all \(u \in E\) and \(\theta \in [2, 2_s^*]\). Moreover, for any \(R > 0\) and \(\theta \in [2, 2_s^*]\), the embedding \(E \hookrightarrow L^\theta(B_R(0))\) is compact.
Lemma 2.2. Suppose that \((V_1)\) and \((V_2)\) are fulfilled. If \(\{v_j\}\) is a bounded sequence in \(E\), then there exists \(v \in E \cap L^\vartheta(\mathbb{R}^N)\) such that up to a subsequence,

\[ v_j \rightharpoonup v \text{ strongly in } L^\vartheta(\mathbb{R}^N) \]
as \(j \to \infty\), for any \(\vartheta \in [2, 2^*_s)\).

**Proof.** Since \(v_j\) is bounded in \(E\), by lemma 2.1 we have \(v_j\) is bounded in \(L^\vartheta(\mathbb{R}^N)\). Then by the reflexivity of \(E\), up to a subsequence, we get that \(v_j \rightharpoonup v\) weakly in \(E \cap L^\vartheta(\mathbb{R}^N)\) as \(j \to \infty\). Next we prove that

\[ v_j \rightharpoonup v \text{ strongly in } L^\vartheta(\mathbb{R}^N). \]

Now, for any \(\varepsilon > 0\), there exists \(R_1 > 0\) such that

\[
\int_{\mathbb{R}^N \setminus B_R(0)} \frac{1}{V(x)} \, dx < \varepsilon
\]

for all \(R \geq R_1\), since \(\frac{1}{V(x)} \in L^1(\mathbb{R}^N)\) by assumption \((V_2)\). Then, by Hölder inequality, we can get that

\[
\int_{\mathbb{R}^N \setminus B_R(0)} \frac{V(x)^{\frac{1}{2}}}{V(x)^{\frac{1}{2} - \frac{\vartheta}{2}}} |v_j - v|^2 \, dx
\]

\[
\leq \left( \int_{\mathbb{R}^N \setminus B_R(0)} V(x) |v_j - v|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N \setminus B_R(0)} \frac{1}{V(x)} \, dx \right)^{\frac{1}{2} - \frac{\vartheta}{2}} \leq \varepsilon.
\]

for all \(R \geq R_1\).

Fix \(R_1 > 0\), we have

\[ v_j \rightharpoonup v \text{ weakly in } L^{2\vartheta}(B_{R_1}(0)) \cap H^s(B_{R_1}(0)). \]

by Theorem 6.7 of [17]. Since \(2 \leq \vartheta < 2^*_s\), by Corollary 7.2 of [17], we obtain \(v_j \rightharpoonup v\) strongly in \(L^\vartheta(B_{R_1}(0))\), i.e. for above \(\varepsilon > 0\), there exists \(N_1 > 0\) such that

\[
\int_{B_{R_1}(0)} |v_j - v|^{\vartheta} \, dx < \varepsilon
\]

for all \(j \geq N_1\). Combining (7) and (8), for all \(j \geq N_1\), by interpolation inequality we have

\[
\int_{\mathbb{R}^N} |v_j - v|^\vartheta \, dx
\]

\[
= \int_{B_{R_1}(0)} |v_j - v|^\vartheta \, dx + \int_{\mathbb{R}^N \setminus B_{R_1}(0)} |v_j - v|^\vartheta \, dx
\]

\[
\leq \varepsilon + \left( \int_{\mathbb{R}^N \setminus B_{R_1}(0)} V(x)^{\frac{1}{2}} |v_j - v|^2 \, dx \right)^{\frac{\vartheta}{2}} \left( \int_{\mathbb{R}^N \setminus B_{R_1}(0)} \frac{1}{V(x)} \, dx \right)^{\frac{1}{2} - \frac{\vartheta}{2}}
\]

