Hom-Lie superalgebra structures on exceptional simple Lie superalgebras of vector fields

Abstract: According to the classification by Kac, there are eight Cartan series and five exceptional Lie superalgebras in infinite-dimensional simple linearly compact Lie superalgebras of vector fields. In this paper, the Hom-Lie superalgebra structures on the five exceptional Lie superalgebras of vector fields are studied. By making use of the \(\mathbb{Z}\)-grading structures and the transitivity, we prove that there is only the trivial Hom-Lie superalgebra structures on exceptional simple Lie superalgebras. This is achieved by studying the Hom-Lie superalgebra structures only on their 0-th and \((-1)\)-th \(\mathbb{Z}\)-components.

Keywords: Exceptional Lie superalgebra, Hom-Lie superalgebra structure, Gradation, Transitivity

MSC: 17B40, 17B65, 17B66

1 Introduction

The motivations to study Hom-Lie algebras and related algebraic structures come from physics and deformations of Lie algebras, in particular, Lie algebras of vector fields. Recently, this kind of investigation has become rather popular [1–10], partially due to the prospect of having a general framework in which one can produce many types of natural deformations of Lie algebras, in particular, \(q\)-deformations. The notion of Hom-Lie algebras was introduced by J. T. Hartwig, D. Larsson and S. D. Silvestrov in [1] to describe the structures on certain deformations of Witt algebras and Virasoro algebras, which are applied widely in the theoretical physics [11–14]. Later, W. J. Xie and Q. Q. Jin gave a description of Hom-Lie algebra structures on semi-simple Lie algebras [9, 10]. In 2010, F. Ammar and A. Makhlouf generalized Hom-Lie algebras to Hom-Lie superalgebras [15]. In 2013 and 2015, B. T. Cao, L. Luo, J. X. Yuan and W. D. Liu studied the Hom-structures on finite dimensional simple Lie superalgebras [16, 17].

In the well-known paper [18], Kac classified the infinite-dimensional simple linearly compact Lie superalgebras of vector fields, including eight Cartan series and five exceptional simple Lie superalgebras. In 2014, the Hom-Lie superalgebra structures on the eight infinite-dimensional Cartan series were investigated by J. X. Yuan, L. P. Sun and W. D. Liu [19]. In the present paper, we characterize the Hom-Lie superalgebra structures on the five infinite-dimensional exceptional simple Lie superalgebras of vector fields. Taking advantage of the \(\mathbb{Z}\)-grading structures and the transitivity, we analyse the Hom-Lie superalgebra structures through computing the Hom-Lie superalgebra structures on the 0-th and \((-1)\)-th \(\mathbb{Z}\)-components. The main result of this study shows that the Hom-Lie superalgebra structures on the five infinite-dimensional exceptional simple Lie superalgebras of vector fields must be trivial.
2 Preliminaries

Throughout the paper, we will use the following notations. $F$ denotes an algebraically closed field of characteristic zero. $\mathbb{Z}_2 := \{0, 1\}$ is the additive group of two-elements. The symbols $|x|$ and $zd(x)$ denote the $\mathbb{Z}_2$-degree of a $\mathbb{Z}_2$-homogeneous element $x$ and the $\mathbb{Z}$-degree of a $\mathbb{Z}$-homogeneous element $x$, respectively. Notation $\{v_1, \ldots, v_k\}$ is used to represent the linear space spanned by $v_1, \ldots, v_k$ over $F$. Now, let us review the five infinite dimensional exceptional simple Lie superalgebras and their $\mathbb{Z}$-grading structures adopted in this paper. More detailed descriptions can be found in [18, 20].

We construct a Weisfeiler filtration of $L$ by using open subspaces:

$$L = L_{-h} \supset L_{-h+1} \supset \cdots \supset L_0 \supset L_1 \supset \cdots$$

where $L$ is a simple linearly compact Lie superalgebra. Let $g_i = L_i/L_{i+1}$, $Gr L = \bigoplus_{i=-h}^{\infty} g_i$. In [18], all possible choices for nonpositive part $g_{\leq 0} = \bigoplus_{i=-h}^{0} g_i$ of the associated graded Lie superalgebra of $L$ were derived. The five exceptional simple Lie superalgebras are as follows (the subalgebra $g_{\leq 0}$ is written below as the $h + 1$-tuple($g_{-h}, g_{-h+1}, \ldots, g_{-1}, g_0$)):

- $E(5, 10) : (F^{5*}, \wedge^2(F^5), sl(5))$
- $E(3, 6) : (F^{3*}, F^3 \otimes F^2, gl(3) \oplus sl(2))$
- $E(3, 8) : (F^2, F^{3*}, F^3 \otimes F^2, gl(3) \oplus sl(2))$
- $E(1, 6) : (F, F^n, \mathfrak{co}(n), n \geq 1, n \neq 2)$
- $E(4, 4) : (F^{4|4}, \tilde{P}(4))$

As we know, the five exceptional simple Lie superalgebras are transitive and irreducible. In particular, $g_{-i} = g_{-i}^0$ for $i \geq 1$ and the transitivity will be used frequently in this paper:

transitivity : if $x \in g_i, i \geq 0$, then $[x, g_{-1}] = 0$ implies $x = 0$. (1)

To study the Hom-Lie superalgebra structures, we recall the following definitions [16].

A Hom-Lie superalgebra is a triple $(g, \cdot, \sigma)$ consisting of a $\mathbb{Z}_2$-graded vector space $g$, an even bilinear mapping $\cdot : g \times g \to g$ and an even linear mapping: $\sigma : g \to g$ satisfying:

$$\sigma[x, y] = [\sigma(x), \sigma(y)],$$

$$[x, y] = -(1)^{|x||y|}[y, x]$$

$$(-1)^{|x||z|}\sigma(x, [y, z]) + (-1)^{|y||x|}\sigma(y, [z, x]) + (-1)^{|z||y|}\sigma(z, [x, y]) = 0.$$

for all homogeneous elements $x, y, z \in g$.

