Triple solutions for a Dirichlet boundary value problem involving a perturbed discrete $p(k)$-Laplacian operator

Abstract: Triple solutions are obtained for a discrete problem involving a nonlinearly perturbed one-dimensional $p(k)$-Laplacian operator and satisfying Dirichlet boundary conditions. The methods for existence rely on a Ricceri-local minimum theorem for differentiable functionals. Several examples are included to illustrate the main results.

Keywords: Discrete boundary value problem, $p(k)$–Laplacian, Three solutions, Variational methods, Critical point theory

MSC: 39A10, 39A12, 39A70, 34B15, 65Q10

1 Introduction

There is an increasing interest in the existence of solutions to boundary value problems for finite difference equations with the $p(x)$–Laplacian operator in the last decades. These kinds of problems like (1) play a fundamental role in different fields of research, because of their applications in many fields, they can model various phenomena arising from the study of elastic mechanics [33], electrorheological fluids [15] and image restoration [14] and other fields such as biological neural networks, cybernetics, ecology, control systems, economics, computer science, physics, finance, artificial and many others.

Important tools in the study of nonlinear difference equations are fixed point methods in cone; see [2, 19], and upper and lower solution techniques; see, for instance, [24, 25] and references therein. It is well known that critical point theory is an important tool to deal with the problems for differential equations. For background and recent results for nonlinear discrete boundary value problems, we refer the reader to [5–7, 10, 12, 13, 16, 17, 20, 27] and the seminal papers [3, 4] and references therein.

The aim of this paper is to establish the existence of three solutions for the following discrete boundary-value problem

$$\begin{cases}
-\Delta(w(k-1))\phi_{p(k-1)}(\Delta u(k-1)) + q(k)\phi_{p(k)}(u(k)) = \lambda f(k, u(k)) + \mu g(k, u(k)), \\
u(0) = u(T + 1) = 0,
\end{cases}$$

for every $k \in [1, T]$, where $T \geq 2$ is a fixed positive integer, $[1, T]$ is the discrete interval $\{1, \ldots, T\}$, $f, g : [1, T] \times \mathbb{R} \to \mathbb{R}$ are two continuous functions in the second variable, $\lambda > 0$ and $\mu \geq 0$ are two parameters,
We note that problem (1) is the discrete variant of a type of the variable exponent anisotropic problem
\[ p(k) \text{ satisfies certain properties, and} \]
and the function \( p \) is bounded, we denote for short
\[ p^+ := \max_{k \in [0,T+1]} p(k) \quad \text{and} \quad p^- := \min_{k \in [0,T+1]} p(k), \]
and the function \( q : [0, T+1] \to [1, \infty) \) is bounded such that
\[ q^+ := \max_{k \in [0,T+1]} q(k). \]

We note that problem (1) is the discrete variant of a type of the variable exponent anisotropic problem
\[
\begin{cases}
-\sum_{i=1}^{N} \frac{\partial}{\partial \nu_i} \left( w_i(x) \frac{\partial u}{\partial \nu_i} \right) & \quad p_i(x) \geq 1 \quad \text{and} \quad q_i(x) \geq 1 \quad \text{are continuous functions on} \ \overline{\Omega}, \\
u = 0, x \in \partial \Omega,
\end{cases}
\]
where \( \Omega \subset \mathbb{R}^N, N \geq 3, \) is a bounded domain with smooth boundary, \( f, g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R}) \) are two given functions that satisfy certain properties, and \( p_i(x), w_i(x) \geq 1 \) and \( q(x) \geq 1 \) are continuous functions on \( \overline{\Omega}, \) with \( 2 \leq p_i(x), \)
for each \( x \in \Omega \) and every \( i \in \{1,2,\ldots,N\}, \lambda > 0 \) and \( \mu \) are real numbers.

In [31] the authors studied (1) depending on a parameter with \( \lambda = -1 \) and \( \mu = 0 \) by the mountain pass method. This method has been applied for (1) with \( q(k) = 0 \) and \( \mu = 0 \) in [18]. In 2003, by using the upper and lower solution method, Atici and Cabada [1] considered Eq. (1) with \( w(k) = 1, \) \( p(k) = 2, \) and \( \lambda = -1 \) subject to the boundary value conditions \( u(0) = u(T), \quad \Delta u(0) = \Delta u(T). \) In [11] the authors studied Eq. (1) with \( p(x) = 2 \) and \( w(k) = 1 \) and \( \mu = 0 \) and Neumann boundary condition \( \Delta u(0) = \Delta u(T) = 0. \) In [23, 26], the authors considered Eq. (1) in the special case, \( w(k) = 1 \) and \( p(x) = p. \) In [21], the authors applying variational methods, studied the existence of solutions of the following Kirchhoff-type discrete boundary value problems
\[
\begin{cases}
-M(\|u\|^p) \Delta \phi_p(u(k)) + q(k) \phi_p(u(k)) = f(k, u(k)), \quad k \in [1, T] \\
\Delta u(0) = \Delta u(T + 1) = 0,
\end{cases}
\]
where \( M : [0, \infty) \to \mathbb{R} \) is a continuous function.

In this paper, based on a local minimum theorem (Theorem 2.5) due to Ricceri [30], we ensure an exact interval of the parameter \( \lambda, \) in which the problem (1) admits at least three solutions.

As an example, here, we point out the following special case of our main results.

**Theorem 1.1.** Assume \( T \geq 2 \) is a fixed integer number and there exists a positive constant \( d < 1 \) such that,
\[(A0) \int_0^1 f(t) dt > d^2(2T^2 + 6T + 4),\]
\[(A1) \max \left\{ \limsup_{|t| \to +\infty} \frac{\int_0^t f(\xi) \, d\xi}{\xi^2}, \limsup_{|t| \to 0} \frac{\int_0^t f(\xi) \, d\xi}{\xi^2} \right\} < 1.\]
Then, there exists \( r > 0 \) with the following property: for every continuous function \( g : [1, T] \times \mathbb{R} \to \mathbb{R}, \) there exists \( \delta > 0 \) such that, for each \( \mu \in [0, \delta], \) the problem
\[
\begin{cases}
-\Delta^2 (u(k) - 1) + u(k) = \frac{1}{4T(T+1)} f(u(k)) + \mu g(k, u(k)), \quad k \in [1, T], \\
u(0) = u(T + 1) = 0,
\end{cases}
\]
has at least three solutions whose norms are less than \( r. \)

The rest of this paper is arranged as follows. In Section 2, we recall some basic definitions and the main tool (Theorem 2.5.), and in Section 3, we provide our main results that contain several theorems, and finally, we illustrate the results by giving examples.
2 Preliminaries

Let us introduce some notations that will be used later. Let \( T \geq 2 \) be a fixed positive integer, \([1, T]\) is the discrete interval \( \{1, \ldots, T\} \). In order to give the variational formulation of the problem (1), we introduce, \( T \)-dimensional Banach space

\[
W := \{ u : [0, T + 1] \to \mathbb{R} : u(0) = u(T + 1) = 0 \},
\]
equipped with the norm

\[
\|u\| := \left\{ \sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p^-} + q(k)|u(k)|^{p^-} \right\}^{1/p^-}.
\]

