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On topological properties of spaces obtained by the double band matrix

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Abstract: Let \( \lambda \) denote any one of the spaces \( \ell_{\infty} \) and \( \ell_p \) and \( \lambda(\vec{T}) \) be the domain of the band matrix \( \vec{T} \). We study \( \ell_p(\vec{T}) \) for \( 1 \leq p \leq \infty \) and give some inclusions and its topological properties. Also, we define the alpha–, beta– and gamma– duals of the space \( \ell_p(\vec{T}) \). Finally, we give some matrix mappings.

Keywords: Sequence spaces, Matrix transformations, \( \alpha\)–, \( \beta\)– and \( \gamma\)– duals

MSC: 46A45, 40C05, 11B39

1 Introduction

Let \( \omega, \ell_{\infty}, c, c_0 \) and \( \ell_p \) denote the sequence spaces of all real or complex valued, bounded, convergent, null and absolutely \( p\)– summable sequences, respectively, where \( 1 \leq p < \infty \). Besides, let \( bs \) and \( cs \) denote the spaces of all bounded and convergent series, respectively. \( bv_p \) is the space consisting of all sequences \( (x_k) \) such that \( (x_k - x_{k+1}) \in \ell_p \). We assume that \( \frac{1}{p} + \frac{1}{q} = 1 \) for \( p, q > 1 \).

A sequence space \( X \) is called a \( K \)– space if \( p_n : X \to \mathbb{C} \) defined by \( p_n(x) = x_n \) is continuous for all \( n \in \mathbb{N} \). A \( K \)– space \( X \) is called an \( FK \)– space if \( X \) is a complete linear metric space. If \( FK \)– space is normable then it is called a \( BK \)– space.

\( \ell_{\infty}, c \) and \( c_0 \) have the same norm given by

\[ \|x\|_{\ell_{\infty}} = \sup_k |x_k| \]

for all \( k \in \mathbb{N} \) and these spaces are \( BK \)– spaces. \( \ell_p \) is a \( BK \)– space with

\[ \|x\|_{\ell_p} = \sum_k |x_k|^p \]

for \( 0 < p < 1 \) and

\[ \|x\|_{\ell_p} = \left( \sum_k |x_k|^p \right)^{1/p} \]

for \( 1 \leq p < \infty \).

Let \( A = (a_{nk}) \) be an infinite matrix and \( X, Y \) be sequence spaces. If \( Ax = (a_n(x)) \) exists for all \( x \in X \) and lies in \( Y \), then \( A \) defines a matrix mapping from \( X \) into \( Y \), where

\[ A_n(x) = \sum_k a_{nk}x_k \quad (n \in \mathbb{N}) \tag{1} \]

provided the series on the right side of (1) converges for each \( n \in \mathbb{N} \).

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For a sequence space $X$, the matrix domain of an infinite matrix $A$ in $X$ is defined by

$$X_A = \{ x \in \omega : Ax \in X \}.$$  \hfill (2)

Also, (2) is a sequence space.

## 2 The sequence space $\ell_p(\tilde{T})$

We introduce the sequence space $\ell_p(\tilde{T})$ where $1 \leq p \leq \infty$ by using the band matrix $\tilde{T}$. Then, we analyze some topological properties of $\ell_p(\tilde{T})$ and give some inclusion relations related to this space.

Let us define $\ell_p(\tilde{T})$ and $\ell_\infty(\tilde{T})$ as follows:

$$\ell_p(\tilde{T}) = \left\{ x = (x_n) \in \omega : \sum_n |rt_n x_n + s \frac{1}{t_n} x_{n-1}|^p < \infty \right\} \quad (1 \leq p < \infty),$$

$$\ell_\infty(\tilde{T}) = \left\{ x = (x_n) \in \omega : \sup_n |rt_n x_n + s \frac{1}{t_n} x_{n-1}| < \infty \right\},$$

where $\tilde{T} = (t_{nk})$ is in $\ell_p$ and $\ell_\infty$, respectively, also, is the double band matrix defined as follows:

$$t_{nk} = \begin{cases} rt_n, & k = n \\ \frac{s}{t_n}, & k = n - 1 \\ 0, & k > n \text{ or } 0 \leq k < n - 1 \end{cases}$$

for all $k, n \in \mathbb{N}$ where $r, s \in \mathbb{R} \setminus \{0\}$ and $t = (t_n) \in c \setminus c_{00}$ with $t_n > 0$ for all $n \in \mathbb{N}$. If we rewrite $\ell_p(\tilde{T})$ and $\ell_\infty(\tilde{T})$ by using (2), we have

$$\ell_p(\tilde{T}) = (\ell_p)_{\tilde{T}}, \quad (1 \leq p < \infty) \quad \text{and} \quad \ell_\infty(\tilde{T}) = (\ell_\infty)_{\tilde{T}}.$$  \hfill (3)

