Open Mathematics

Research Article

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On pairs of equations in unlike powers of primes and powers of 2

https://doi.org/10.1515/math-2017-0125
Received June 30, 2017; accepted November 3, 2017.

Abstract: In this paper, we obtained that when \( k = 455 \), every pair of large even integers satisfying some necessary conditions can be represented in the form of a pair of unlike powers of primes and \( k \) powers of 2.

Keywords: Circle method, Linnik problem, Powers of 2

MSC: 11P32, 11P05, 11P55

1 Introduction

In 1951 and 1953, Linnik established the following “almost Goldbach” result that each large even integer \( N \) is a sum of two primes \( p_1, p_2 \) and a bounded number of powers of 2, namely

\[
N = p_1 + p_2 + 2^{v_1} + \cdots + 2^{v_k}.
\]

(1)

In 2002, Heath-Brown and Puchta [1] applied a rather different approach to this problem and showed that \( k = 13 \) and, on the GRH, \( k = 7 \). In 2003, Pintz and Ruzsa [10] established this latter result and announced that \( k = 8 \) is acceptable unconditionally. This paper is yet to appear in print. Elsholtz, in an unpublished manuscript, showed that \( k = 12 \); this was proved independently by Liu and Lü [9].

In 1999, Liu, Liu and Zhan [6] proved that every large even integer \( N \) can be written as a sum of four squares of primes and a bounded number of powers of 2, namely

\[
N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2^{v_1} + \cdots + 2^{v_k}.
\]

(2)


\[
N = p_1^3 + p_2^3 + p_3^3 + p_4^3 + 2^{v_1} + \cdots + 2^{v_k}.
\]

(3)

The acceptable value was determined by Platt and Trudgian [11]. In 2011, Liu and Lü [8] considered a hybrid problem of (1.1), (1.2) and (1.3),

\[
N = p_1 + p_2^3 + p_3^3 + p_4^3 + 2^{v_1} + \cdots + 2^{v_k}.
\]

(4)

They showed that \( k = 161 \) is acceptable and Platt and Trudgian [11] revised it to 156.

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Very recently, Kong [2] first considered the result on pairs of linear equations in four prime variables and powers of 2, in the form

\[
\begin{aligned}
N_1 &= p_1 + p_2 + 2^{v_1} + \cdots + 2^{v_k}, \\
N_2 &= p_3 + p_4 + 2^{v_1} + \cdots + 2^{v_k},
\end{aligned}
\]

where \( k \) is a positive integer. She proved that the simultaneous equations (1.5) are solvable for \( k = 63 \). Then Platt and Trudgian [11] revised it to 62.

In this paper, we shall consider the simultaneous representation of pairs of positive even integers \( N_2 \gg N_1 > N_2 \), in the form

\[
\begin{aligned}
N_1 &= p_1 + p_2 + p_3 + p_4^2 + 2^{v_1} + \cdots + 2^{v_k}, \\
N_2 &= p_5 + p_6^2 + p_7^3 + 2^{v_1} + \cdots + 2^{v_k},
\end{aligned}
\]

where \( k \) is a positive integer. Our result is stated as follows.

**Theorem 1.1.** For \( k = 455 \), the equations (1.6) are solvable for every pair of sufficiently large positive even integers \( N_1 \) and \( N_2 \) satisfying \( N_2 \gg N_1 > N_2 \).

We establish Theorem 1.1 by means of the circle method in combination with some new methods of using the method of L"{u}i [8].

**Notation.** Throughout this paper, the letter \( e \) denotes a positive constant which is arbitrarily small but may not be the same at different occurrences. And \( p \) and \( \nu \) denote a prime number and a positive integer, respectively.

## 2 Outline of the method

Here we give an outline for the proof of Theorem 1.1.

In order to apply the circle method, we set

\[ P_i = N_i^{3/9 - 2\epsilon}, \quad Q_i = N_i^{8/9 + \epsilon} \]

for \( i = 1, 2 \). For any integers \( a_1, a_2, q_1, q_2 \) satisfying

\[ 1 \leq a_1 \leq q_1 < P_1, (a_1, q_1) = 1, \]
\[ 1 \leq a_2 \leq q_2 < P_2, (a_2, q_2) = 1, \]

we define the major arcs \( \mathcal{M}_1, \mathcal{M}_2 \) and minor arcs \( C(\mathcal{M}_1), C(\mathcal{M}_2) \) as usual, namely

\[
\mathcal{M}_i = \bigcup_{q_i \leq P_i} \bigcup_{\substack{a_i \in \mathbb{Z} \cap (0, q_i) \atop (a_i, q_i) = 1}} \mathcal{M}_i(a_i, q_i), \quad C(\mathcal{M}_i) = \left\{ \frac{1}{Q_i}, 1 + \frac{1}{Q_i} \right\} \setminus \mathcal{M}_i, \tag{7}
\]

where \( i = 1, 2 \) and

\[ \mathcal{M}_i(a_i, q_i) = \left\{ a_i \in [0, 1] : \left| a_i - \frac{a_i}{q_i} \right| \leq \frac{1}{q_i Q_i} \right\}. \]