If

\[
\|u\|_+ = \left\{ \sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p^+} + q(k)|u(k)|^{p^+} \right\}^{1/p^+},
\]
then by Weighted Hölder’s inequality, one can conclude that

\[
L \|u\|_+ \leq \|u\| \leq 2 \frac{p^+ - p^-}{p^+} L \|u\|_+,
\]
where,

\[
L = \left( (T + 1) \max\{w^+, q^+\} \right)^{\frac{p^+ - p^-}{p^+}}.
\]

In the space \( W \) we can also consider the Luxemburg norm [8],

\[
\|u\|_{p(\cdot)} = \inf \left\{ \mu > 0 : \sum_{k=1}^{T+1} w(k-1) \left| \frac{\Delta u(k-1)}{\mu} \right|^{\rho(k-1)} + q(k) \left| \frac{u(k)}{\mu} \right|^{\rho(k)} \leq 1 \right\}.
\]

Since \( W \) has finite dimension, the two last norms are equivalent. Therefore there exist constants \( L_1 > 0 \) and \( L_2 > 1 \) such that

\[
L_1 \|u\|_{p(\cdot)} \leq \|u\| \leq L_2 \|u\|_{p(\cdot)}.
\]

Now, let \( \varphi : W \to \mathbb{R} \) be given by the formula

\[
\varphi(u) := \sum_{k=1}^{T+1} \left[ w(k-1)|\Delta u(k-1)|^{\rho(k-1)} + q(k)|u(k)|^{\rho(k)} \right].
\]

It is easy to check that for any \( u \in W \) the following properties hold:

\[
\|u\|_{p(\cdot)} < 1 \Rightarrow \|u\|_{p(\cdot)}^{\rho} \leq \varphi(u) \leq \|u\|_{p(\cdot)}^{\rho},
\]

\[
\|u\|_{p(\cdot)} > 1 \Rightarrow \|u\|_{p(\cdot)}^{\rho} \leq \varphi(u) \leq \|u\|_{p(\cdot)}^{\rho}.
\]

Let, \( F(k, t) := \int_0^t f(k, \xi) d\xi, \ G(k, t) := \int_0^t g(k, \xi) d\xi \), for every \( k \in [1, T] \) and \( t \in \mathbb{R} \).

In the sequel, we will use the following inequality

**Lemma 2.1.** Let \( u \in W \), Then

\[
\|u\|_{\infty} \leq c_1 \|u\|,
\]

where

\[
\|u\|_{\infty} = \max_{k \in [1, T]} |u(k)|, \quad c_1 = (2T + 2) \frac{p^+ - 1}{p^+}.
\]
Proof. Let $u$ be in $W$. Then by the discrete Hölder inequality, for every $k \in [1, T]$, we get
\[
|u(k)| \leq \sum_{k=1}^{T+1} |u(k)| + \sum_{k=1}^{T+1} |\Delta u(k-1)|
\]
\[
\leq (T + 1)^{\frac{p-1}{p}} \left\{ \sum_{k=1}^{T+1} |u(k)|^p \right\}^{\frac{1}{p}} + (T + 1)^{\frac{p-1}{p}} \left\{ \sum_{k=1}^{T+1} |\Delta u(k-1)|^p \right\}^{\frac{1}{p}}
\]
\[
\leq (T + 1)^{\frac{p-1}{p}} \left\{ \sum_{k=1}^{T+1} q(k)|u(k)|^p \right\}^{\frac{1}{p}} + \left\{ \sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^p \right\}^{\frac{1}{p}}
\]
\[
\leq 2 \frac{p-1}{p} (T + 1)^{\frac{p-1}{p}} \left\{ \sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p-1} + q(k)|u(k)|^{p-1} \right\}^{\frac{1}{p-1}}
\]
\[
= (2T + 2)^{\frac{p-1}{p}} \|u\|,
\]
for all $k \in [1, T]$.

\[\square\]

**Lemma 2.2.** 1. There exists a positive constant $C_1$ such that for all $u \in W$ with $\|u\| > 1$,  
\[
\sum_{k=1}^{T+1} \left[ \frac{w(k-1)}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} + \frac{q(k)}{p(k)} |u(k)|^{p(k)} \right] \geq \frac{\|u\|^{p^+}}{p^+} - C_1.
\]

2. for all $u \in W$ with $\|u\| < 1$,  
\[
\sum_{k=1}^{T+1} \left[ \frac{w(k-1)}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} + \frac{q(k)}{p(k)} |u(k)|^{p(k)} \right] \geq \frac{\|u\|^{p^-}}{p^-} + \|u\|^{p^-}.
\]

**Proof.** Let $u \in W$ be fixed. By a similar argument as in [28], we set  
\[A^\leq := \{ k \in [0, T + 1], |\Delta u(k)| < 1 \}, \quad B^\leq := \{ k \in [0, T + 1], |u(k)| < 1 \}, \]
\[A^\geq := \{ k \in [0, T + 1], |\Delta u(k)| > 1 \}, \quad B^\geq := \{ k \in [0, T + 1], |u(k)| > 1 \}, \]
\[A^= := \{ k \in [0, T + 1], |\Delta u(k)| = 1 \}, \quad B^= := \{ k \in [0, T + 1], |u(k)| = 1 \}.
\]

We define for each $k \in [0, T + 1]$,  
\[\alpha_k := \begin{cases} p^+, & \text{if } k \in A^\leq, \\ p^-, & \text{if } k \in A^\geq, \\ 1, & \text{if } k \in A^=, \end{cases} \quad \beta_k := \begin{cases} p^+, & \text{if } k \in B^\leq, \\ p^-, & \text{if } k \in B^\geq, \\ 1, & \text{if } k \in B^=.
\]