We have the inverse matrix $\tilde{T}^{-1} = (t_{nk}^{-1})$ as follows:

$$t_{nk}^{-1} = \begin{cases} \frac{1}{t} (-\frac{s}{t})^{n-k} t_k \prod_{j=k}^{n} \frac{1}{t_j} & 0 \leq k \leq n \\ 0, & k > n \end{cases}$$

for all $k, n \in \mathbb{N}$.

Additionally, we will frequently use $y = (y_n)$ by the $\tilde{T}$-transform of a sequence $x = (x_n)$, i.e.,

$$y_n = \hat{T}_n(x) = \begin{cases} r t_0 x_0, & n = 0 \\ r t_n x_n + s \frac{1}{t_n} x_{n-1}, & n \geq 1 \end{cases}$$

for all $n \in \mathbb{N}$.

We can obtain the following statements according to the special cases of $t = (t_n)$, $r, s$ :

(i) If $t_n = 1$ for all $n \in \mathbb{N}$, $r = 1$ and $s = -1$, then $\ell_p(\tilde{T}) = 1_{v_p}$.

(ii) If $r = 1$ and $s = -1$, then $\ell_p(\tilde{T}) = \ell_p(T)$ and $\ell_\infty(\tilde{T}) = \ell_\infty(T)$. (see [1])

(iii) If $t_n = \frac{f_{n+1}}{f_n}$ for all $n \in \mathbb{N}$, then $\ell_p(\tilde{T}) = \ell_p(F(r, s))$ and $\ell_\infty(\tilde{T}) = \ell_\infty(F(r, s))$. (see [2])

(iv) If $t_n = \frac{f_{n+1}}{f_n}$ for all $n \in \mathbb{N}$, $r = 1$ and $s = -1$ then $\ell_p(\tilde{T}) = \ell_p(F)$ and $\ell_\infty(\tilde{T}) = \ell_\infty(F)$. (see [3])

(v) If $t_n = e$ for all $n \in \mathbb{N}$, $r = 1$ and $s = -1$, then $\tilde{T}$ is the difference matrix $\Delta$. (see [4])

(vi) If $t_n = 1$ for all $n \in \mathbb{N}$, then $\ell_p(\tilde{T}) = \ell_p$ and $\ell_\infty(\tilde{T}) = \ell_\infty$. (see [5])

By using a new double band matrix, many sequence spaces have recently been defined by several authors, see for instance [6-9].

We now may begin the following theorem.
Theorem 2.1. \( \ell_p(\hat{T}) \) is a Banach space with the norm \( \|x\|_{\ell_p(\hat{T})} = \|\hat{T}x\|_{\ell_p} \), i.e.,

\[
\|x\|_{\ell_p(\hat{T})} = \left( \sum_n \left( \frac{\hat{T}_n(x)}{r} \right)^p \right)^{1/p}, \quad 1 \leq p < \infty
\]

\[
\sup_n |\hat{T}_n(x)|, \quad p = \infty.
\]

Theorem 2.2. It is obvious that \( \ell_p(\hat{T}) \) is a BK-space for \( 1 \leq p \leq \infty \).

Proof. The proof is easy. \( \square \)

Theorem 2.3. The sequence space \( \ell_p(\hat{T}) \) is linearly isomorphic to the space \( \ell_p \), i.e., \( \ell_p(\hat{T}) \cong \ell_p \) for \( 1 \leq p \leq \infty \).

Proof. By using (3), we define transformation \( \hat{T} : \ell_p(\hat{T}) \to \ell_p \). The linearity and injectivity of \( \hat{T} \) is clear.