It follows from \( 2P_i \leq Q_i \) that the arcs \( \mathcal{M}_1(a_1, q_1) \) and \( \mathcal{M}_2(a_2, q_2) \) are mutually disjoint respectively.

As in [12], let \( \delta = 10^{-\kappa} \), and

\[ U_i = \left( \frac{N_i}{16(1 + \delta)} \right)^{1/3}, \quad V_i = U_i^{5/6} \tag{8} \]

for \( i = 1, 2 \). We set

\[
f(a_i, N_i) = \sum_{p \leq N_i} (\log p) e(p a_i), \quad g(a_i, N_i) = \sum_{p^2 \leq N_i} (\log p) e(p^2 a_i), \tag{9}
\]
where $i = 1, 2$, $e(x) := \exp(2\pi i x)$ and $L = \log_2 N_1$.

Let

$$R(N_1, N_2) = \sum \log p_1 \log p_2 \cdots \log p_8$$

be the weighted number of solutions of (1.8) in $(p_1, \cdots, p_8, v_1, \cdots, v_k)$ with

$$p_1 \leq N_1, \quad p_2 \leq N_1, \quad p_3 \sim U_1, \quad p_4 \sim V_1, \quad p_5 \leq N_2, \quad p_6 \leq N_2, \quad p_7 \sim U_2, \quad p_8 \sim V_2, \quad v_j \leq L,$$

for $j = 1, 2, \cdots, k$. Then $R(N_1, N_2)$ can be written as

$$R(N_1, N_2) = \int_0^1 \int_0^1 f(\alpha_1, N_1)g(\alpha_2, N_1)S(\alpha_1, U_1)T(\alpha_2, V_1)f(\alpha_2, N_2)g(\alpha_2, N_2)$$

$$\times S(\alpha_2, U_2)T(\alpha_2, V_2)G^k(\alpha_1 + \alpha_2)e(-\alpha_1 N_1 - \alpha_2 N_2)da_1 da_2$$

$$= \left\{ \int_{\mathcal{N}_1} + \int_{\mathcal{L}_1} \right\} \left\{ \int_{\mathcal{N}_2} + \int_{\mathcal{L}_2} \right\} \int_{\mathcal{N}_1} \int_{\mathcal{N}_2} f(\alpha_1, N_1)g(\alpha_1, N_1)S(\alpha_1, U_1)T(\alpha_2, V_1)f(\alpha_2, N_2)g(\alpha_2, N_2)$$

$$\times S(\alpha_2, U_2)T(\alpha_2, V_2)G^k(\alpha_1 + \alpha_2)e(-\alpha_1 N_1 - \alpha_2 N_2)da_1 da_2$$

$$:= \sum_{s=1}^3 \sum_{t=1}^3 R_{st}(N_1, N_2),$$

where $R_{st}(N_1, N_2)$ denotes the combination of $s$-th term in the first bracket and the $t$-th term in the second bracket.

We will establish Theorem 1.1 by estimating the term $R_{st}(N_1, N_2)$ for all $1 \leq s, t \leq 3$. We need to show that $R(N_1, N_2) > 0$ for every pair of sufficiently large odd positive integers $N_2 \gg N_1 > N_2$.

We need the following lemmas to prove Theorem 1.1.

For Dirichlet character $\chi \mod q$, let

$$C_1(\chi, a) = \sum_{h=1}^q \chi(h) e\left(\frac{ah}{q}\right), \quad C_1(q, a) = C_1(\chi^0, a),$$

$$C_2(\chi, a) = \sum_{h=1}^q \chi(h) e\left(\frac{ah^2}{q}\right), \quad C_2(q, a) = C_2(\chi^0, a),$$

$$C_3(\chi, a) = \sum_{h=1}^q \chi(h) e\left(\frac{ah^3}{q}\right), \quad C_3(q, a) = C_3(\chi^0, a),$$