Thus, if $\|u\| > 1$, we have  
\[
\sum_{k=1}^{T+1} \left[ \frac{w(k-1)}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} + \frac{q(k)}{p(k)} |u(k)|^{p(k)} \right]
\]
\[
\geq \frac{1}{p^+} \sum_{k=1}^{T+1} \left[ w(k-1) |\Delta u(k-1)|^{p(k-1)} + q(k) |u(k)|^{p(k)} \right]
\]
\[
= \frac{1}{p^+} \left( \sum_{k \in A^\leq} + \sum_{k \in A^\geq} + \sum_{k \in A^=} \right) w(k-1) |\Delta u(k-1)|^{p(k-1)}
\]
\[
+ \left( \sum_{k \in B^\leq} + \sum_{k \in B^\geq} + \sum_{k \in B^=} \right) q(k) |u(k)|^{p(k)}
\]
\[
\geq \frac{1}{p^+} \sum_{k=1}^{T+1} w(k-1) |\Delta u(k-1)|^{p(k-1)}
\]
\[
+ \sum_{k=1}^{T+1} q(k)|u(k)|^{p_k}
\]
\[\geq \frac{1}{p^+} \left[ \sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p^-} \right]
- \sum_{k \in A^-} \left( w(k-1)(|\Delta u(k-1)|^{p^-} - |\Delta u(k-1)|^{p^+}) \right)
+ \sum_{k=1}^{T+1} q(k)|u(k)|^{p^-} - \sum_{k \in B^-} \left( q(k)|u(k)|^{p^-} - |u(k)|^{p^+} \right)\]
\[\geq \frac{1}{p^+} \left[ \sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p^-} \right]
- w^+ \sum_{k \in A^-} \left( |\Delta u(k-1)|^{p^-} - |\Delta u(k-1)|^{p^+} \right)
+ \sum_{k=1}^{T+1} q(k)|u(k)|^{p^-} - q^+ \sum_{k \in B^-} \left( |u(k)|^{p^-} - |u(k)|^{p^+} \right)\]
\[\geq \frac{1}{p^+} \left[ \sum_{k=1}^{T+1} w(k-1)|\Delta u(k-1)|^{p^-} - (T+1)w^+ \right]
+ \sum_{k=1}^{T+1} q(k)|u(k)|^{p^-} - q^+ (T+1)\]
\[= \frac{\|u\|^{p^-}}{p^+} - \frac{(T+1)(w^+ + q^+)}{p^+}.\]

If \(\|u\| < 1\), then \(A^- = A^\ast = B^- = B^\ast = \emptyset\). It follows that \(|\Delta u(k-1)|, |u(k)| < 1\) for each \(k \in [1, T+1]\).

Hence, we have

\[
\sum_{k=1}^{T+1} \frac{w(k-1)}{p(k-1)}|\Delta u(k-1)|^{p(k-1)} + \frac{q(k)}{p(k)}|u(k)|^{p(k)}
\geq \frac{1}{p^+} \sum_{k=1}^{T+1} \left[ w(k-1)|\Delta u(k-1)|^{p^+} + q(k)|u(k)|^{p^+} \right]
= \frac{1}{p^+} \|u\|^{p^+}
\geq \frac{2^{p^+ - p^-}}{p^+ L^{p^+}} \|u\|^{p^+}.
\]

To study the problem (1), we consider the functional \(I_{\lambda, \mu} : W \to \mathbb{R}\) defined by

\[
I_{\lambda, \mu}(u) = \sum_{k=1}^{T+1} \frac{w(k-1)}{p(k-1)}|\Delta u(k-1)|^{p(k-1)} + \frac{q(k)}{p(k)}|u(k)|^{p(k)} - \lambda \sum_{k=1}^{T} F(k, u) - \mu \sum_{k=1}^{T} G(k, u).
\]

An easy computation ensures that \(I_{\lambda, \mu}\) is of class \(C^1\) on \(W\) with

\[
I'_{\lambda, \mu}(u)(v) = \sum_{k=1}^{T+1} \left[ w(k-1)\phi_{p(k-1)}(|\Delta u(k-1)|^{p(k-1)}) \Delta v(k-1) + q(k)|u(k)|^{p(k)-2}u(k)v(k) \right]
- \sum_{k=1}^{T} [\lambda f(k, u(k)) + \mu g(k, u(k))] v(k),
\]
for all \(u, v \in W\).
Lemma 2.3. The critical points of $I_{\lambda, \mu}$ are exactly the solutions of the problem (1).

Proof. Let $\overline{u}$ be a critical point of $I_{\lambda, \mu}$ in $W$. Then $\overline{u}(0) = \overline{u}(T + 1) = 0$ and for all $v \in W$, $I'_{\lambda, \mu}(\overline{u})(v) = 0$. Thus, for every $v \in W$, and taking $v(0) = v(T + 1) = 0$ and summation by parts into account, one has

$$0 = I'_{\lambda, \mu}(\overline{u})(v) = \sum_{k=1}^{T+1} w(k-1)\phi_{p(k-1)}(\Delta \overline{u}(k-1))\Delta v(k-1)$$

$$+ \sum_{k=1}^{T} \left[ q(k)|\overline{u}(k)|^{p(k)-2}\overline{u}(k)v(k) - \lambda f(k, \overline{u}(k))v(k) - \mu g(k, \overline{u}(k))v(k) \right]$$

$$= w(T+1)\phi_{p(k-1)}(\Delta \overline{u}(T+1))v(T+1) - w(0)\phi_{p(k-1)}(\Delta \overline{u}(0))v(0)$$

$$- \sum_{k=1}^{T+1} \Delta(w(k-1)\phi_{p(k-1)}(\Delta \overline{u}(k-1)))v(k)$$

$$+ \sum_{k=1}^{T} \left[ q(k)|\overline{u}(k)|^{p(k)-2}\overline{u}(k)v(k) - \lambda f(k, \overline{u}(k))v(k) - \mu g(k, \overline{u}(k))v(k) \right]$$

$$= - \sum_{k=1}^{T} \left[ \Delta(w(k-1)\phi_{p(k-1)}(\Delta \overline{u}(k-1))) \right.\right.$$

$$- q(k)|\overline{u}(k)|^{p(k)-2}\overline{u}(k) + \lambda f(k, \overline{u}(k)) + \mu g(k, \overline{u}(k)) \right] v(k).$$

Since $v \in W$ is arbitrary, one has

$$-\Delta(w(k-1)\phi_{p(k-1)}(\Delta \overline{u}(k-1))) + q(k)\phi_{p(k)}(\overline{u}(k)) = \lambda f(k, \overline{u}(k)) + \mu g(k, \overline{u}(k)).$$

for every $k \in [1, T]$. Therefore, $\overline{u}$ is a solution of (1). So by bearing in mind that $\overline{u}$ is arbitrary, we conclude that every critical point of the functional $I_{\lambda, \mu}$ in $W$ is exactly a solution of the problem (1).

Also, if $\overline{u} \in W$ is a solution of problem (1), one can show that $\overline{u}$ is a critical point of $I_{\lambda, \mu}$. Indeed, by multiplying the difference equation in problem (1) by $v(k)$ as an arbitrary element of $W$ and summing and using the fact that

$$\sum_{k=1}^{T+1} w(k-1)\phi_{p(k-1)}(\Delta \overline{u}(k-1))\Delta v(k-1) = - \sum_{k=1}^{T} \Delta\left(w(k-1)\phi_{p(k-1)}(\Delta \overline{u}(k-1))\right) v(k),$$

we have $I'_{\lambda, \mu}(\overline{u})(v) = 0$, hence $\overline{u}$ is a critical point for $I_{\lambda, \mu}$. Thus the vice versa holds and the proof is completed. 

Our main tool is a local minimum theorem (Theorem 2.5) due to Ricceri (see [30, Theorem 2]), which is recalled below. We refer to the papers [22, 30, 32] in which Theorem 2.5 has been successfully employed for the existence of at least three solutions for two-point boundary value problems. First, we give the following definition.