Let \( y = (y_n) \in \ell_p \) and define the sequence \( x = (x_n) \) as follows:

\[ x_n = \frac{1}{r} \sum_{k=0}^n \left( \frac{-s}{r} \right)^{-n-k} \left( \prod_{j=k}^n \frac{1}{t_j^2} \right) t_k y_k \quad (n, k \in \mathbb{N}) \]  

(4)

Then, by using (3) and (4), we obtain

\[
\hat{T}_n(x) = r \frac{1}{n} x_n + \frac{1}{l_n} x_{n-1}
\]

\[
= r \frac{1}{n} \sum_{k=0}^n \left( \frac{-s}{r} \right)^{-n-k} \left( \prod_{j=k}^n \frac{1}{t_j^2} \right) t_k y_k + \frac{1}{l_n} \sum_{k=0}^{n-1} \left( \frac{-s}{r} \right)^{-n-k} \left( \prod_{j=k}^{n-1} \frac{1}{t_j^2} \right) t_k y_k
\]

\[
= l_n \left[ \frac{1}{n} x_n + \sum_{k=0}^{n-1} \left( \frac{-s}{r} \right)^{-n-k} \left( \prod_{j=k}^{n-1} \frac{1}{t_j^2} \right) t_k y_k \right] - \frac{1}{l_n} \sum_{k=0}^{n-1} \left( \frac{-s}{r} \right)^{-n-k} \left( \prod_{j=k}^{n-1} \frac{1}{t_j^2} \right) t_k y_k
\]

\[
= y_n
\]

for all \( n \in \mathbb{N} \). This shows that \( \hat{T}x = y \). Since \( y \in \ell_p \), we obtain \( \hat{T}x \in \ell_p \) and hence \( x \in \ell_p(\hat{T}) \). Thus \( \hat{T} \) is surjective.

Moreover for any \( x \in \ell_p(\hat{T}) \), we have

\[
\|y\|_{\ell_p} = \|\hat{T}x\|_{\ell_p} = \|x\|_{\ell_p(\hat{T})}
\]

which means that \( \hat{T} \) preserves the norm in the case of \( 1 \leq p \leq \infty \). Hence \( \hat{T} \) is an isometry. Consequently, \( \hat{T} \) is a linear bijection which shows that the spaces \( \ell_p(\hat{T}) \) and \( \ell_p \) are linearly isomorphic. \( \square \)

Lemma 2.4 ([10]). A product \( \prod_n (1 + a_n) \) with positive terms \( a_n \) is convergent if and only if the series \( \sum_n a_n \) converges.

Theorem 2.5. The inclusion \( \ell_p(\hat{T}) \subset \ell_q(\hat{T}) \) is strictly satisfied for \( 1 \leq p < q < \infty \).

Proof. Let \( 1 \leq p < q < \infty \). Then, it follows by the inclusion \( \ell_p \subset \ell_q \) that the inclusion \( \ell_p(\hat{T}) \subset \ell_q(\hat{T}) \) holds. Further, because of the inclusion \( \ell_p \subset \ell_q \) is strict, there is a sequence \( x \in \ell_q \setminus \ell_p \). Let \( y = (y_n) \) be as follows:

\[ y_n = \frac{1}{r} \sum_{k=0}^n \left( \frac{-s}{r} \right)^{-n-k} \left( \prod_{j=k}^n \frac{1}{t_j^2} \right) t_k x_k \quad (n \in \mathbb{N}). \]

Thus, we get

\[ \hat{T}_n(y) = x_n \]

for all \( n \in \mathbb{N} \) which means that \( \hat{T}y = x \) and since \( x \in \ell_q \setminus \ell_p \), we have \( \hat{T}y \in \ell_q \setminus \ell_p \). This implies that the inclusion \( \ell_p(\hat{T}) \subset \ell_q(\hat{T}) \) is strict. \( \square \)
Theorem 2.6. The inclusion $\ell_p(\tilde{T}) \subset \ell_\infty(\tilde{T})$ is strictly satisfied for $1 \leq p < \infty$.