where the Ramanujan sum $C_1(q, a) = \mu(q), (a, q) = 1$. If $\chi_1, \chi_2, \chi_3$ and $\chi_4$ are characters mod $q$, then we write

$$B(n, q; \chi_1, \chi_2, \chi_3, \chi_4) = \sum_{(a, q)=1}^q C_1(\chi_1, a)C_2(\chi_2, a)C_3(\chi_3, a)C_3(\chi_4, a)e\left(-\frac{an}{q}\right),$$

$$B(n, q) = B(n, q; \chi^0, \chi^0, \chi^0, \chi^0),$$

$$A(n, q) = \frac{B(n, q)}{\varphi(q)}, \quad \mathcal{O}(n) = \sum_{q=1}^\infty A(n, q).$$
Lemma 2.1. We have  
\[ \text{meas}(E_A) \ll N_2^{-E_A}, \]
with \( E(0.9457) > 109/126 + 10^{-10} \).

Proof. This is Lemma 4.4 in Liu and Lü [8].

Lemma 2.2. Let \( N_i \) be as in (2.1). Then for \( N_i/2 \leq n \leq N_i \), we have  
\[ \int_{N_i} f(a_i, N_i) g(a_i, N_i) S(a_i, U_i) T(a_i, V_i)e(-an)da = \frac{1}{2 \cdot 3^k} \mathcal{G}(n) + O(N_i^{1/9}L^{-1}). \]

Here the singular series \( \mathcal{G}(n) \) satisfies \( \mathcal{G}(n) \gg 1 \) for \( n \equiv 0 (\text{mod } 2) \). \( f(n) \) is defined as  
\[ f(n) := \sum_{n_1, n_2 \leq \text{sup} (N_i, k)} m_1^{1/2}(m_1 m_i)^{-2/3}, \]
and satisfies \( N_i^{10/9} \ll f(n) \ll N_i^{10/9} \).

Proof. This is Lemma 2.1 in Liu and Lü [8].

Lemma 2.3. For all integers \( n \equiv 0 (\text{mod } 2) \), we have \( \mathcal{G}(n) \gg 0.2448 \).

Proof. This result can be found in Section 3 in Liu and Lü [8].

Lemma 2.4. Let \( \mathcal{B}(N_i, k) = \{ n_i \geq 2 : n_i = N_i - 2^{v_1} - \cdots - 2^{v_k} \} \) with \( k \geq 2 \). Then for \( N_1 \equiv N_2 \equiv 0 (\text{mod } 2) \), we have  
\[ \sum_{n_1 \in \mathcal{B}(N_i, k)} f(n_1) f(n_2) \geq 5.4671 N_1^{10/9} N_2^{10/9} N_i^{10/9}. \]

Proof. Using the Lemma 4.2 in [8], we have  
\[ \sum_{n_1 \in \mathcal{B}(N_i, k)} f(n_1) f(n_2) \geq (2.3381)^2 N_1^{10/9} N_2^{10/9} \frac{1}{(v_1 \cdots v_k)}, \]
where \((v)\) means that \( v_1, \ldots, v_k \) satisfies  
\[ 1 \leq v_1, \ldots, v_k \leq \log_2(N_i/KL), \quad 2^{v_1} + \cdots + 2^{v_k} \equiv N_i (\text{mod } 2). \]
Then following the argument of Lemma 4.1 in [8], we have  
\[ \sum_{(v)} 1 \geq (1 - e)L^k. \]
Then we get the proof of this lemma.

Lemma 2.5. Let \( f(a_i, N_i), g(a_i, N_i), S(a_i, N_i), T(a_i, V_i) \) be defined by (2.3) and (2.4), \( C(M_i) \) by (2.1). Then  
\[ \sup_{a \in C(M_i)} |f(a_i, N_i)| \ll N_i^{17/18 + \epsilon}, \quad \sup_{a \in C(M_i)} |g(a_i, N_i)| \ll N_i^{6/9 + \epsilon}, \]
\[ \sup_{a \in C(M_i)} |S(a_i, U_i)| \ll N_i^{5/18 + \epsilon}, \quad \sup_{a \in C(M_i)} |T(a_i, V_i)| \ll N_i^{13/42 + \epsilon}. \]

Proof. The proof of this lemma can be found in [8], which is based on the estimate of exponential sums over primes.
Lemma 2.6. Let \(f(a_i, N_i), g(a_i, N_i), S(a_i, N_i)\) and \(T(a_i, V_i)\) be defined by (2.3) and (2.4), \(G(a_i)\) by (2.5). Then we have
\[
\int_0^1 |f(a_i, N_i)g(a_i, N_i)S(a_i, U_i)T(a_i, V_i)|^2 \, da_i \leq 170.1881 N_i^{10/9} L^2.
\]