Definition 2.4. If $X$ is a real Banach space, we denote by $\mathcal{W}_X$ the class of all functionals $\Phi : X \to \mathbb{R}$ possessing the following property: if $\{u_n\}$ is a sequence in $X$ converging weakly to $u \in X$ and $\liminf_{n \to \infty} \Phi(u_n) \leq \Phi(u)$, then $\{u_n\}$ has a subsequence converging strongly to $u$. For instance, if $X$ is uniformly convex and $g : [0, +\infty] \to \mathbb{R}$ is a continuous, strictly increasing function, then, by a classical result, the functional $u \to g(\|u\|)$ belongs to the class $\mathcal{W}_X$.

Our main tool reads as follows:

Theorem 2.5 ([30, Theorem 2]). Let $X$ be a separable and reflexive real Banach space and $\Phi : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous $C^1$ functional, belonging to $\mathcal{W}_X$, bounded on each bounded subset of $X$ and whose derivative admits a continuous inverse on $X^*; J : X \to \mathbb{R}$ a $C^1$ functional with compact derivative. Assume that $\Phi$ has a strict local minimum $x_0$ with $\Phi(x_0) = J(x_0) = 0$. Finally, setting

$$\alpha = \max \left\{ 0, \limsup_{\|x\| \to \infty} \frac{J(x)}{\Phi(x)}, \limsup_{\|x\| \to 0} \frac{J(x)}{\Phi(x)} \right\},$$

...
Theorem 3.1. Assume that there exist constants \( \alpha < \beta \). Then, for each compact interval \([a, b]\) there exists \( r > 0\) with the following property: for every \( \lambda \in [a, b] \) and every \( C^1 \) functional \( \Phi : X \to \mathbb{R} \) with compact derivative, there exists \( \delta > 0 \) such that, for each \( \mu \in [0, \delta] \), the equation \( \Phi'(x) = \lambda J'(x) + \mu \Phi'(x) \) has at least three solutions whose norms are less than \( r \).

Also we state the following consequence of the Strong comparison principle [3, Lemma 2.3] (see also [5, Theorem 2.2]) which we will use in the sequel in order to obtain positive solutions to the problem (1), i.e. \( u(k) > 0 \) for each \( k \in [1, T] \).

Lemma 2.6. If \( u \in W \) and

\[
-\Delta (w(k-1)\phi_{p(k-1)}(\Delta u(k-1))) + q(k)\phi_{p(k)}(u(k)) \geq 0, \quad k \in [1, T],
\]

\[
u(0) \geq 0, \quad u(T+1) \geq 0,
\]

then either \( u \) is positive or \( u \equiv 0 \).

Proof. First, we can prove that \( u \geq 0 \). Assume that \( u(k) < 0 \) for any \( k \in [0, T+1] \). Then, there would exist \( k_0 \in [0, T+1] \) such that \( \min_{k \in [1, T]} u(k) = u(k_0) < 0 \) and \( \Delta u(k_0 - 1) \leq 0 \). Hence from (7), one has

\[
0 > w(k_0 - 1)\phi_{p(k_0-1)}(\Delta u(k_0 - 1)) + q(k_0)|u(k_0)|^{p(k_0)}u(k_0) \geq w(k_0)\phi_{p(k_0)}(\Delta u(k_0)).
\]

Thus \( u(k_0 + 1) < u(k_0) \). This is a contradiction. Therefore one can conclude \( u(k) \geq 0 \) for any \( k \in [0, T+1] \). Finally, arguing as in the proof of [3, Lemma 2.2], we observe that if \( u(k) = 0 \) satisfying (7), then by positivity of \( w(k) \) and \( w(k-1) \) and non-negativity of \( u(k+1) \), \( u(k-1) \), one has

\[
0 \leq -w(k)|u(k+1)|^{p(k)-2}u(k+1) - w(k-1)|u(k-1)|^{p(k-1)-2}u(k-1) \leq 0,
\]

from which it follows that, \( u(k+1) = u(k-1) = 0 \). Thus either \( u \) is positive or \( u \equiv 0 \). Hence, the result follows.

By a similar argument used in [5, Theorem 2.3], we obtain the next result which guarantees the same conclusion of the preceding Strong maximum principle, independently of the sign of the operator.

Lemma 2.7. Fix \( u \in W \) such that, if \( u(k) \leq 0 \), it follows that

\[
-\Delta (w(k-1)\phi_{p(k-1)}(\Delta u(k-1))) + q(k)\phi_{p(k)}(u(k)) = 0.
\]

Then either \( u \) is positive or \( u \equiv 0 \).

3 Main results

Theorem 3.1. Assume that there exist constants \( M > 0 \), \( m > 0 \), \( d \in (0, 1) \), and \( \bar{a}, \bar{b} > 0 \) and \( s > p^+ \) such that,

(B0) \( p^+ T \max\{\hat{b}c^p_1, Lp^+ 2p^+ - p^-\bar{a}c^p_1\} < \frac{p^- \sum_{k=1}^{T} F(k, d)}{d^p (w(0) + w(T) + \sum_{k=1}^{T} q(k))} \).

(B1) \( F(k, t) < \bar{a}|t|^s + |t|^{p^+} \); for every \( (k, |t|) \in [1, T] \times [0, m] \).

(B2) \( F(k, t) < \bar{b}(1 + |t|^{p^-}) \); for every \( (k, |t|) \in [1, T] \times [M, \infty] \).

Then, for each compact interval \([a, b] \subset \Lambda := \lambda_1, \lambda_2 \] where

\[
\lambda_1 = \frac{d^p (w(0) + w(T) + \sum_{k=1}^{T} q(k))}{p^- \sum_{k=1}^{T} F(k, d)},
\]

\[
\lambda_2 = \frac{d^p (w(0) + w(T) + \sum_{k=1}^{T} q(k))}{p^- \sum_{k=1}^{T} F(k, d)},
\]

we have that any positive solution \( u \) to the problem (1) satisfies

\[
u(0) \geq 0, \quad u(T+1) \geq 0,
\]

and

\[
\min_{k \in [1, T]} u(k) > 0.
\]

Furthermore, any solution \( u \) is strictly positive on \([1, T] \).
there exists \( r > 0 \) with the following property: for every \( \lambda \in [a, b] \) and for every continuous function \( g : [1, T] \times \mathbb{R} \to \mathbb{R} \), there exists \( \delta > 0 \) such that, for each \( \mu \in [0, \delta] \), the problem (1) has at least three solutions whose norms are less than \( r \).