An even linear mapping $\sigma$ on a Lie superalgebra $g$ is called a Hom-Lie superalgebra structure on $g$ if $(g, \cdot, \sigma)$ is a Hom-Lie superalgebra. In particular, $\sigma$ is called trivial if $\sigma = id|_g$.

It is obvious that $\sigma$ is graded and if $\sigma$ is a Hom-Lie superalgebra structure on a simple Lie superalgebra $g$, then $\sigma$ must be a monomorphism.

Lemma 2.1. Let $g = \bigoplus_{i \geq -h} g_i$ be a transitive Lie superalgebra. If $\sigma$ is a Hom-Lie superalgebra structure on $g$ and $\sigma|_{g_{-1}} = id|_{g_{-1}}$, then $\sigma(x) - x \in g_{-k}$, where $x \in g_0, k \geq 1$.

Proof. For any $y \in g_{-1}$, from $\sigma|_{g_{-1}} = id|_{g_{-1}}$ and (2), one can deduce

$$[\sigma(x), y] = [\sigma(x), \sigma(y)] = \sigma[x, y] = [x, y].$$

Then $[\sigma(x) - x, y] = 0$. So, $[\sigma(x) - x, g_{-1}] = 0$. The transitivity(1) of $g$ and the gradation of $\sigma$ imply that $\sigma(x) - x \in g_{-k}, k \geq 1$. 

Lemma 2.2. Let $g = \bigoplus_{i \geq -h} g_i$ be a transitive and irreducible Lie superalgebra. If $\sigma$ is a Hom-Lie superalgebra structure on $g$, and satisfies
\[ \sigma|_{g_0 \oplus \theta_{-1}} = \text{id}|_{g_0 \oplus \theta_{-1}}, \]
then
\[ \sigma = \text{id}|_{g_0}. \]

Proof. Let $i \geq 1$. Equation (2) and $g_{-i} = g_{-i}'$ imply
\[ \sigma|_{g_{-i}} = \text{id}|_{g_{-i}}. \]  \hspace{1cm} (5)

Suppose $x \in g_i$, $y, z \in g_{\leq 0} = \bigoplus_{i = -h}^0 g_i$. By (4) and (5), one can deduce
\[ [\sigma(x), [y, z]] = [x, [y, z]]. \]

Since $g$ is transitive and irreducible, then $g_{-1} = [g_0, g_{-1}]$. Together with Equation (4), it implies
\[ [\sigma(x) - x, g_{\leq 0}] = 0. \]

Then $\sigma(x) - x = 0$, $\sigma|_{g_0} = \text{id}|_{g_0}$. Thus, we have $\sigma = \text{id}|_{g_0}$, that is, $\sigma$ is trivial. \hfill \Box

According to Lemma 2.2, to study the Hom-Lie superalgebra structures on five exceptional simple Lie superalgebras, we can begin with their 0-th and (−1)-th components. In the remaining of this paper, we will directly use Equation (4) without further remarks.

3 Hom-Lie superalgebra structures on five exceptional simple Lie superalgebras

3.1 $E(4,4)$

From [20], there is a unique irreducible gradation over $E(4,4)$, such that $E(4,4)$ is the only inconsistent gradated algebra (its (−1)-th component is not purely odd) of the five exceptional simple Lie superalgebras. The 0-th component $E(4,4)_0$ is isomorphic to $\tilde{P}(4)$, which is the unique nontrivial central extension of $P(4)$. As $E(4,4)_0$-module, $E(4,4)_{-1}$ is isomorphic to the natural module $\mathbb{P}[4]$. Throughout the rest of the paper, we will use the notations which are introduced in [21] for $E(4,4)$.

Let $C = (c_{ij}) \in g_4^{\mathfrak{g}(\mathbb{F})}$ be a skew-symmetric matrix. $\tilde{C} = (\tilde{c}_{ij}) = (\epsilon_{ijk}c_{kl})$ stands for the Hodge dual of $C$, where $\epsilon_{ijk}$ is the symbol of permutation (1234) $\mapsto (ijk)$. For short, write $i' := i + 4$, $i = 1, 2, 3, 4$. Fix a basis of $\tilde{P}(4) : A \cup B \cup C \cup I$, where
\begin{align*}
A := & \{E_{ij} - E_{j'i'}, \ 1 \leq i \neq j \leq 4\}, \\
B := & \{E_{ij'} + E_{j'i}, \ 1 \leq i \leq j \leq 4\}, \\
C := & \{E_{i'j} - E_{j'i} - (\tilde{E}_{i'j} - \tilde{E}_{j'i}), \ 1 \leq i < j \leq 4\}, \\
I := & \{I, E_{ii} - E_{i+1,i+1} - (E_{i'j'} - E_{i'+1,j'+1}), \ 1 \leq i \leq 3\}. \\
\end{align*}
$I$ is the unit matrix.

Suppose $E(4,4)_{-1} = \{v_i, v_{i'} | i = 1, \ldots, 4\}$, where $|v_i| = \tilde{0}$, $|v_{i'}| = \tilde{1}$.