Proof. Let $x \in \ell_p(\tilde{T})$, then $\tilde{T} x \in \ell_p$. Since $\ell_p \subset \ell_\infty$, $\tilde{T} x \in \ell_\infty$. Hence $x \in \ell_\infty(\tilde{T})$ which means that $\ell_p(\tilde{T}) \subset \ell_\infty(\tilde{T})$. If we define $y = (y_n)$ by

$$y_n = \sum_{k=0}^{n} (-1)^k \left( \frac{s}{n} \right)^{n-k} t_k \left( \prod_{i=k}^{n} \frac{1}{t_i^2} \right) \quad (n \in \mathbb{N})$$

Hence, we obtain for every $n \in \mathbb{N}$ that

$$
\tilde{T}_n(y) = r_n y_n + s \frac{1}{t_n} y_{n-1} = (-1)^n
$$

which means that $\tilde{T} y \in \ell_\infty \setminus \ell_p$ and hence $y \in \ell_\infty(\tilde{T}) \setminus \ell_p(\tilde{T})$. Hence, the inclusion $\ell_p(\tilde{T}) \subset \ell_\infty(\tilde{T})$ is strict. This concludes the proof.

Lemma 2.7. The inclusion $\ell_p \subset \ell_p(\tilde{T})$ is strictly satisfied for $1 \leq p \leq \infty$.

Proof. The proof is similar to [1].

Lemma 2.8. Neither of the spaces $\ell_\infty$ and $\ell_p(\tilde{T})$ includes the other one, where $1 \leq p < \infty$.

Proof. The proof is similar to [1].

3 The $\alpha$, $\beta$- and $\gamma$- duals of the space $\ell_p(\tilde{T})$ and $\ell_\infty(\tilde{T})$

The $\alpha$, $\beta$- and $\gamma$- duals of sequence space $\lambda$ are introduced as follows:

$$
\lambda^\alpha = \{ a = (a_k) \in \omega : a x = (a_k x_k) \in \ell_1 \text{ for all } x = (x_k) \in \lambda \},
$$

$$
\lambda^\beta = \{ a = (a_k) \in \omega : a x = (a_k x_k) \in cs \text{ for all } x = (x_k) \in \lambda \},
$$

$$
\lambda^\gamma = \{ a = (a_k) \in \omega : a x = (a_k x_k) \in bs \text{ for all } x = (x_k) \in \lambda \}.
$$

In present section, we give the $\alpha$, $\beta$- and $\gamma$- duals of the space $\ell_p(\tilde{T})$, where $1 \leq p \leq \infty$.

Throughout the present section, let $F$ be the collection of all nonempty and finite subsets of $\mathbb{N}$.

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell_p$</td>
<td>1. 2. 3.</td>
</tr>
<tr>
<td>$\ell_1$</td>
<td>4. 5. 6.</td>
</tr>
<tr>
<td>$\ell_\infty$</td>
<td>7. 8. 9.</td>
</tr>
</tbody>
</table>

Table 1. The characterization of the class $(\lambda_1 : \lambda_2)$ with $\lambda_1 \in \{ \ell_p, \ell_1, \ell_\infty \}$ and $\lambda_2 \in \{ \ell_\infty, c, \ell_1 \}$

The following known results from [11, 12] are fundamental for our investigation.

$$
\sup_n \sum_k |a_{nk}|^q < \infty. \quad (6)
$$

$$
\lim_{n \to \infty} a_{nk} \text{ exists for all } k \in \mathbb{N}. \quad (7)
$$

$$
\sup_{K \in F} \sum_k \left( \sum_{n \in K} |a_{nk}|^q \right) < \infty. \quad (8)
$$
Lemma 3.1. The necessary and sufficient conditions for $A \in (\lambda : \mu)$ when $\lambda \in \{\ell_p, \ell_1, \ell_\infty\}$ and $\mu \in \{\ell_\infty, c, \ell_1\}$ can be read from Table 1, where 1. (6), 2. (6) and (7), 3. (8), 4. (9), 5. (7) and (9), 6. (10), 7. (6) with $q = 1$. (7) and (11). 9. (8) with $q = 1$.

Theorem 3.2. Define the sets $\hat{d}_1(r, s), \hat{d}_2(r, s), \hat{d}_3(r, s), \hat{d}_4(r, s), \hat{d}_5(r, s)$ and $\hat{d}_6(r, s)$ as follows:

\[
\hat{d}_1(r, s) = \left\{ a = (a_k) \in \omega : \sup_{K \in F} \sum_{k \in K} \left| \sum_{n \in K} \frac{1}{r} \left( -x \right)^{n-k} \prod_{j=k}^{n} \frac{1}{t_j} a_n \right|^q < \infty \right\},
\]