\[
\leq \varepsilon + \frac{1}{\sqrt{2\vartheta}} \left( \int_{\mathbb{R}^N \setminus B_{R_1}(0)} V(x)^{\frac{1}{2}} |v_j - v|^2 \, dx \right)^{\frac{\vartheta}{2}} \left( \int_{\mathbb{R}^N \setminus B_{R_1}(0)} \frac{1}{V(x)} \, dx \right)^{\frac{1}{2} - \frac{\vartheta}{2}}
\]

\[
\times \left( \int_{\mathbb{R}^N \setminus B_{R_1}(0)} \frac{1}{V(x)} \, dx \right)^{\frac{\vartheta}{2} (2^*_s - 2)} \leq C \varepsilon.
\]
where \( C \) denotes various positive constants, and \( \theta \in (0, 1) \) such that
\[
\frac{1}{\theta} = \frac{\theta}{2} + \frac{1 - \theta}{2^*}.
\]

Therefore, \( v_j \to v \) strongly in \( L^0(\mathbb{R}^N) \).

**Proposition 2.3.** Let \( (E, \| \cdot \|_E) \) be a Banach space and its dual space \( (E^*, \| \cdot \|_{E^*}) \) and \( I \in C^1(E, \mathbb{R}^1) \)

(1) For \( c \in \mathbb{R}^1 \), we say that \( I \) satisfies the \( C_c \) condition if for any sequence \( \{x_n\} \subset E \) with
\[
I(x_n) \to c, \quad \| I'(x_n) \|_{E^*}(1 + \| x_n \|_E) \to 0.
\]

(2) For \( c \in \mathbb{R}^1 \), we say that \( I \) satisfies the \( (PS) \) condition if for any sequence \( \{x_n\} \subset E \) with
\[
I(x_n) \to c, \quad I'(x_n) \to 0 \text{ in } E^*.
\]

The following critical points theorem was established in [1, 2].

**Lemma 2.4** ([1, 2]). Let \( (E, \| \cdot \|) \) be a separable and reflexive real Banach space, and let \( \Phi, \Psi : E \to \mathbb{R} \) be two continuously Gâteaux differentiable functionals. Assume that there exists \( z_0 \in E \) such that \( \Phi(z_0) = \Psi(z_0) = 0 \) and \( \inf_{z \in E} \Phi(z) \geq 0 \) and that there exist \( z_1 \in E, \varrho > 0 \) such that

(i) \( \varrho < \Phi(z_1) \);

(ii) \( \sup_{\Phi(z) < \varrho} \Psi(z) < \varrho \frac{\Psi(z_1)}{\Phi(z_1)} \). Further, put
\[
\bar{a} = \frac{\zeta \varrho}{\psi(z_1)} - \frac{\sup_{\Phi(z) < \varrho} \Psi(z)}{\Phi(z_1)},
\]
with \( \zeta > 1 \), assume that the functional
\[
J(z) = \Phi(z) - \lambda \Psi(z), \quad (\forall z \in E)
\]
is sequentially weakly lower semicontinuous, satisfies the \( (PS) \) condition, and

(iii) \( \lim_{\| z \| \to +\infty} J(z) = +\infty \), for every \( \lambda \in [0, \bar{a}] \).

Then there is an open interval \( \Lambda \subset [0, \bar{a}] \) and a number \( \kappa > 0 \) such that for each \( \lambda \in \Lambda \), the equation \( J'(z) = 0 \) admits at least three solutions in \( E \) having norm less than \( \kappa \).

### 3 The main results and its proofs

In this section, we are ready to prove the Theorem 1.1. In the sequel, for the sake of clarity, we divide the proof of the theorem into several steps. We write the functional \( J \) as follows:
\[
J(u) = \Phi(u) - \lambda \Psi(u),
\]
where
\[
\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} \, dx \, dy + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2 \, dx,
\]
and
\[
\Psi(u) = \int_{\mathbb{R}^N} F(x, u) \, dx.
\]

We first give two preliminary lemmas.
Lemma 3.1. If $(f_1), (f_2)$ and $(V_1)$ hold, then we have that

\[
\lim_{\theta \to 0^+} \frac{\sup_{u \in S_\theta} \int_{\mathbb{R}^N} F(x,u) \, dx}{\theta} = 0
\]

where $S_\theta = \Phi^{-1}(-\infty, \theta)$.