Proposition 3.1. If $\sigma$ is a Hom-Lie superalgebra structure on $E(4,4)$, then
\[ \sigma|_{E(4,4)_{-1}} = \text{id}|_{E(4,4)_{-1}}. \]
Comparing (9) with (10), one gets
\[
[\sigma(v_i), v_j] = [\sigma(v_i), [E_{kj} - E_{j'k'}, v_j]] = 0. \tag{6}
\]

Similarly, one has
\[
[\sigma(v_i), v_{k'}] = -[\sigma(v_i), [E_{jk} - E_{k'j'}, v_{j'}]] = 0 \tag{7}
\]
and
\[
[\sigma(v_k), v_{k'}] = -[\sigma(v_k), [E_{k'j'} - E_{j'k'}, v_{j'}]] = -[\sigma(v_j), v_{j'}].
\]
The arbitrariness of \(k, j\) in the last equation shows
\[
[\sigma(v_i), v_{j'}] = 0, \quad i = 1, \ldots, 4. \tag{8}
\]

It follows from Equations (6)-(8) that
\[
[\sigma(E(4, 4)_{-1}), E(4, 4)_{-1}] = 0.
\]

Using the transitivity (1), one gets \(\sigma(E(4, 4)_{-1}) \leq E(4, 4)_{-1}\). Recall that \(|\sigma| = 0\), one may suppose
\[
\sigma(v_i) = \sum_{m=1}^{8} \lambda_m v_m, \lambda_m, \in \mathbb{F}.
\]

Now, for distinct \(i, j, k, l = 1, 2, 3, 4\), suppose \((1234) \mapsto (ijkl)\) is an even permutation and
\[
c = E_{k'j'} - E_{j'k} - (E_{ll'} - E_{l'l}) \in C, \quad a = E_{jj} - E_{j'j'} \in A.
\]

Obviously, \([a, v_i] = 0\), \([v_j, c] = 0\). Then
\[
0 = [\sigma(v_i), [c, a]] = [\sigma(v_i), E_{k'l'} - E_{l'k} - (E_{ll'} - E_{l'l})] = -\lambda_i v_{k'} + \lambda_k v_{l'}.
\]

Hence, \(\lambda_k = \lambda_i = 0\), and then \(\sigma(v_i) = \lambda_i v_i, i = 1, 2, 3, 4\).

The equation
\[
0 = [\sigma(v_{j'}), [E_{kj} - E_{j'k'}, E_{ll'} + E_{l'l}]] = [\sigma(v_{j'}), E_{l'l'} + E_{k'k}] = \lambda_{l'} v_k + \lambda_k v_l
\]
is implied \(\lambda_{k'} = \lambda_{l'} = 0\). One has \(\sigma(v_{j'}) = \lambda_{l'} v_{l'}\).

At last, let us prove \(\sigma(v_j) = v_j\) and \(\sigma(v_{l'}) = v_{l'}\). For distinct \(i, j, k\), put
\[
h = E_{jj} - E_{j+1,j+1} - (E_{kk} - E_{k+1,k+1}) \in T, \quad a = E_{ji} - E_{j'j'} \in A.
\]

Clearly,
\[
[\sigma(h), v_j] = [\sigma(h), [a, v_i]] = -[\sigma(v_i), a] = \lambda_i v_j. \tag{9}
\]

Recall that \(\sigma\) is monomorphic. Let \(\sigma^{-1}\) denote a left inverse of \(\sigma\). Then
\[
\sigma^{-1}[\sigma(h), v_j] = [h, \sigma^{-1}(v_j)] = \lambda^{-1}_j [h, v_j] = \lambda^{-1}_j v_j.
\]

It implies that
\[
[\sigma(h), v_j] = \sigma(\lambda^{-1}_j v_j) = v_j. \tag{10}
\]

Comparing (9) with (10), one gets \(\lambda_i = 1\). Analogously, one gets \(\sigma(v_{l'}) = v_{l'}\).

Summing up the above, we have proved \(\sigma|_{E(4,4)_{-1}} = \text{id}|_{E(4,4)_{-1}}\).

\[\square\]

**Proposition 3.2.** If \(\sigma\) is a Hom-Lie superalgebra structure on \(E(4, 4)\), then
\[
\sigma|_{E(4,4)_0} = \text{id}|_{E(4,4)_0}.
\]
Proof. Put \( x \in E(4,4)_0, \) \( v \in E(4,4)_{-1}. \) Using Propposition 3.1 and Lemma 2.1, one can deduce \( \sigma(x) - x \in E(4,4)_{-1}. \) Note that \( \mathcal{A} \) and \( \mathcal{I} \) are generated (Lie product) by \( \mathcal{B} \) and \( \mathcal{C}, \) it is sufficient to prove the cases for \( x \in \mathcal{B} \) and \( x \in \mathcal{C}. \)

Case 1: Let \( x = E_{ij'} + E_{ji'} \in \mathcal{B}. \) Noting that \( |\sigma| = 0, \) one may suppose

\[
\sigma(x) = x + \sum_{m=1}^{4} \lambda_{m'} v_{m'}, \quad \lambda_{m'} \in \mathbb{F}.
\]

Put

\[
c = E_{k'|l} - E_{l'|k} - (E_{k'|l} - E_{l'|k}) \in \mathcal{C}, \quad b = E_{k'k} \in \mathcal{B},
\]

where \( k, l \neq i, j. \) Then

\[
0 = [\sigma(x), [b, c]] = \left[ E_{ij'} + E_{ji'} + \sum_{m=1}^{4} \lambda_{m'} v_{m'}, E_{k'l} - E_{l'k'} \right] = \lambda_{k'} v_{l'}.
\]

It implies that \( \lambda_{k'} = 0, k \neq i, j. \) Then,

\[
\sigma(x) = x + \lambda_{i'} v_{i'} + \lambda_{j'} v_{j'}.
\]

(11)

Now, put

\[
c = E_{i'k} - E_{k'l} - (E_{i'k} - E_{k'l}) \in \mathcal{C}, \quad b = E_{ij} \in \mathcal{B},
\]

Using Equation (11), one may suppose \( \sigma(b) = b + \mu_{i'} v_{i'}, \mu_{i'} \in \mathbb{F}, \) then

\[
0 = [\sigma(x), a] = [E_{ij'} + E_{ji'} + \lambda_{i'} v_{i'} + \lambda_{j'} v_{j'}, E_{ik} - E_{k'j}] = \lambda_{i'} v_{k'}.
\]

So \( \lambda_{i'} = 0. \) That is, \( \sigma(x) = x + \lambda_{j'} v_{j'} \). Similarly, put

\[
c = E_{j'k} - E_{k'i} - (E_{j'k} - E_{k'i}) \in \mathcal{C}, \quad b = E_{jj} \in \mathcal{B}.
\]