**Proof.** Our aim it to apply Theorem 2.5 to our problem. To this end, we observe that due to \((B0)\) the interval \([\lambda_1, \lambda_2]\) is non-empty, so fix \( \lambda \in [a, b] \subset [\lambda_1, \lambda_2] \), take \( X = W \), and put \( \Phi, \Psi \) as follows:

\[
\Phi(u) := \sum_{k=1}^{T+1} \left[ \frac{w(k-1)}{p(k-1)} \Delta u(k-1) |\Delta v(k-1)|^{\rho(k)-1} + \frac{q(k)}{p(k)} |u(k)|^{\rho(k)} \right],
\]

\[
J(u) := \sum_{k=1}^{T} F(k, u(k)), \quad \text{and} \quad \Psi(u) := \sum_{k=1}^{T} G(k, u(k)),
\]

for every \( u \in W \). So \( I_{\lambda, \tilde{\mu}} = \Phi - \tilde{\lambda}J - \tilde{\mu}\Psi \). An easy computation ensures that \( \Phi, J \) and \( \Psi \) turn out to be of class \( C^1 \) on \( W \) with

\[
\Phi'(u)(v) = \sum_{k=1}^{T+1} \left[ \frac{w(k-1)}{p(k-1)} \Delta u(k-1) \Delta v(k-1) + q(k) |u(k)|^{\rho(k)-2} u(k) v(k) \right],
\]

\[
= - \sum_{k=1}^{T} \left[ \Delta(w(k-1)) \phi_{p(k-1)}(\Delta u(k-1)) v(k) - q(k) |u(k)|^{\rho(k)-2} u(k) v(k) \right],
\]

and

\[
J'(u)(v) = \sum_{k=1}^{T} f(k, u(k)) v(k), \quad \Psi'(u)(v) = \sum_{k=1}^{T} g(k, u(k)) v(k),
\]

for all \( u, v \in W \). By Lemma 2.3, the solutions of the equation \( I_{\lambda, \tilde{\mu}}' = \Phi' - \tilde{\lambda}J' - \tilde{\mu}\Psi' = 0 \) are exactly the solutions for problem (1). Hence, to prove our result, it is enough to apply Theorem 2.5. We know \( \Phi \) is a nonnegative continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on \( X^* \), and \( \Psi \) is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Also the functional \( \Phi \) is coercive. Indeed, by (3) for \( \|u\| > L_2 \), we have

\[
\|u\|_{L^p} > \frac{\|u\|_{L^p}}{L^2} > \frac{\|u\|_{L^p}}{p+L^2} \to +\infty \text{ as } \|u\| \to +\infty.
\]

In addition, \( \Phi \) has a strict local minimum 0 with \( \Phi(0) = J(0) = 0 \).

We show that for \( \tilde{\lambda} \in (a, b) \subset \Lambda := [\lambda_1, \lambda_2] \) to be fixed, \( \alpha < \frac{1}{2} \). Let \( d \in (0, 1) \) be fixed and put \( \bar{v}(k) = d \) for every \( k \in [1, T] \) and \( \bar{v}(0) = \bar{v}(T+1) = 0 \). Clearly \( \bar{v} \in W \) and

\[
0 < \Phi(\bar{v}) < \frac{\|\bar{v}\|_{L^p}}{p} \left( \frac{w(0)}{p} + \frac{w(T)}{p} + \sum_{k=1}^{T} d \frac{q(k)}{p(k)} \right),
\]

\[
< \frac{d^{\frac{p-1}{p}}}{p} \left( w(0) + w(T) + \sum_{k=1}^{T} q(k) \right),
\]

and

\[
J(\bar{v}) = \sum_{k=1}^{T} F(k, \bar{v}(k)) = \sum_{k=1}^{T} F(k, d).
\]
Therefore,
\[ \beta = \sup_{u \in \Phi^{-1}(0, \infty)} \frac{J(u)}{\Phi(u)} > \frac{J(\bar{v})}{\Phi(\bar{v})} \]
\[ > \frac{p^- \sum_{k=1}^{T} F(k, d)}{d p^- \left(w(0) + w(T) + \sum_{k=1}^{T} q(k)\right)} \]
\[ = \frac{1}{\lambda_2} \]
\[ > \frac{1}{\lambda} \]  
(11)

On the other hand, from the conditions (B1) and (B2), and bearing (6) in mind, we get
\[ \sum_{k=1}^{T} F(k, u(k)) < T \tilde{a} c_1^s \|u\|^s + \tilde{\alpha} T c_1^{p^+} \|u\|^{p^+}, \text{ for } \|u\| \text{ small enough,} \]
\[ \sum_{k=1}^{T} F(k, u(k)) < T \tilde{b} + \tilde{\beta} T c_1^{p^-} \|u\|^{p^-}, \text{ for } \|u\| \text{ large enough.} \]

Hence, by Lemma 2.2(1)
\[ \limsup_{\|u\| \to \infty} \frac{J(u)}{\Phi(u)} < \limsup_{\|u\| \to \infty} \frac{T \tilde{b} + \tilde{\beta} T c_1^{p^-} \|u\|^{p^-}}{\|u\|^{p^-} - C_1} = p^+ T \tilde{b} c_1^{p^-}, \]  
(12)

and by Lemma 2.2(2)
\[ \limsup_{\|u\| \to 0} \frac{J(u)}{\Phi(u)} < \limsup_{\|u\| \to 0} \frac{T \tilde{a} c_1^s \|u\|^s + \tilde{\alpha} T c_1^{p^+} \|u\|^{p^+}}{2 p^- \rho^+} = p^+ T L \rho^+ \frac{p^+ - p^-}{\rho^+ - \rho^-} \tilde{a} c_1^{p^+}. \]  
(13)

By (12) and (13), we get,
\[ \alpha = \max \left\{ 0, \limsup_{\|u\| \to \infty} \frac{J(u)}{\Phi(u)}, \limsup_{\|u\| \to 0} \frac{J(u)}{\Phi(u)} \right\} \]
\[ < p^+ T \max\{\tilde{b} c_1^{p^-}, L \rho^+ \frac{p^+ - p^-}{\rho^+ - \rho^-} \tilde{a} c_1^{p^+}\} = \frac{1}{\lambda_2} < \frac{1}{\lambda}. \]  
(14)

By (11) and (14), we deduce that \( \alpha < \beta. \) All the assumptions of Theorem 2.5 are satisfied, so by applying that theorem, the conclusion follows. Hence, the proof is complete.

Now, we present an example to illustrate the results of Theorem 3.1.