Proof. By $(f_1)$ and $(f_2)$, there is a positive constant $\delta$ such that

\[
\begin{cases}
|f(x,t)| \leq \frac{\min(1, V_0) e |t|}{2C^2_x} + \frac{c \min(1, V_0) \frac{\theta}{2} |t|^\theta - 1}{C^\theta_x} \\
|F(x,t)| \leq \frac{\min(1, V_0) e |t|^2}{4C^2_x} + \frac{c \min(1, V_0) \frac{\theta}{2} |t|^\theta}{C^\theta_x}
\end{cases}
\]

for a fixed $\theta \in [2, 2^*]$ and for all $t \in \mathbb{R}^N$.

Moreover, by using Lemma 2.1, we have

\[
\sup_{u \in S_\theta} \int_{\mathbb{R}^N} \hat{F}(x,u) \, dx \leq \int_{\mathbb{R}^N} \frac{\min(1, V_0) e \|u\|^2}{4C^2_x} + \frac{c \min(1, V_0) \frac{\theta}{2} \|u\|^\theta}{C^\theta_x} \, dx
\]

\[
= \frac{\min(1, V_0) e \|u\|^2}{4C^2_x} + \frac{c \min(1, V_0) \frac{\theta}{2} \|u\|^\theta}{C^\theta_x}
\]

\[
\leq \epsilon + \epsilon^2 \theta \|u\|^2
\]

Thus, there exists $\varphi(\epsilon) > 0$ such that, for every $0 < \varphi < \varphi(\epsilon)$, we have

\[
0 \leq \frac{\sup_{u \in S_\theta} \int_{\mathbb{R}^N} \hat{F}(x,u) \, dx}{\theta} \leq \frac{\epsilon}{2} + \epsilon^2 \theta \|u\|^2 \leq \epsilon.
\]

The proof is complete. \qed

Lemma 3.2. Let $\sigma \in [0, 1]$, $t_0 \in \mathbb{R}$, and

\[
\Gamma(t) = \int_0^{+\infty} z^{t-1} e^{-z} \, dz, \quad (\forall t > 0)
\]

be the usual Gamma function, then we have

\[
\|u^0_{t_0}\|_2 \leq \frac{\alpha^2 t_0^2 \pi^{\frac{N}{2}} \Gamma(\frac{N}{2})}{(1 - \sigma)^2 (1 - \sigma^N)} S_0 \tag{9}
\]

where

\[
u^0_{t_0} = \begin{cases}
0, & \text{if } x \in \mathbb{R}^N \setminus B(x_0, \tau), \\
\frac{t_0}{(1 - \sigma) \tau} (\tau - |x - x_0|), & \text{if } x \in B(x_0, \tau) \setminus B(x_0, \sigma \tau), \\
t_0, & \text{if } x \in B(x_0, \sigma \tau),
\end{cases}
\]

and

\[
S_0 = \max\{(2\pi)^N (1 + \frac{1}{\lambda_1}), \max_{x \in B(x_0, \tau)} V(x) \frac{\lambda_1}{\lambda_1} \}.\]
Proof. Computing the standard seminorm of the function $u_0^\sigma$ in $H^1(\mathbb{R}^N)$, we have

$$[u_0^\sigma]_{H^1(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} |\nabla u_0^\sigma|^2\,dx = \int_{B(x_0, \tau) \setminus B(x_0, \sigma \tau)} \frac{t^2_0}{(1 - \sigma)^{2}} \,dx = \frac{t^2_0}{(1 - \sigma)^{2}} \left[\text{meas}(B(x_0, \tau)) - \text{meas}(B(x_0, \sigma \tau))\right] = \frac{t^2_0}{(1 - \sigma)^{2}} \frac{\pi \frac{N}{2} (1 - \sigma)^N}{\Gamma(1 + \frac{N}{2})},$$

where $\text{meas}(B(x_0, \tau))$ and $\text{meas}(B(x_0, \sigma \tau))$ denote the Lebesgue measure of $B(x_0, \tau)$ and $B(x_0, \sigma \tau)$.