One obtains \( \lambda_{j'} = 0. \) Thus, we have \( \sigma(x) = x \) for any \( x \in \mathcal{B}. \)

Case 2: Let \( x = E_{ij} - E_{ji} - (E_{ik} - E_{k'i}) \in \mathcal{C}. \) where \( \epsilon_{ijkl} = 1. \) Noting that \( |\sigma| = 0, \) one may assume \( \sigma(x) = x + \sum_{m=1}^{4} \mu_{m'} v_{m'}, \mu_{m'} \in \mathbb{F}. \)

Firstly, put

\[
c = E_{k'l} - E_{l'k} - (E_{k'l} - E_{l'k}) \in \mathcal{C}, \quad b = E_{kk'} \in \mathcal{B},
\]

where \( \epsilon_{ijkl} = 1. \) Equation (4) and the result \( \sigma(b) = b \) obtained in Case 1 imply

\[
0 = [\sigma(x), [c, b]] = [x + \sum_{m=1}^{4} \mu_{m'} v_{m'}, E_{k'l} - E_{l'k}] = c_0 + \mu_{k'} v_{l'}, \quad c_0 \in \mathbb{C}.
\]

The equation above shows \( \mu_{k'} = 0. \) By the arbitrariness of \( k \neq i, j, \) one has

\[
\sigma(x) = x + \mu_{i'} v_{i'} + \mu_{j'} v_{j'}.
\]

(12)

Secondly, put

\[
c = E_{i'k} - E_{k'i} - (E_{i'k} - E_{k'i}) \in \mathcal{C}, \quad b = E_{jj}.
\]

Using Equation (12), one may suppose

\[
\sigma(c) = c + \gamma_{i'} v_{i'} + \gamma_{j'} v_{j'}, \quad \gamma_{i'}, \gamma_{j'} \in \mathbb{F}.
\]

On one hand,

\[
[\sigma(x), [b, c]] = -[\sigma(b), [c, x]] - [\sigma(c), [x, b]] = E_{i'j} - E_{j'i} - (E_{k'j} - E_{j'k}) - \gamma_{j'} v_{i'}.
\]
On the other hand,

\[ [\sigma(x), [b, c]] = [E_i, E_j] = (E_{k'l'} - E_{k'j'}) + \mu_{i'j'}(E_{k'l'}) + \mu_{i'j'}(E_{k'j'}) = E_{i'j'} - E_{j'i'} \]

Since \( i \neq j \), one has \( \gamma_{j'} = \mu_{j'} = 0 \). Then \( \sigma(x) = x + \lambda_{i'}y_{j'} \).

At last, put

\[ c = E_i - E_{k'} - (E_{k'} - E_{j'}) \in \mathbb{C}, \quad b = E_i \in \mathbb{B}. \]

We can obtain \( \lambda_{i'} = 0 \) in the same way. Thus, \( \sigma(x) = x \) is proved.

We write each element consistent \( \mathbb{E} \) and denote \( \mathbb{E} \). We can obtain the \( \mathbb{E} \)

Before studying the Hom-Lie superalgebra structures on \( E(3,6), E(5,10), E(3,8) \), we would like to review their algebraic structures briefly \([18, 20, 22, 23]\). From \([18]\) we know that the even part \( E(5,10)_0 \) of \( E(5,10) \) is isomorphic to the Lie algebra \( S_5 \), which consists of divergence 0 polynomial vector fields on \( \mathbb{F}^5 \), i.e., polynomial vector fields annihilating the volume form \( \Delta \). As \( S_5 \)-module, the odd part \( E(5,10)_1 \) of \( E(5,10) \) is isomorphic to \( \mathbb{F}^2 \), the space of closed polynomial differential 2-forms on \( \mathbb{F}^5 \). In the following, we keep the notations from \([22]\):

\[ d_{ij} := dx_i \wedge dx_j, \quad \partial_i := \partial_i \partial x_i. \]

We write each element \( D \) of \( E(5,10) \) as

\[ D = \sum_{i=1}^{5} f_i \partial_i, \quad \text{where} \quad f_i \in \mathbb{F}[x_1, x_2, \ldots, x_5], \quad \sum_{i=1}^{5} \partial_i(f_i) = 0, \]

and denote \( E \in E(5,10)_1 \) through

\[ E = \sum_{i,j=1}^{5} f_{ij} d_{ij}, \quad \text{where} \quad f_{ij} \in \mathbb{F}[x_1, x_2, \ldots, x_5], \quad dE = 0. \]  \hspace{1cm} (13)

The bracket in \( E(5,10)_1 \) is defined by

\[ [f d_{ij}, g d_{kl}] = \varepsilon_{ijkl} f g \partial_i, \]

where \( \varepsilon_{ijkl} \) is the sign of permutation \( (ijkl) \) when \( \{ijkl\} = \{12345\} \) and zero otherwise. The bracket of \( E(5,10)_0 \) with \( E(5,10)_1 \) is defined by the usual action of vector fields on differential forms. The irreducible consistent \( \mathbb{Z} \)-gradation over \( E(5,10) \) is defined by letting (see \([22]\))

\[ zd(x_i) = 2, \quad zd(d) = -\frac{5}{2}, \quad zd(dx_i) = -\frac{1}{2}. \]