**Example 3.2.** Choose \( m = 0.198, M = 250, d = \ln(1 + \sqrt{2}) \in (0, 1), s = 4.5, p(k) = \frac{2k}{11} + 2, \tilde{a} = e^{-19}, \)
\( \tilde{b} = e^{-11}, T = 10, q(k) = 1, w(k) = e^{k(10-k)}, \text{ hence } w(0) = 1, w(10) = 1, w^+ = e^{4.5}, L = \sqrt{\frac{1}{11} e^{4.5}}, \)
\( \sum_{k=1}^{T} q(k) = 10, p^+ = 4, p^- = 2, c_1 = (2T + 2) \frac{p^+ - p^-}{\rho^+ - \rho^-} = \sqrt{22}. \) Let
\[ f(k, t) = \frac{2 \cosh t}{\sinh t} e^{-\frac{1}{\sinh t}}, \ f(k, 0) = 0, \ \forall t \neq 0, k \in [1, 10], \]
\[ F(k, t) = e^{-\frac{1}{\sinh t}}, \]
for any \( k \in [1, 10], \) where \( t^+ = \max\{0, t\}. \) Therefore,
\[ p^+ T \max\{\tilde{b} c_1^{p^-}, L \rho^+ \frac{p^+ - p^-}{\rho^+ - \rho^-} \tilde{a} c_1^{p^+}\} = 40 \times 22^3 e^{-14.5} \approx 0.2148117565, \]
\[ \frac{p^- \sum_{k=1}^{T} F(k, d)}{p^- (w(0) + w(T) + \sum_{k=1}^{T} q(k))} = \frac{5}{3e(\ln(1 + \sqrt{2}))^2} \approx 0.7892856465, \]
it follows that (B0) holds. Also by the graph of the function
\[ h(x) := e^{-\frac{1}{\sinh(x)^2}} - \frac{|x|^{4.5} + x^4}{e^{19}}, \]
on \([-0.198, 0.198]\), where it is under x axis (see Figure 1), one can conclude (B1) holds. It is clear that (B2) holds.

Fig. 1. The graph of the function \( h \)

We see that all assumptions of Theorem 3.1 are satisfied, hence Theorem 3.1 implies that for each compact interval
\[ [a, b] \subset \left[ 3e \left( \frac{\ln(1 + \sqrt{2})}{5} \right)^2, \frac{14.5 e^{14.5}}{40 \times 22^3} \right], \]
there exists \( r > 0 \) with the following property: for every \( \lambda \in [a, b] \) and every positive continuous function \( g : [1, 10] \times \mathbb{R} \rightarrow \mathbb{R}, \) there exists \( \delta > 0 \) such that, for each \( \mu \in [0, \delta], \) the problem
\[
\begin{align*}
-\Delta(e^{-\frac{e^{k(k-1)}}{p+1}} \phi_{p(k-1)}(\Delta u(k-1)) + \phi_{p(k)}(u(k))) &= \lambda f(k, u^+(k)) + \mu g(k, u(k)), \\
u(0) &= u(11) = 0,
\end{align*}
\]
k \( \in [1, 10], \) has at least three solutions whose norms are less than \( r, \) hence by Lemma 2.6 whose signs are positive.

The next result reads as follows.

**Theorem 3.3.** Assume that there exists a constant \( d > 1 \) such that,
\[
(C0) \quad p^+ TC_4^{p^+} \rho_+ \frac{p^+ - p_-}{p} L p^+ < \frac{p^- \sum_{k=1}^{T} F(k, d)}{d p^+ (w(0) + w(T) + \sum_{k=1}^{T} q(k))}.
\]
\[
(C1) \quad \max \left\{ \limsup_{|t| \to +\infty} \frac{F(k, t)}{|t|^{p^+}}, \limsup_{|t| \to 0} \frac{F(k, t)}{|t|^{p^+}} \right\} < 1; \text{ for every } k \in [1, T].
\]
Then, for each compact interval \( [a, b] \subset \Lambda := [\lambda_1, \lambda_2], \) where
\[
\lambda_1 = \frac{d p^+ (w(0) + w(T) + \sum_{k=1}^{T} q(k))}{p^- \sum_{k=1}^{T} F(k, d)},
\]
\[
\lambda_2 = \frac{1}{p^+ TC_4^{p^+} \rho_+ \frac{p^+ - p_-}{p} L p^+},
\]
there exists \( r > 0 \) with the following property: for every \( \lambda \in [a, b] \) and every continuous function \( g : [1, T] \times \mathbb{R} \rightarrow \mathbb{R}, \) there exists \( \delta > 0 \) such that, for each \( \mu \in [0, \delta], \) the problem (1) has at least three solutions whose norms are less than \( r. \)
Proof. Our aim is to apply Theorem 2.5 to our problem. To this end, we observe that due to (C0) the interval \( [\lambda_1, \lambda_2] \) is non-empty, so fix \( \bar{\lambda} \) in \( [\lambda_1, \lambda_2] \), take \( Y = W \), and put \( \Phi, \Psi \) and \( J \), as given in (8) and (9). So \( I_{\bar{\lambda}, \mu} = \Phi - \bar{\lambda}J - \mu \Psi \). By Lemma 2.3, the solutions of the equation \( I'_{\bar{\lambda}, \mu} = \Phi' - \bar{\lambda}J' - \mu \Psi' \) are exactly the solutions for problem (1). We know that \( \Phi, \Psi \) and \( J \) satisfy the regularity assumptions of Theorem 2.5. Hence, to prove our result, it is enough to apply Theorem 2.5.

We show that for \( \lambda \in [a, b] \subset \Lambda := [\lambda_1, \lambda_2] \) to be fixed, \( \alpha < \frac{1}{\lambda}, \beta > \frac{1}{\lambda} \). Let \( d > 1 \) be fixed and put \( \bar{v}(k) = d \) for every \( k \in [1, T] \) and \( \bar{v}(0) = \bar{v}(T + 1) = 0 \). Clearly \( \bar{v} \in W \) and

\[
0 < \Phi(\bar{v}) < \frac{\alpha}{p} \left( w(0)d^p + w(T)d^p + \sum_{k=1}^{T} d^{p}(q(k)) \right) < \frac{d^{p}}{p} (w(0) + w(T) + \sum_{k=1}^{T} q(k)).
\]

Therefore

\[
\beta = \sup_{u \in \Phi^{-1}(0, \infty)} \frac{J(u)}{\Phi(u)} = \frac{p \sum_{k=1}^{T} F(k, d)}{d^{p}(w(0) + w(T) + \sum_{k=1}^{T} q(k))} = \frac{1}{\lambda_1} > \frac{1}{\lambda}.
\]

In view of (C1), there exist \( 0 < r_1 < 1 < r_2 \) such that

\[
F(k, t) < |t|^{p^+}, \text{ for any } (k, t) \in [1, T] \times [-r_1, r_1],
\]

\[
F(k, t) < |t|^{p^-} < |t|^{p^+}, \text{ for any } (k, t) \in [1, T] \times \mathbb{R} \setminus ([r_2, r_2]).
\]