Now, by standard arguments on the Fourier transform, we have

$$u_0^\sigma \in L^2(\mathbb{R}^N) \text{ if and only if } \mathcal{J}u_0^\sigma \in L^2(\mathbb{R}^N),$$

and

$$\|u_0^\sigma\|^2_{L^2(\mathbb{R}^N)} = (2\pi)^{-N} \|\mathcal{J}u_0^\sigma\|^2_{L^2(\mathbb{R}^N)},$$

as well as

$$|\nabla u_0^\sigma| \in L^2(\mathbb{R}^N) \text{ if and only if } |\xi| |\mathcal{J}u_0^\sigma| \in L^2(\mathbb{R}^N),$$

and

$$\|\nabla u_0^\sigma\|^2_{L^2(\mathbb{R}^N)} = (2\pi)^{-N} \|\xi| \mathcal{J}u_0^\sigma\|^2_{L^2(\mathbb{R}^N)}.$$ 

Then, we have

$$\|u_0^\sigma\|^2 = \int_{\mathbb{R}^N} |\xi|^2 |\mathcal{J}u_0^\sigma|^2\,d\xi + \int_{\mathbb{R}^N} V(x)|u_0^\sigma|^2\,dx$$

$$< \int_{\mathbb{R}^N} (1 + |\xi|^2) |\mathcal{J}u_0^\sigma|^2\,d\xi + \int_{\mathbb{R}^N} V(x)|u_0^\sigma|^2\,dx$$

$$= (2\pi)^{-N} \|\mathcal{J}u_0^\sigma\|^2_{L^2(\mathbb{R}^N)} + \|u_0^\sigma\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} V(x)|u_0^\sigma|^2\,dx$$

$$\leq (2\pi)^{-N} \left(1 + \frac{1}{\lambda_1}\right)\|\mathcal{J}u_0^\sigma\|^2_{L^2(\mathbb{R}^N)} + \max_{x \in B(x_0, \tau)} V(x) \max_{x \in B(x_0, \sigma \tau)} V(x) \|u_0^\sigma\|^2_{L^2(\mathbb{R}^N)}$$

$$< \max \{2\pi)^{-N} \left(1 + \frac{1}{\lambda_1}\right)\|\mathcal{J}u_0^\sigma\|^2_{L^2(\mathbb{R}^N)}\} \cdot \|u_0^\sigma\|_{H^1(\mathbb{R}^N)}^2,$$

where

$$\lambda_1 = \inf_{u \in H^1(B(x_0, \tau)) \setminus \{0\}} \frac{\|\nabla u\|^2_{L^2(\mathbb{R}^N)}}{\|u\|^2_{L^2(\mathbb{R}^N)}},$$

Thus (9) holds.

Now, we prove Theorem 1.1.

Proof of Theorem 1.1. Step 1. We prove for every $\lambda \in \mathbb{R}$, the functional $J$ is coercive and satisfies the compactness $(PS)$ condition.

Let us fix $\lambda \in \mathbb{R}$. By $(f_2)$, there is a positive constant $\delta$ such that

$$|f(t)| \leq \frac{\min(1, V_0)|t|}{C_*^2(1 + |\lambda|)},$$

for every $|t| \geq \delta$.

So, we get

$$|F(t)| \leq \frac{\min(1, V_0)|t|^2}{2C_*^2(1 + |\lambda|)} + \max_{|t| \leq \delta} |f(t)||t|.$$
for every $t \in \mathbb{R}$.