Then \( E(5,10) = \oplus_{i \geq -2} \mathfrak{g}_i \), where

\[ \mathfrak{g}_0 \simeq \mathfrak{sl}(5) = \{ x_i \partial_j \}, \quad x_k \partial_k - x_{k+1} \partial_{k+1} \mid i, j = 1, \ldots, 5, i \neq j, k = 1, \ldots, 4 \}, \]

\[ \mathfrak{g}_{-1} \simeq \mathbb{F}^5 = \{ d_{ij} \mid 1 \leq i < j \leq 5 \}, \]

\[ \mathfrak{g}_{-2} \simeq \mathbb{F}^5 \] of \( \mathbb{F}^5 \) is

The exceptional simple Lie superalgebra \( E(3,6) \) is a subalgebra of \( E(5,10) \). The irreducible consistent \( \mathbb{Z} \)-gradation over \( E(3,6) \) is induced by the \( \mathbb{Z} \)-gradation of \( E(5,10) \) above. Let \( h_1 = x_1 \partial_1 - x_2 \partial_2, h_2 = x_2 \partial_2 - x_3 \partial_3, h_3 = x_4 \partial_4 - x_5 \partial_5, h_4 = -x_2 \partial_2 - x_3 \partial_3 + 2x_5 \partial_5 \). Then \( E(3,6) = \oplus_{i \geq -2} \mathfrak{g}_i \), where

\[ \mathfrak{g}_0 \simeq \mathfrak{gl}(3) \oplus \mathfrak{sl}(2) \]
Therefore, by Lemma 3.3
\[ g_{-1} \simeq \mathbb{F}^3 \otimes \mathbb{F}^2 = \langle d_{ij} \mid i = 1, 2, 3, j = 4, 5 \rangle, \]
\[ g_{-2} \simeq \mathbb{F}^3^* = \langle \partial_i \mid i = 1, 2, 3 \rangle. \]

The exceptional simple Lie superalgebra \( E(3, 8) \), which is strikingly similar to \( E(3, 6) \), carries a unique irreducible consistent \( \mathbb{Z} \)-gradation [23] defined by
\[ zd(x_i) = -zd(\partial_i) = zd(dx_i) = 2, \quad i = 1, 2, 3; \quad zd(x_4) = zd(x_5) = -3. \]

Then \( E(3, 8) = \bigoplus_{i \geq -3} g_i \), where
\[ g_j = E(3, 6)_j, \quad j = 0, -1, -2; \quad g_{-3} \simeq \mathbb{F}^2 \simeq \langle dx_4, dx_5 \rangle. \] (14)

Hereafter, \( g \) denotes \( E(5, 10), E(3, 6) \) or \( E(3, 8) \) unless otherwise noted. We establish a technical lemma, which can be verified directly.

**Lemma 3.3.** If \( \sigma \) is a Hom-Lie superalgebra structure on \( g \), then

1. \[ [\sigma(d_{ij}), d_{il}] = \begin{cases} 0, & \text{if } d_{ij}, d_{il} \in E(5, 10), \\ -[\sigma(d_{il}), d_{ij}], & \text{if } d_{ij}, d_{il} \in E(3, 6) \text{ or } E(3, 8); \end{cases} \]
2. \[ [\sigma(d_{ij}), d_{kj}] = 0, \quad \text{in particular, } [\sigma(d_{ij}), d_{ij}] = 0; \]
3. \[ [\sigma(d_{ij}), d_{kl}] = [d_{ij}, \sigma(d_{kl})] \text{ for distinct } i, j, k, l. \]

**Proposition 3.4.** Let \( g \) be Lie superalgebra \( E(5, 10), E(3, 6) \) or \( E(3, 8) \). If \( \sigma \) is a Hom-Lie superalgebra structure on \( g \), then
\[ \sigma|_{g_{-1}} = \text{id}|_{g_{-1}}. \]

**Proof.** First, let us prove \( \sigma|_{g_{-1}} = \lambda \text{id}|_{g_{-1}} \), where \( \lambda \in \mathbb{F} \).

**Case 1 :** \( g = E(5, 10) \). Noting that the gradation over \( E(5, 10) \) is consistent and \( |\sigma| = 0, |d_{ij}| = 1 \), one may suppose
\[ \sigma(d_{ij}) = \sum_{1 \leq m < n \leq 5} f_{mn}d_{mn}, \quad \text{where } f_{mn} \in \mathbb{F}[[x_1, \ldots, x_5]]. \]

By Lemma 3.3 (1), one has
\[ [\sigma(d_{ij}), d_{il}] = \left[ \sum_{1 \leq m < n \leq 5} f_{mn}d_{mn}, d_{il} \right] = \left[ \sum_{m \neq i, l} f_{mn}d_{mn}, d_{il} \right] \]
\[ = \sum_{m \neq i, l} [f_{mn}d_{mn}, d_{il}] = \sum_{m \neq i, l} \epsilon_{qmnil} f_{mn}d_{il} = 0. \]

The arbitrariness of \( l \) shows
\[ f_{mn} = 0, \quad \text{where } m, n \neq i. \]

Similarly, by Lemma 3.3 (2), one gets
\[ f_{mn} = 0, \quad \text{where } m, n \neq j. \]

Thus,
\[ \sigma(d_{ij}) = f_{ij}d_{ij}. \]

In view of \( \sigma(d_{ij}) \in E(5, 10)_1 \simeq d \Omega^4(5) \) and (13), one has
\[ 0 = d(\sigma(d_{ij})) = d(f_{ij}d_{ij}) = \sum_{m=1}^5 \partial_m(f_{ij})dx_m \wedge dx_i \wedge dx_j. \]

Therefore, \( \partial_m(f_{ij}) = 0 \) for \( m \neq i, j \), that is \( f_{ij} \in \mathbb{F}[[x_i, x_j]] \). So one may suppose \( \sigma(d_{kl}) = f_{kl}d_{kl} \), where \( f_{kl} \in \mathbb{F}[[x_k, x_l]] \). Then
\[ [\sigma(d_{ij}), d_{kl}] = [f_{ij}d_{ij}, d_{kl}] = \epsilon_{qijkl} f_{ij}d_{kl}. \]
\[ [\sigma(d_{ki}), d_{ij}] = [f_{ki}d_{ki}, d_{ij}] = \varepsilon_{qkij} f_{ki} \partial_q. \]

For distinct \(i, j, k, l\), Lemma 3.3 (3) implies \(f_{ij} = f_{kl} \in \mathbb{F}\). Noting that there exists \(t\) such that \(t \neq i, j, k, l\), one may obtain \(f_{ki} = f_{ij}\) in the same way. Then \(f_{ij} = \lambda \in \mathbb{F}\) for any \(i, j = 1, 2, 3, 4, 5\) and \(i \neq j\). Thus, we proved \(\sigma(d_{ij}) = \lambda d_{ij}\).