Since \( F \) is continuous, then \( F(k, t) \) is bounded for any \( (k, t) \in [1, T] \times ([r_1, r_2] \cup [-r_2, -r_1]) \), so we can choose \( C_2 > 0 \) and \( s > p^+ \) such that

\[
F(k, t) < |t|^{p^+} + C_2 |t|^{s}, \text{ for all } (k, t) \in [1, T] \times \mathbb{R},
\]

and bearing (6) in mind, we get

\[
J(u) = \sum_{k=1}^{T} F(k, u(k)) < Tc_1^{p^+} \|u\|^{p^+} + TC_2c_1^{s} \|u\|^{s}, \text{ for all } u \in W,
\]

and by Lemma 2.2(2)

\[
\lim_{\|u\| \to 0} \sup \frac{J(u)}{\Phi(u)} < \lim_{\|u\| \to 0} \frac{Tc_1^{p^+} \|u\|^{p^+} + TC_2c_1^{s} \|u\|^{s}}{2^{p} \|u\|^{p}} = p^+ 2^{p^+} \|u\|^{p} + p^+ Tc_1^{p^+}. \tag{16}
\]

Again since \( F \) is continuous, then \( F(k, t) \) is bounded for any \( (k, t) \in [1, T] \times [-r_2, r_2] \), so we can choose \( C_3 > 0 \) such that

\[
F(k, t) < |t|^{p^-} + C_3, \text{ for all } (k, t) \in [1, T] \times \mathbb{R},
\]

and bearing (6) in mind, we get

\[
J(u) = \sum_{k=1}^{T} F(k, u(k)) < Tc_1^{p^-} \|u\|^{p^-} + TC_3, \text{ for all } u \in W,
\]
hence, by Lemma 2.2(1)
\[
\limsup_{\|u\| \to \infty} \frac{J(u)}{\Phi(u)} < \limsup_{\|u\| \to \infty} \frac{Tc_1^{p^+}\|u\|^{p^+} + TC_3}{\frac{\|u\|}{p^+} - C_1} = p^+ Tc_1^{p^+}.
\] (17)

By (16) and (17), we get
\[
\alpha = \max \{0, \limsup_{\|u\| \to \infty} \frac{J(u)}{\Phi(u)} \} \leq p^+ T \max \{c_1^{p^+} - \frac{p^+}{p^-} L, c_1^{p^+} \} = p^+ T c_1^{p^+} - \frac{p^+}{p^-} L p^+ = \frac{1}{\lambda_2} < \frac{1}{\lambda}.
\] (18)

By (15) and (18), we deduce that \( \alpha < \beta \). All the assumptions of Theorem 2.5 are satisfied, so by applying that theorem, the conclusion follows. Hence, the proof is complete.

Now, we present an example to illustrate the results of Theorem 3.3.

**Example 3.4.** Let \( T = 10 \), and for every \( k \in [1, 10] \), put
\[
f(k, t) = \begin{cases} t^3, & \text{if } t = 0, \\
3^2 \sin(\ln |t|) + \cos(\ln |t|), & \text{if } |t| \in (0, 1], \\
t^2 + (2 + e^{6\pi}) t + e^{6\pi}, & \text{if } |t| \in [-e^{6\pi}, -1], \\
(t^2 + (2 + e^{6\pi}) t) e^{-6\pi}, & \text{if } |t| \in [1, e^{6\pi}]. \\
t \sin(\ln |t|) + \cos(\ln |t|), & \text{if } |t| \in \{e^{6\pi}, \infty\}. \\
\end{cases}
\]
as an odd continuous function on \( \mathbb{R} \). Let \( d = e^{\frac{\pi}{2}}, p(k) = \frac{k^2}{17} k + 4, q(k) = 1, w(k) = e^{\frac{k(10-k)}{2} + 5k + 16} \) for \( k = 1, 2, 3, ..., 10 \). Hence \( p^- = 2, p^+ = 4, \sum_{k=1}^{10} q(k) = 10, q^+ = 1, w^+ = e^{0.8}, L = \frac{4}{11} e^{0.8} \) and \( c_1 = (2T + 2)^{n-1} \frac{1}{p^-} = \frac{1}{22} \). Simple calculations show that
\[
F(k, t) = \begin{cases} t^3, & \text{if } t = 0, \\
3^2 \sin(\ln |t|) + 5 \cos(\ln |t|), & \text{if } |t| \in (0, 1], \\
\frac{t^2}{2} + (2 + e^{6\pi}) t^2 - e^{6\pi} t - \frac{2}{3} + \frac{1}{3} e^{6\pi} + \frac{3}{3^2}, & \text{if } |t| \in [-e^{6\pi}, -1], \\
\frac{t^2}{2} [3 \sin(\ln |t|) + \cos(\ln |t|)] + \frac{1}{6} e^{18\pi} - \frac{1}{3} e^{12\pi}, & \text{if } |t| \in [1, e^{6\pi}]. \\
\frac{2}{3} e^{6\pi} - \frac{2}{3} + \frac{3}{3^2}, & \text{if } |t| \in \{e^{6\pi}, \infty\}. \\
\end{cases}
\]

\[
\limsup_{|t| \to +\infty} \frac{F(k, t)}{|t|^2} = \frac{\sqrt{10}}{5} < 1,
\]
\[
\limsup_{|t| \to 0} \frac{F(k, t)}{|t|^4} = \frac{\sqrt{34}}{17} < 1.
\]

By using software Maple, one can conclude that
\[
p^- \sum_{k=1}^{10} F(k, d) = 20 \frac{e^{\frac{5\pi}{3}} + e^{\frac{13\pi}{2}} - e^{\frac{7\pi}{2}} - 25}{12e^{\frac{1}{2}}} + 25 \frac{e^{6\pi}}{2} \approx 7873816.449
\]
\[
> p^+ T c_1^{p^+} - \frac{p^+}{p^-} L p^+ = 40 \cdot 22^3 e^{0.8} \approx 947902.4.
\]

Hence, by using Theorem 3.3, for each compact interval \([a, b] \in [0.00000012, 0.000001]\) there exists \( r > 0 \) with the following property: for every \( \lambda \in [a, b] \) and every continuous function \( g : [1, 10] \times \mathbb{R} \to \mathbb{R} \), there exists \( \delta > 0 \) such that, for each \( \mu \in [0, \delta] \), the problem
\[
-\Delta(e^{\frac{k(10-k)}{2} + 6\pi} \phi_{p(k-1)}(\Delta u(k - 1)) + \phi_{p(k)}(u(k)) = \lambda f(k, u(k)) + \mu g(k, u(k)),
\]
\[
u(0) = u(11),
\]
for every \( k \in [1, 10] \), has at least three solutions whose norms are less than \( r \).
Remark 3.5. By Lemma 2.6, the ensured solutions in the conclusions of Theorems 3.1 and 3.3 are either zero or positive. Now, let \( f(k, 0) + g(k, 0) = 0 \) for all \( k \in [0, T] \), by putting

\[
    f^*(k, t) + g^*(k, t) = \begin{cases} 
        f(k, t) + g(k, t), & t > 0, \\ 
        0, & t \leq 0,
    \end{cases}
\]

and due to Lemma 2.7, the ensured solutions are positive (see [5, Remark 2.1]).

Finally we present the problem (1), in which the function \( f(k, u) \) has separable variables and \( \lambda = 1 \).