Thus
\[ J(u) = \frac{1}{2} \|u\|_E^2 - \lambda \int_{\mathbb{R}^N} F(x, u) \, dx \]
\[ \geq \frac{1}{2} \|u\|_E^2 - |\lambda| \int_{\mathbb{R}^N} |F(x, u)| \, dx \]
\[ \geq \frac{1}{2} \|u\|_E^2 - |\lambda| \int_{\mathbb{R}^N} |F(x, u)| \, dx \]
\[ \geq \frac{1}{2(1 + |\lambda|)} \|u\|_E^2 - |\lambda| \int_{\mathbb{R}^N} \max_{|t| \leq \delta} |f(t)| \|u\| \, dx \]
\[ \geq \frac{1}{2(1 + |\lambda|)} \|u\|_E^2 - |\lambda| \int_{\mathbb{R}^N} \frac{V(x)^{1/2}}{V(x)^2} |f(t)| \|u\| \, dx \]
\[ \geq \frac{1}{2(1 + |\lambda|)} \|u\|_E^2 - C |\lambda| \max_{|t| \leq \delta} |f(t)| \|u\|_E \]

for every $u \in E$.

Then the functional $J$ is bounded from below and $J(u) \to +\infty$ when $\|u\|_E \to +\infty$. Hence $J$ is coercive.

Now we prove that $J$ satisfies the $(PS)$ condition. Let $\{u_n\} \subset E$ be a $(PS)$ sequence for $J(u)$, that is
\[ J(u_n) \to c, \quad J'(u_n) \to 0 \text{ in } E^* . \]

Taking into account the coercivity of $J$, the sequence $\{u_n\}$ is necessarily bounded in $E$. Assume without loss of generality that $\{u_n\}$ converges to $u$ weakly in $E$, and by Lemma 2.2, we may assume that
\[ \begin{cases} u_n(x) \to u(x) & \text{ a.e. in } \mathbb{R} \\ u_n \to u & \text{ in } L^\theta(\mathbb{R}^N) \end{cases} \quad (10) \]

where $\theta \in [2, 2s^*).$

To prove that $\{u_n\}$ converges strongly to $u$ in $E$, we first introduce a simple notation. Let $\varphi \in E$ be fixed and denote by $B_{\varphi}$ the linear functional on $E$ defined by
\[ B_{\varphi}(v) = \int_{\mathbb{R}^{2N}} \frac{(\varphi(x) - \varphi(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy \]
for all $v \in E$.

Due to $(f_1)$ and $(f_2)$, there exists $C_\varepsilon > 0$ such that
\[ |f(t)| \leq \varepsilon |t| + C_\varepsilon |t|^\theta - 1 . \]

Then, by (10), we get
\[ \int_{\mathbb{R}^N} (f(x, u_n) - f(x, u))(u_n - u) \, dx \]
\[ \leq \int_{\mathbb{R}^N} (\varepsilon |u_n| + C_\varepsilon |u_n|^\theta - 1 + \varepsilon |u| + C_\varepsilon |u|^\theta - 1)(u_n - u) \, dx \]
\[ \leq \varepsilon \|u_n\|_E \|u_n - u\|_2 + \varepsilon \|u\|_2 \|u_n - u\|_2 + C_\varepsilon \|u_n\|_E^{\theta - 1} \|u_n - u\|_\theta 
+ C_\varepsilon \|u\|_E^{\theta - 1} \|u_n - u\|_\theta \]
\[ \to 0 . \]
Obviously, $(J'(u_n) - J'(u), u_n - u) \to 0$ as $n \to \infty$, since $u_n \rightharpoonup u$ in $E$ and $J'(u_n) \to 0$ in $E^*$. Hence, (10) and (11) give as $n \to \infty$

$$o(1) = (J'(u_n) - J'(u), u_n - u)$$

$$= B_{u_n}(u_n - u) - B_u(u_n - u) + \int_{\mathbb{R}^N} V(x)(u_n - u)^2 \, dx$$

$$- \int_{\mathbb{R}^N} (f(x, u_n) - f(x, u))(u_n - u) \, dx$$

$$= B_{u_n}(u_n - u) - B_u(u_n - u) + \int_{\mathbb{R}^N} V(x)(u_n - u)^2 \, dx + o(1).$$

That is

$$\|u_n - u\|^2_E = B_{u_n}(u_n - u) - B_u(u_n - u) + \int_{\mathbb{R}^N} V(x)(u_n - u)^2 \, dx \to 0$$

as $n \to \infty$.