**Case 2:** \(g = E(3, 6)\) or \(E(3, 8)\). Noting that the gradation over \(g\) is consistent and \(|\sigma| = 0\), \(|d_{ij}| = 1\), one may suppose

\[
\sigma(d_{ij}) = \sum_{m,n} f_{mn} d_{mn}, \quad \text{where } m = 1, 2, 3, n = 4, 5, \quad f_{mn} \in \mathbb{F}[[x_1, \ldots, x_5]].
\]

By Lemma 3.3 (2), one can deduce

\[
[\sigma(d_{ij}), d_{kj}] = \left[ \sum_{m,n} f_{mn} d_{mn}, d_{kj} \right] = \left[ \sum_{m \neq k, n \neq j} f_{mn} d_{mn}, d_{kj} \right] = 0.
\]

It implies \(f_{mj} = 0, m \neq k, t \neq j\). By the arbitrariness of \(k\), one may suppose

\[
\sigma(d_{ij}) = \sum_{m=1}^{3} f_{mj} d_{mj}, \quad \sigma(d_{li}) = \sum_{m=1}^{3} f_{ml} d_{ml}, \quad f_{mj}, f_{ml} \in \mathbb{F}[[x_1, \ldots, x_5]].
\]

Noting that \([\sigma(d_{ij}), d_{kj}] = [\sigma(d_{li}), [x_k \partial_i, d_{ij}]] = -[\sigma(d_{ij}), d_{kj}]\), the simple computations show \(f_{mj} = f_{ml}\) for \(m = 1, 2, 3\). In view of \(\sigma(d_{ij}) \in \mathfrak{g}_1 \subseteq \mathcal{D}_{-1}^1(5)\) and (13), one could further deduce

\[
f_{mj} = f_{ml} \in \mathbb{F}[[x_1, x_2, x_3]]. \quad \text{(15)}
\]

For distinct \(i, k, q = 1, 2, 3\),

\[
0 = [\sigma(d_{ij}), [x_i \partial_q, x_q \partial_k]] = [\sigma(d_{ij}), x_i \partial_k] = \left[ \sum_{m=1}^{3} f_{mj} d_{mj}, x_i \partial_k \right]
\]

\[
= -\sum_{m=1}^{3} \left( x_i \partial_k (f_{mj}) d_{mj} + f_{mj} x_i \partial_k (d_{mj}) \right)
\]

\[
= - (x_i \partial_k (f_{ij}) + f_{kj}) d_{ij} - x_i \partial_k (f_{kj}) d_{kj} - x_i \partial_k (f_{qj}) d_{qj}.
\]

Therefore,

\[
f_{kj} = -x_i \partial_k (f_{ij}). \quad \text{(16)}
\]

\[
\partial_k (f_{kj}) = \partial_k (f_{qj}) = 0. \quad \text{(17)}
\]

From (15)-(17) and the arbitrariness of \(k\), one has \(f_{mj} = 0, m \neq i\). Thus, one may suppose

\[
\sigma(d_{ij}) = f_i d_{ij}, \quad \sigma(d_{ki}) = f_k d_{ki}, \quad \text{where } f_i \in \mathbb{F}[[x_i]], f_k \in \mathbb{F}[[x_k]].
\]

Then, for distinct \(i, j, k, l, q\),

\[
[\sigma(d_{ij}), d_{kl}] = [f_i d_{ij}, d_{kl}] = \varepsilon_{qijkl} f_i \partial_q,
\]

\[
[d_{ij}, \sigma(d_{kl})] = [d_{ij}, f_k d_{kl}] = \varepsilon_{qijkl} f_k \partial_q.
\]

By Lemma 3.3 (3), \(f_i = f_k \in \mathbb{F}\). Thus, we have proved \(\sigma(d_{ij}) = \lambda d_{ij}\) for some \(\lambda \in \mathbb{F}\).

Now, let us prove \(\lambda = 1\) for \(\sigma|_{g_{-1}} = \lambda d_{ij}|_{g_{-1}}\). Suppose \(x = x_i \partial_i, y = x_j \partial_j, z = d_{qj}\). Then

\[
[\sigma(x), d_{ij}] = [\sigma(x), [y, z]] = -[\sigma(y), [z, x]] - [\sigma(z), [x, y]] = \lambda d_{ij}.
\]

Noting that \(\sigma\) is a monomorphism, one may suppose \(\sigma^{-1}\) is a left linear inverse of \(\sigma\). Then the equation above implies

\[
\sigma^{-1}[\sigma(x), d_{ij}] = [x, \lambda^{-1} d_{ij}] = \lambda^{-1} d_{ij} = d_{ij}.
\]

Thus, \(\lambda = 1\), that is, \(\sigma(d_{ij}) = d_{ij}\). The proof is complete. \(\square\)
Consequently, suppose Comparing (19) with (20), one gets

\[ \sigma(x_i \partial_j) = x_i \partial_j + \sum_{m=1}^{5} \lambda_m \partial_m, \quad \lambda_m \in \mathbb{F}. \]

For distinct \( i, j, q, k, l, \)

\[ 0 = [\sigma(x_i \partial_j), [x_q \partial_k, x_k \partial_l]] = [x + \sum_{m=1}^{5} \lambda_m \partial_m, x_q \partial_l] = \lambda_q \partial_l. \]

Hence, \( \lambda_q = 0. \) The arbitrariness of \( q \neq i, j \) shows

\[ \sigma(x_i \partial_j) = x_i \partial_j + \lambda_j \partial_j. \]