\[
\hat{d}_2(r, s) = \left\{ a = (a_k) \in \omega : \sup_{n \in K} \sum_{k \in K} \left| \sum_{i=k}^{n} \frac{1}{r} \left( -x \right)^{i-k} \prod_{j=k}^{i} \frac{1}{t_j} a_i \right|^q < \infty \right\},
\]

\[
\hat{d}_3(r, s) = \left\{ a = (a_k) \in \omega : \lim_{n \to \infty} \sum_{i=k}^{n} \frac{1}{r} \left( -x \right)^{i-k} \prod_{j=k}^{i} \frac{1}{t_j} a_i \text{ exists for all } k \in \mathbb{N} \right\},
\]

\[
\hat{d}_4(r, s) = \left\{ a = (a_k) \in \omega : \lim_{n \to \infty} \sum_{i=k}^{n} \frac{1}{r} \left( -x \right)^{i-k} \prod_{j=k}^{i} \frac{1}{t_j} a_i \in \mathbb{R} \right\},
\]

\[
\hat{d}_5(r, s) = \left\{ a = (a_k) \in \omega : \sum_{k} \left| \sum_{i=k}^{n} \frac{1}{r} \left( -x \right)^{i-k} \prod_{j=k}^{i} \frac{1}{t_j} a_i \right|^q < \infty \right\},
\]

\[
\hat{d}_6(r, s) = \left\{ a = (a_k) \in \omega : \lim_{n \to \infty} \sum_{i=k}^{n} \frac{1}{r} \left( -x \right)^{i-k} \prod_{j=k}^{i} \frac{1}{t_j} a_i < \infty \right\}.
\]

Let $1 < p < \infty$. Then,

(i) $\left( \ell_p(\hat{T}) \right)^\alpha = \hat{d}_1(r, s)$, $\left( \ell_1(\hat{T}) \right)^\alpha = \hat{d}_2(r, s)$ and $\left( \ell_\infty(\hat{T}) \right)^\alpha = \hat{d}_1(r, s)$ with $q = 1$.

(ii) $\left( \ell_p(\hat{T}) \right)^\beta = \hat{d}_3(r, s) \cap \hat{d}_4(r, s)$, $\left( \ell_1(\hat{T}) \right)^\beta = \hat{d}_5(r, s) \cap \hat{d}_6(r, s)$ and $\left( \ell_\infty(\hat{T}) \right)^\beta = \hat{d}_5(r, s) \cap \hat{d}_6(r, s)$.

(iii) $\left( \ell_p(\hat{T}) \right)^\gamma = \hat{d}_3(r, s)$ and $\left( \ell_\infty(\hat{T}) \right)^\gamma = \hat{d}_3(r, s)$ with $q = 1$.

Proof. Let us prove it only for the space $\ell_p(\hat{T})$, where $1 < p < \infty$. Let us take any $a = (a_n) \in \omega$. We have with the notation (4) that

\[
a_n x_n = a_n \sum_{k=0}^{n} \left( -x \right)^{n-k} \prod_{j=k}^{n} \frac{1}{t_j} y_k \\
= \sum_{k=0}^{n} \left[ \prod_{j=k}^{n} \frac{1}{t_j} \right] y_k \\
= \sum_{k=0}^{n} b_{nk}(r, s) y_k
\]
On topological properties of spaces obtained by the double band matrix

By $(b_{nk}(r,s))$ is defined by

$$b_{nk}(r,s) = \begin{cases} \frac{1}{\tau} \left( \frac{-s}{r} \right)^{n-k} \prod_{j=k}^{n} \frac{1}{r} t_j a_n, & 0 \leq k \leq n \\ 0, & k > n \end{cases} \quad (k,n \in \mathbb{N}).$$

Hence, we have that $ax = (a_{nk}x_n) \in \ell_1$ whenever $x = (x_n) \in \ell_p(\tilde{T})$ iff $By \in \ell_1$ whenever $y = (y_n) \in \ell_p$. From the Table 1, we have the desired result that

$$\left( \ell_p(\tilde{T}) \right)^{\alpha} = \hat{d}_1(r,s).$$

Now, we obtain

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} a_k \left\{ \sum_{i=0}^{k} \left[ \frac{1}{r} \left( \frac{-s}{r} \right)^{i-k} \prod_{j=i}^{k} \frac{1}{r} t_j y_i \right] \right\}$$