Therefore, $J$ satisfies the $(PS)$ condition.

Step 2. We claim that the functional $J(u)$ is weakly lower semicontinuous on $E$.

The functional

$$u \mapsto \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} \, dx + \int_{\mathbb{R}^N} V(x)u^2 \, dx,$$

is sequentially weakly lower semicontinuous on $E$.

Thus it is enough to prove that the map

$$u \mapsto \int_{\mathbb{R}^N} F(x, u) \, dx,$$

is sequentially weakly continuous on $E$. To this aim fix $u_n \in E$ and $u \in E$ such that $u_n \rightharpoonup u$ in $E$ as $n \to \infty$.

Then, by Lemma 2.2, without loss of generality, we can assume that $u_n \to u$ strongly in $L^\theta(\mathbb{R}^N)$ for $2 \leq \theta < 2s^*$

and a.e. in $\mathbb{R}^N$. It is dominated by some function $h_\theta \in L^\theta(\mathbb{R}^N)$, that is,

$$|u_n| \leq h_\theta(x) \text{ a.e. } x \in \mathbb{R}^N$$

for any $n \in \mathbb{N}$ and for any $\theta \in [2, 2s^*)$.

By $(f_1)$ and $(f_2)$, there exists $C_\varepsilon$ such that

$$|f(t)| \leq \varepsilon|t| + C_\varepsilon|t|^\theta - 1.$$ 

Then, by the continuity of $F$, it follows that

$$|F(x, u_n)| \leq h(x),$$

a.e. $x \in \mathbb{R}^N$, for some $h \in L^1(\mathbb{R}^N)$, and

$$F(x, u_n) \to F(x, u),$$

a.e. in $\mathbb{R}^N$.

Hence, by Lebesgue dominated convergence theorem, we have

$$\lim_{n \to +\infty} \int_{\mathbb{R}^N} F(x, u_n) \, dx = \int_{\mathbb{R}^N} F(x, u) \, dx,$$

that is the map

$$u \mapsto \int_{\mathbb{R}^N} F(x, u) \, dx.$$
is weakly continuous in $E$. Thus, $J$ is weakly semicontinuous in $E$.

Step 3. We show that there exists a $u_0 \in E$, $\rho > 0$ such that $\rho < \Phi(u_0)$ and 
$\sup_{\Phi(u) < \rho} \Psi(u) < \frac{\Phi(u_0)}{\rho} \left( \frac{\Psi(u_0)}{\Phi(u_0)} \right)$.

By $(f_3)$, there exists $t_0 \in \mathbb{R}^N$ such that $F(t_0) > 0$. Further, let $\sigma_0 \in [0, 1]$ be such that

$$F(t_0)\sigma_0^N - (1 - \sigma_0^N) \max_{|t| \leq |t_0|} |F(t)| > 0.$$ 

Indeed, since

$$\|u_{\sigma}^0\|_{\infty} = \max_{x \in B(x_0, \tau)} |u_{\sigma}^0| \leq |t_0|,$$

(recalled the definition of $u_{\sigma}^0$), it follows that

$$\int_{B(x_0, \tau) \setminus B(x_0, \sigma_0 \tau)} F(u_{\sigma}^0) dx \geq - \max_{|t| \leq |t_0|} |F(t)| \int_{B(x_0, \tau) \setminus B(x_0, \sigma_0 \tau)} dx$$

$$= -(1 - \sigma_0^N) \max_{|t| \leq |t_0|} |F(t)| \tau^N \omega_N.$$

where $\omega_N$ denotes the volume of the unit ball in $\mathbb{R}^N$.