Similarly, \( 0 = [\sigma(x_i \partial_j), [x_i \partial_l, x_j \partial_k]] = [x + \lambda_i \partial_l + \lambda_j \partial_j, x_i \partial_k] = \lambda_i \partial_k \)

implies \( \lambda_i = 0. \) Therefore,

\[ \sigma(x_i \partial_j) = x_i \partial_j + \lambda_j \partial_j. \quad (18) \]

On one hand,

\[ [\sigma(x_i \partial_j), [x_j \partial_l, x_i \partial_k]] = [x_i \partial_l + \lambda_j \partial_j, x_j \partial_k] = x_i \partial_k + \lambda_j \partial_k. \quad (19) \]

On the other hand, using (4) and (18), one can derive

\[ [\sigma(x_i \partial_j), [x_j \partial_l, x_i \partial_k]] = -[\sigma(x_i \partial_k), x_i \partial_l - x_j \partial_j] = x_i \partial_k. \quad (20) \]

Comparing (19) with (20), one gets \( \lambda_j = 0. \) By (18), we have \( \sigma(x_i \partial_j) = x_i \partial_j. \)

Case 2 : \( \mathfrak{g} = E(3, 6) \) or \( E(3, 8) \). Firstly, let us show \( \sigma(x_i \partial_j) = x_i \partial_j \) for any \( x_i \partial_j \in \mathfrak{g}_0. \) As before, one may suppose

\[ \sigma(x_i \partial_j) = x_i \partial_j + \sum_{m=1}^{3} \lambda_m \partial_m, \quad \lambda_m \in \mathbb{F}. \]

Case 2.1 : Let \( i, j = 1, 2, 3. \) For distinct \( i, j, k, \) by (4),

\[ 0 = [\sigma(x_i \partial_j), [x_k \partial_l, x_i \partial_j]] = [\sigma(x_i \partial_j), x_i \partial_j] = [x_i \partial_j + \sum_{m=1}^{3} \lambda_m \partial_m, x_i \partial_j] = \lambda_i \partial_j. \]

Consequently, \( \lambda_i = 0. \) One may suppose \( \sigma(x_k \partial_j) = x_k \partial_j + \sum_{m \neq k}^{3} \mu_m \partial_m. \) On one hand,

\[ [\sigma(x_i \partial_j), x_k \partial_l] = [x_i \partial_j + \sum_{m \neq k}^{3} \lambda_m \partial_m, x_k \partial_l] = -x_k \partial_j + \lambda_k \partial_l. \]

On the other hand, from (4) one deduces

\[ [\sigma(x_i \partial_j), x_k \partial_l] = [\sigma(x_i \partial_j), [x_k \partial_l, x_j \partial_i]] \]

\[ = -[\sigma(x_k \partial_j), [x_j \partial_i, x_i \partial_j]] - [\sigma(x_j \partial_i), [x_i \partial_j, x_k \partial_l]]. \]
Comparing the two equations above, one gets \( \mu_j = 0 \) and \( \lambda_k = \mu_i \). In view of the arbitrariness of \( i, j, k \), one may assume

\[
\sigma(x_i \partial_j) = x_i \partial_j + \lambda_k \partial_k, \quad \sigma(x_k \partial_j) = x_k \partial_j + \lambda_i \partial_i, \quad i, j, k \text{ are distinct.}
\]

Then

\[
\sigma(x_i \partial_j) = \sigma[x_i \partial_k, x_k \partial_j] = \sigma(x_i \partial_k), \quad \sigma(x_k \partial_j) = x_j \partial_j - \lambda_k \partial_k.
\]

The two equations above imply \( \lambda = 0 \), and then \( \sigma(x_i \partial_j) = x_j \partial_j, \quad i, j = 1, 2, 3 \).

**Case 2.2:** Let \( i, j = 4, 5 \). For distinct \( k, l, q = 1, 2, 3, \)

\[
0 = [\sigma(x_i \partial_j), [x_k \partial_q, x_q \partial_j]] = [\sigma(x_i \partial_j), x_k \partial_j] = \left[ x_i \partial_j + \sum_{m=1}^3 \lambda_m \partial_m, x_k \partial_j \right] = \lambda_k \partial_l.
\]

Therefore, \( \lambda_k = 0 \). The arbitrariness of \( k \) implies \( \sigma(x_i \partial_j) = x_j \partial_j \).

Next, we will prove \( \sigma(h_m) = h_m \) for \( m = 1, 2, 3, 4 \). Similarly, suppose \( \sigma(h_m) = h_m + \sum_{k=1}^3 \lambda_k \partial_k \). Noting that \( h_m \) is an element of the basis of \( g_0 \)'s Cartan subalgebra, one may assume \( [h_m, x_i \partial_j] = \gamma x_i \partial_j, \quad \gamma \in F, \quad i, j = 1, 2, 3 \). By the result obtained in Case 2.1, one can deduce

\[
\sigma[h_m, x_i \partial_j] = \gamma \sigma(x_i \partial_j) = \gamma x_i \partial_j.
\]

On the other hand,

\[
\sigma[h_m, x_i \partial_j] = [\sigma(h_m), \sigma(x_i \partial_j)] = \left[ h_m + \sum_{k=1}^3 \lambda_k \partial_k, x_i \partial_j \right] = \gamma x_i \partial_j + \lambda_i \partial_j.
\]

The two equations above imply that \( \lambda_i = 0 \) for \( i = 1, 2, 3 \). Thus, we have \( \sigma(h_m) = h_m \) for \( m = 1, 2, 3, 4 \).