$$= \sum_{k=0}^{n} \sum_{i=k}^{n} \frac{1}{r} \left( \frac{-s}{r} \right)^{i-k} \prod_{j=k}^{i} \frac{1}{r} t_j a_i y_k$$

$$= \sum_{k=0}^{n} \sum_{i=k}^{n} c_{ik}(r,s) a_i y_k$$

$$= \sum_{k=0}^{n} d_{nk}(r,s) y_k$$

$$= (Dy)_n \quad (n \in \mathbb{N})$$

$D = (d_{nk}(r,s))$ defined by

$$d_{nk}(r,s) = \begin{cases} \sum_{i=k}^{n} c_{ik}(r,s) a_i, & 0 \leq k \leq n \\ 0, & k > n \end{cases}$$

where

$$c_{ik}(r,s) = \begin{cases} \frac{1}{\tau} \left( \frac{-s}{r} \right)^{i-k} \prod_{j=k}^{i} \frac{1}{r} t_j, & 0 \leq k \leq n \\ 0, & k > n \end{cases}$$

for all $k,n \in \mathbb{N}$. Therefore, we have that $ax = (a_{nk}x_n) \in \ell_1$ whenever $x = (x_n) \in \ell_p(\tilde{T})$ iff $By \in \ell_1$ whenever $y = (y_n) \in \ell_p$. From the Table 1, we have that

$$\left( \ell_p(\tilde{T}) \right)^{\beta} = \hat{d}_3(r,s) \cap \hat{d}_4(r,s).$$

As this, we deduce by (12) that $ax = (a_{nk}x_n) \in \ell_1$ whenever $x = (x_n) \in \ell_p(\tilde{T})$ iff $By \in \ell_1$ whenever $y = (y_n) \in \ell_p$. Thus, we obtain from the Table 1 that

$$\left( \ell_p(\tilde{T}) \right)^{\gamma} = \hat{d}_3(r,s).$$

This concludes the proof.

4 Certain matrix mappings on the space $\ell_p(\tilde{T})$

In the present section, we analyze the matrix mappings from the space $\ell_p(\tilde{T})$ to the spaces $\ell_\infty$, $c$, $c_0$ and $\ell_1$, where $1 \leq p \leq \infty$. 

Theorem 4.1. Let $\lambda$ be an arbitrary subset of $\omega$ and $1 \leq p \leq \infty$. Then, we have $A = (a_{nk}) \in (\ell_p(\hat{\mathcal{T}}), \lambda)$ if and only if

$$E^{(m)} = \left( e^{(m)}_{nk} \right) \in (\ell_p, c) \text{ for all } n \in \mathbb{N},$$

$$E = (e_{nk}) \in (\ell_p, \lambda),$$

where

$$e^{(m)}_{nk} = \left\{ \begin{array}{ll}
\sum_{i=k}^{m} \frac{1}{r} \left( \frac{s}{r} \right)^{i-k} \prod_{j=k}^{i} \frac{1}{r_j} t_j a_{ni}, & 0 \leq k \leq m \\
0, & k > m
\end{array} \right.$$

and

$$e_{nk} = \sum_{i=k}^{\infty} \frac{1}{r} \left( \frac{s}{r} \right)^{i-k} \prod_{j=k}^{i} \frac{1}{r_j} t_j a_{ni}$$

for every $k, m, n \in \mathbb{N}$.

Proof. Suppose that $A = (a_{nk}) \in (\ell_p(\hat{\mathcal{T}}), \lambda)$ and $x = (x_k) \in \ell_p(\hat{\mathcal{T}})$. Then, we have from (4)

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m} a_{nk} \sum_{i=0}^{k} \frac{1}{r} \left( \frac{s}{r} \right)^{k-i} \prod_{j=i}^{k} \frac{1}{r_j} t_j y_i$$

$$= \sum_{k=0}^{m} \sum_{i=k}^{m} \frac{1}{r} \left( \frac{s}{r} \right)^{i-k} \prod_{j=k}^{i} \frac{1}{r_j} t_j a_{ni} y_k$$

$$= \sum_{k=0}^{m} e^{(m)}_{nk} y_k$$

$$= E^{(m)}(y)$$

for every $m, n \in \mathbb{N}$. Since $Ax$ exists and $Ax \in c$, $E^{(m)} \in (\ell_p, c)$. Letting $m \to \infty$ in the last equality, we have $Ax = Ey$. Then, we have that $E \in (\ell_p, \lambda)$.