Then, we have that

$$\Psi(u_{\sigma}^0) = \int_{B(x_0, \tau)} F(u_{\sigma}^0) dx$$

$$= \int_{B(x_0, \sigma_0 \tau)} F(u_{\sigma}^0) dx + \int_{B(x_0, \tau) \setminus B(x_0, \sigma_0 \tau)} F(u_{\sigma}^0) dx$$

$$\geq F(t_0)\sigma_0^N \tau^N \omega_N + \int_{B(x_0, \tau) \setminus B(x_0, \sigma_0 \tau)} F(u_{\sigma}^0) dx$$

$$\geq [F(t_0)\sigma_0^N - (1 - \sigma_0^N) \max_{|t| \leq |t_0|} |F(t)|] \tau^N \omega_N.$$

By Lemma 3.2, we have

$$\Phi(u_{\sigma}^0) \leq C,$$

where

$$C = \frac{\sigma_0^2}{2(1 - \sigma)^2} \frac{\pi^N \tau^{N-2} (1 - \sigma^N)}{\Gamma(1 + \frac{N}{2}) \omega_N}.$$ 

By Lemma 3.1, there exists $\rho > 0$ such that the function $u_{\sigma}^0 \in E$ verifies the following conditions:

$$u_{\sigma}^0 \in E \setminus \sigma_0 \omega_N,$$

and

$$\sup_{u \in \sigma_0 \omega_N} \Phi(u) < \frac{\sigma_0^N \tau^N \omega_N - (1 - \sigma_0^N) \max_{|t| \leq |t_0|} |F(t)|} {C}.$$ 

Then by (14), we have

$$\rho < \Phi(u_{\sigma}^0).$$

as well as, by (12) and (13), it follows that:

$$\frac{[F(t_0)\sigma_0^N - (1 - \sigma_0^N) \max_{|t| \leq |t_0|} |F(t)|] \tau^N \omega_N}{C} \leq \frac{\Psi(u_{\sigma}^0)}{\Phi(u_{\sigma}^0)}.$$ 

Hence, (15) and (16) give

$$\sup_{u \in \sigma_0 \omega_N} \Phi(u) \leq \rho \frac{\Psi(u_{\sigma}^0)}{\Phi(u_{\sigma}^0)}.$$
By choosing \( u_0 = u_0^0 \), we get that \( \varrho < \Phi(u_0) \), and \( \sup_{\Phi(u) < \varrho} \Psi(u) < \frac{\Psi(u_0)}{\Phi(u_0)} \).

Set

\[
\tilde{a} = \frac{(1 + \varrho)\varrho}{\Psi(u_0) - \sup_{\Phi(u) < \varrho} \Psi(u)}.
\]

Clearly, \( 1 + \varrho > 1 \), and \( \tilde{a} \) is a positive constant.

It’s easy to see that \( \inf_{x \in E} \Phi(x) = 0 \), and by choosing \( u_1 = 0 \), we have

\[
\Phi(u_1) = \Psi(u_1) = 0.
\]

Then, by Step 1, Step 2, and Step 3, all the assumptions of Lemma 2.4 are verified. Thus there is an open interval \( \Lambda \subseteq [0, \tilde{a}] \) and a number \( \kappa > 0 \) such for all \( \lambda \in \Lambda \), the functional \( J \) admits three solutions in \( E \) having norm less than \( \kappa \). Since one of them may be a trivial one, we have at least two distinct, nontrivial weak solutions of problem (2).

**Remark 3.3.** Our hypotheses are similar to those employed by Bisci-Rădulescu [24] of (2) on bounded subset \( \Omega \) of \( \mathbb{R}^N \). Moreover, \((f_2)\) is weaker than the condition \((h)\). So Theorem 1.1 extends Theorem 1 of [24] and Theorem 2 of [25].

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