Summing up, we have proved \( \sigma(x) = x \) for any \( x \in g_0 \), that is, \( \sigma|_{g_0} = \text{id}|_{g_0} \). \( \square \)

### 3.3 E(1,6)

The exceptional simple Lie superalgebra \( E(1, 6) \) is a subalgebra of \( K(1, 6) \). The principal gradation over \( K(1, 6) \) (see [20]) induces an irreducible consistent \( \mathbb{Z} \)-gradation over \( E(1, 6) \). Moreover, \( E(1, 6) \) has the same non-positive \( \mathbb{Z} \)-graded components as \( K(1, 6) \): \( E(1, 6) = K(1, 6), \quad j \leq 0 \). Let \( x \) be even and \( \xi_i, \quad i = 1, \ldots, 6, \) be odd indeterminates. Then

\[
E(1, 6)_0 \simeq \mathfrak{osp}(6) = \{ x. \xi_i \xi_j \mid i, j = 1, \ldots, 6, \ i \neq j \},
\]

\[
E(1, 6)_{-1} \simeq \mathbb{F}^6 = \{ \xi_i \mid i = 1, \ldots, 6 \},
\]

\[
E(1, 6)_{-2} \simeq \mathbb{F}.
\]

For \( f, g \in E(1, 6) \), the bracket product is defined as

\[
[f, g] = \left( 2f - \sum_{i=1}^{6} \xi_i \partial_i(f) \right) \partial_x(g) - (-1)^{|f||g|} \left( 2g - \sum_{i=1}^{6} \xi_i \partial_i(g) \right) \partial_x(f)
\]

\[
+ (-1)^{|f|} \sum_{i=1}^{6} \partial_i(f) \partial_j(g).
\]

**Proposition 3.6.** If \( \sigma \) is a Hom-Lie superalgebra structure on \( E(1, 6) \), then

\[
\sigma|_{E(1,6)_{-1}} = \text{id}|_{E(1,6)_{-1}}.
\]
Proof. For $i, j, k = 1, \ldots, 6$ and $j \neq i, k$, by (4),

$$[\sigma(\xi_i), \xi_k] = [\sigma(\xi_i), [\xi_j \xi_k, \xi_j]] = \delta_{k,i} [\sigma(\xi_j), \xi_j].$$

It implies that $[\sigma(\xi_i), \xi_k] = 0$ for any $k, i = 1, \ldots, 6$. From $|\sigma| = 0$ and the transitivity of $E(1, 6)$, one can deduce $\sigma(E(1, 6)_{-1}) \subseteq E(1, 6)_{-1}$. So one may assume $\sigma(\xi_i) = \sum_{k=1}^{6} \lambda_k \xi_k, \lambda_k \in \mathbb{F}$. By using Equation (4), for distinct $i, j, s, t = 1, \ldots, 6$, one can deduce

$$0 = [\sigma(\xi_i), [\xi_j \xi_k, \xi_j]] = [\sigma(\xi_i), \xi_j \xi_k] = \lambda_j \xi_j \xi_k - \lambda_s \xi_j.$$

Then $\lambda_k = 0$ for any $k \neq i$. That is $\sigma(\xi_i) = \lambda_i \xi_i$. By using Equation (4) again, one can deduce

$$-\lambda_j = [\sigma(\xi_i), [\xi_j \xi_i, \xi_j]] = [\sigma(\xi_i), \xi_j \xi_i] = [\lambda_j \xi_j, \xi_i] = -\lambda_j.$$ 

Moreover, one knows $\sigma(\xi_i) = \lambda_i \xi_i$ for any $i = 1, \ldots, 6$. The remaining work is to prove $\lambda = 1$. For distinct $i, j, t,$

$$[\sigma(\xi_i \xi_j), \xi_t] = [\sigma(\xi_i \xi_j), [\xi_i \xi_j, \xi_t]] = [\lambda_i \xi_i \xi_j, \xi_t] = \lambda_i \xi_i \xi_j \xi_t = \lambda_i \xi_j.$$ (21)

Suppose $\sigma^{-1}$ is a left inverse of the monomorphism $\sigma$. Then

$$\sigma^{-1}([\sigma(\xi_i \xi_j), \xi_t]) = [\xi_i \xi_j, \sigma^{-1}(\xi_t)] = \lambda^{-1} \xi_j.$$ (22)

Hence

$$[\sigma(\xi_i \xi_j), \xi_t] = \sigma(\lambda^{-1} \xi_j) = \xi_t.$$ (23)

Equations (21) and (22) imply $\lambda = 1$. The proof is completed. \qed

Proposition 3.7. If $\sigma$ is a Hom-Lie superalgebra structure on $E(1, 6)$, then

$$\sigma|_{E(1, 6)_0} = \text{id}|_{E(1, 6)_0}.$$

Proof. For any $y \in E(1, 6)_0$, by Proposition 3.6 and Lemma 2.1, $\sigma(y) - y \in E(1, 6)_{-2} \simeq \mathbb{F}$. So, one may suppose $\sigma(y) = y + \lambda, \lambda \in \mathbb{F}$. Then

$$[\sigma(y), x] = [y + \lambda, x] = [\lambda, x] = 2\lambda.$$ (23)

If $y = x$, noting that $x$ is an element of a basis of $E(1, 6)_0$'s Cartan subalgebra, one may assume

$$[\sigma(x), x] = \mu \sigma(x) = \mu (x + \lambda), \quad \mu \in \mathbb{F}.$$ (24)

Comparing (23) and (24), one has $\mu = 0$, and then $\lambda = 0$, that is $\sigma(x) = x$.

If $y = \xi_i \xi_j$,

$$[\sigma(\xi_i \xi_j), x] = [\sigma(\xi_i \xi_j), \sigma(x)] = \sigma[\xi_i \xi_j, x] = 0.$$ (25)

From (23) and (25), one gets $\lambda = 0$ again. The proof is complete. \qed

3.4 The main result

Propositions 3.1, 3.2, 3.4-3.7 show that all the Hom-Lie superalgebra structures on the 0-th and $(-1)$-th $\mathbb{Z}$-components of each infinite-dimensional exceptional simple Lie superalgebra are trivial. Then combining this with Lemma 2.2, we immediately have:

Theorem 3.8. There is only the trivial Hom-Lie superalgebra structure on each exceptional simple Lie superalgebra.
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