Conversely, assume that the conditions hold and take an arbitrary $x \in \ell_p(\hat{\mathcal{T}})$. Then, we obtain that $(e_{nk})_{k \in \mathbb{N}} \in \ell_p^\beta$ from our assumption and Theorem 3.2. Since $A_n = (a_{nk})_{k \in \mathbb{N}} \in \left( \ell_p(\hat{\mathcal{T}}) \right)^\beta$ for every $n \in \mathbb{N}$, $Ax$ exists. Then, we have from the above equality as $m \to \infty$ that $Ax = Ey$ and hence $A \in \left( \ell_p(\hat{\mathcal{T}}), \lambda \right)$.

Now, we list the following conditions:

$$\sup_{n} \sum_{k=0}^{n} |\tilde{a}_{nk}|^q < \infty \quad (13)$$

$$\lim_{n \to \infty} \tilde{a}_{nk} \text{ exists for all } k \in \mathbb{N} \quad (14)$$

$$\sup_{n, k} |\tilde{a}_{nk}| < \infty \quad (15)$$

$$\lim_{n \to \infty} \sum_{k=0}^{n} |\tilde{a}_{nk}| = \sum_{k=0}^{n} \left| \lim_{n \to \infty} \tilde{a}_{nk} \right| < \infty \quad (16)$$

$$\lim_{n \to \infty} \tilde{a}_{nk} = 0 \text{ for all } k \in \mathbb{N} \quad (17)$$

$$\lim_{n \to \infty} \sum_{k} |\tilde{a}_{nk}| = 0 \quad (18)$$

where

$$\tilde{a}_{nk} = \sum_{i=k}^{\infty} \frac{1}{r} \left( \frac{s}{r} \right)^{i-k} \prod_{j=k}^{i} \frac{1}{r_j} t_j a_{ni}.$$
Theorem 4.2. Let $1 < p < \infty$. Then we have

(i) $A = (a_{nk}) \in \left( \ell_p(\vec{T}), \ell_\infty \right) \Leftrightarrow (13), (14) \text{ and } (6) \text{ hold.}$

(ii) $A = (a_{nk}) \in \left( \ell_p(\vec{T}), c \right) \Leftrightarrow (13), (14), (6) \text{ and } (7) \text{ hold.}$

(iii) $A = (a_{nk}) \in \left( \ell_p(\vec{T}), c_0 \right) \Leftrightarrow (13), (14), (17) \text{ and } (6) \text{ hold.}$

(iv) $A = (a_{nk}) \in \left( \ell_p(\vec{T}), \ell_1 \right) \Leftrightarrow (13), (14) \text{ and } (8) \text{ hold.}$

Theorem 4.3. (i) $A = (a_{nk}) \in \left( \ell_1(\vec{T}), \ell_\infty \right) \Leftrightarrow (14), (15) \text{ and } (9) \text{ hold.}$

(ii) $A = (a_{nk}) \in \left( \ell_1(\vec{T}), c \right) \Leftrightarrow (14), (15), (7) \text{ and } (9) \text{ hold.}$

(iii) $A = (a_{nk}) \in \left( \ell_1(\vec{T}), c_0 \right) \Leftrightarrow (14), (15), (17) \text{ and } (9) \text{ hold.}$

(iv) $A = (a_{nk}) \in \left( \ell_1(\vec{T}), \ell_1 \right) \Leftrightarrow (14), (15) \text{ and } (10) \text{ hold.}$

Theorem 4.4. (i) $A = (a_{nk}) \in \left( \ell_\infty(\vec{T}), \ell_\infty \right) \Leftrightarrow (14), (16) \text{ and } (6) \text{ with } q = 1 \text{ hold.}$

(ii) $A = (a_{nk}) \in \left( \ell_\infty(\vec{T}), c \right) \Leftrightarrow (14), (16), (7) \text{ and } (11) \text{ hold.}$

(iii) $A = (a_{nk}) \in \left( \ell_\infty(\vec{T}), c_0 \right) \Leftrightarrow (14), (16) \text{ and } (18) \text{ hold.}$

(iv) $A = (a_{nk}) \in \left( \ell_\infty(\vec{T}), \ell_1 \right) \Leftrightarrow (14), (16) \text{ and } (8) \text{ with } q = 1 \text{ hold.}$

References