Theorem 3.6. Let \( f_0 : [1, T] \to \mathbb{R} \) be a non-negative, non-zero and essentially bounded function such that

\[
    \sum_{k=1}^{T} f_0(k) = \frac{1}{p^+ 2 \frac{p^+ - p^-}{p^-} L^{p^+} c_1^{p^+}},
\]

and \( f_1 : \mathbb{R} \to \mathbb{R} \) be a non-negative and continuous function and \( F_1(\xi) = \int_{0}^{\xi} f_1(x)dx \) for every \( \xi \in \mathbb{R} \). Assume that there exists a positive constant \( d < 1 \) such that,

\[
    (D0) \quad F_1(d) > \frac{p^+ 2 \frac{p^+ - p^-}{p^-} L^{p^+} c_1^{p^+}}{p^-} \left( w(0) + w(T) + \sum_{k=1}^{T} q(k) \right)
\]

\[
    (D1) \quad \max\left\{ \limsup_{|t| \to +\infty} \frac{F_1(t)}{|t|^{p^+}}, \limsup_{|t| \to 0} \frac{F_1(t)}{|t|^{p^+}} \right\} < 1.
\]

Then, there exists \( r > 0 \) with the following property: for every continuous function \( g : [1, T] \times \mathbb{R} \to \mathbb{R} \), there exists \( \delta > 0 \) such that, for each \( \mu \in [0, \delta] \), the problem

\[
    \begin{cases} 
        -\Delta (w(k-1)\phi_p(\Delta u(k-1))) + q(k)\phi_p(u(k)) = f_0(k)f_1(u(k)) + \mu g(k, u(k)), \\
        u(0) = u(T + 1) = 0,
    \end{cases}
\]

for every \( k \in [1, T] \), has at least three solutions whose norms are less than \( r \).

Proof. We can choose \( \lambda = 1 \in [a, b] \) such that \( [a, b] \subset \left( \frac{1}{p^+} - \frac{1}{p^-} \right) \). Take \( X = W \), and put \( \Phi, \Psi \) and \( J \) as given in (8) and (9), so \( I_{1, \mu} = \Phi - J - \mu\Psi \) and the solutions of the equation \( I'_{1, \tilde{\mu}} = \Phi' - J' - \tilde{\mu}\Psi' = 0 \) are exactly the solutions for problem (19). Hence, to prove our result, it is enough to apply Theorem 2.5.

We show that, \( \alpha < 1 \), \( \beta > 1 \). Put \( v(k) = d \in (0, 1) \) for every \( k \in [1, T] \) and \( v(0) = v(T + 1) = 0 \). In view of (D0) and (10),

\[
    \beta > J(v) \Phi(v) > \frac{p^- F_1(d) \sum_{k=1}^{T} f_0(k)}{d^{-p^-} (w(0) + w(T) + \sum_{k=1}^{T} q(k))} > p^+ 2 \frac{p^+ - p^-}{p^-} L^{p^+} c_1^{p^+} \sum_{k=1}^{T} f_0(k) = 1.
\]

Applying similar argument as in the proof of Theorem 3.3, and taking into account (D1) and Lemma 2.2

\[
    \limsup_{||u|| \to 0} \frac{J(u)}{\Phi(u)} \leq p^+ 2 \frac{p^+ - p^-}{p^-} L^{p^+} c_1^{p^+} \sum_{k=1}^{T} f_0(k),
\]

\[
    \limsup_{||u|| \to \infty} \frac{J(u)}{\Phi(u)} \leq p^+ c_1^{p^-} \sum_{k=1}^{T} f_0(k),
\]

therefore, considering \( c_1 > \sqrt{6} \) and \( L \geq 1 \), we have

\[
    \alpha = \max \left\{ 0, \limsup_{||u|| \to \infty} \frac{J(u)}{\Phi(u)}, \limsup_{||u|| \to 0} \frac{J(u)}{\Phi(u)} \right\}
\]
By (20) and (21), we deduce that \( \alpha < \beta \). All the assumptions of Theorem 2.5 are satisfied, so by applying that theorem, the conclusion follows. Hence, the proof is complete.

**Remark 3.7.** Theorem 1.1 follows from Theorem 3.6 taking into account that \( p(k) = 2 \), \( f_1(t) = f(t) \) and \( w(k - 1) = 1 \) for every \( k \in [1, T + 1] \). \( q(k) = 1 \), \( f_0(k) = \frac{1}{4T(T + 1)} \) for every \( k \in [1, T] \).

Finally, we present an example of Theorem 1.1.

**Example 3.8.** Let \( T = 4 \) be a fixed positive integer. Consider the problem (2) where

\[
f(t) = \begin{cases} 
0, & \text{if } t = 0, \\
\frac{t}{2} \sin(2 \pi \ln |t|) + \cos(2 \pi \ln |t|), & \text{if } 0 < t \leq e^{-2\pi}, \\
\frac{e^{2\pi} \sin(2 \pi \ln |t|) + t}{2}, & \text{if } e^{-2\pi} \leq t \leq 1, \\
\frac{t}{2} \sin(2 \pi \ln |t|) + \cos(2 \pi \ln |t|), & \text{if } 1 \leq t, \end{cases}
\]

as an odd continuous function on \( \mathbb{R} \). Simple calculations show that

\[
F(t) = \begin{cases} 
\frac{t^2}{2} [\sin(2 \pi \ln |t|)], & \text{if } t < e^{-2\pi}, \\
\frac{e^{2\pi} \sin(2 \pi \ln |t|) - 2\pi t \cos(2 \pi \ln |t|)}{1 + (2\pi)^2} + \frac{t^2}{2} + K, & \text{if } e^{-2\pi} \leq t < 1, \\
\frac{t^2}{2} [\sin(2 \pi \ln |t|)] + L, & \text{if } 1 \leq t, \end{cases}
\]

where

\[
K = \frac{2\pi}{1 + (2\pi)^2} - \frac{e^{-4\pi}}{2},
\]

\[
L = K - \frac{2\pi e^{2\pi}}{1 + (2\pi)^2} + \frac{1}{2}.
\]

Thus

\[
\limsup_{|t| \to 0, +\infty} \frac{F(k,t)}{|t|^2} = \frac{1}{2} < 1.
\]

Hence, (A1) holds. Let \( d = e^{-\frac{1}{2}} \in (e^{-2\pi}, 1) \), so one has,

\[
F(d) = \frac{e^{2\pi} - \frac{1}{2} (2\pi)}{1 + (2\pi)^2} + \frac{e^{-1}}{2} + K \approx 50.75439760 > d^2 (2T^2 + 6T + 4) \approx 36.39183958.
\]

Thus, (A2) holds. Therefore, by using Theorem 1.1, there exists \( r > 0 \) such that for every continuous function \( g : [1, T] \times \mathbb{R} \to \mathbb{R} \), there exists \( \delta > 0 \) such that, for each \( \mu \in [0, \delta] \), the problem (2) has at least three solutions whose norms are less than \( r \).

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**References**