**Proof.** From the definition of \(G(a_i)\), Lemma 10 in [1], Lemma 2.3 and 2.5 in [8], we have
\[
\begin{align*}
\int_0^1 |f(a_i, N_i)G(2a_i)|^2 \, da_i & \leq 12.3238 c_0 N_i L^2, \\
\int_0^1 |g(a_i, N_i)G(2a_i)|^4 \, da_i & \leq c_1 \frac{\pi^2}{16} N_i L^4, \\
\int_0^1 |S(a_i, U_i)T(a_i, V_i)|^4 \, da_i & \leq 0.3591 N_i^{13/9},
\end{align*}
\]
where
\[
c_0 = 0.6601, \quad c_1 \leq \left( \frac{32^4 \cdot 101 \cdot 1.6207}{3} + \frac{8 \cdot \log^2 2}{\pi^2} \right) \cdot (1 + e)^9.
\]

Then we have
\[
\begin{align*}
\int_0^1 |f(a_i, N_i)g(a_i, N_i)S(a_i, U_i)T(a_i, V_i)G^2(2a_i)| \, da_i & \\
\ll \left( \int_0^1 |f(a_i, N_i)G(2a_i)|^2 \, da_i \right)^{\frac{1}{2}} \left( \int_0^1 |g(a_i, N_i)S(a_i, U_i)T(a_i, V_i)|^2 \, da_i \right)^{\frac{1}{2}} \\
\ll \left( \int_0^1 |f(a_i, N_i)G(2a_i)|^2 \, da_i \right)^{\frac{1}{2}} \left( \int_0^1 |g(a_i, N_i)G(2a_i)|^4 \, da_i \right)^{\frac{1}{4}} \left( \int_0^1 |S(a_i, U_i)T(a_i, V_i)|^4 \, da_i \right)^{\frac{1}{4}} \\
\ll 170.1881 N_i^{10/9} L^2.
\end{align*}
\]
Thus we can get the proof of this lemma. \(\square\)

3 Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1.

We begin with the estimate for \(R_{11}(N_1, N_2)\). Applying Lemmas 2.2, 2.3 and 2.4 and introducing the notation \(B(N_1, k)\), we can get
\[
R_{11}(N_1, N_2) = \sum_{n_1 \in \Omega(N_1, N_2)} \int_{N_1} f(a_1, N_1)g(a_1, N_1)S(a_1, U_1)T(a_1, V_1)G^k(a_1) e(-a_1 N_1) \, da_1
\]
\[
\times \int_{N_2} f(a_2, N_2)g(a_2, N_2)S(a_2, U_2)T(a_2, V_2)G^k(a_2) e(-a_2 N_2) \, da_2
\]
\[
= \sum_{n_1 \in \Omega(N_1, N_2)} \sum_{n_2 \in \Omega(N_2, N_1)} \int_{N_1} f(a_1, N_1)g(a_1, N_1)S(a_1, U_1)T(a_1, V_1) e(-a_1 n_1) \, da_1
\]
\[
\times \int_{N_1} f(\alpha_2, N_2) g(\alpha_2, N_2) S(\alpha_2, U_2) T(\alpha_2, V_2) e(-\alpha_2 n_2) d\alpha_2 \\
\geq \left( \frac{1}{2 - 3^2} \right)^2 \sum_{n_1 \in \mathcal{B}(N_1, k)} \sum_{n_2 \in \mathcal{B}(N_2, k)} \mathbb{S}(n_1) \mathbb{S}(n_2) J(n_1)(n_2) + O(N_1^{10/9} N_2^{10/9} L^{k - 1}) \\
\geq \frac{\pi^2}{16} \cdot (0.2448)^2 \cdot 5.4671 N_1^{10/9} N_2^{10/9} L^k,
\]

where we used \( \frac{n}{M} = 1 + O(L^{-1}) \) for \( n_i \in \mathcal{B}(N_i, k) \).

Now we turn to give an upper bound for \( R_{12}(N_1, N_2) \). The estimate for \( R_{21}(N_1, N_2) \) is similar. By Cauchy’s inequality, we can get

\[
|G(\alpha_1 + \alpha_2)| \leq \sqrt{|G(2\alpha_1)G(2\alpha_2)|}.
\]

For \( \alpha \in \mathcal{C}(M_2) \backslash \mathcal{E}_A \) and sufficiently large \( N_1 \), we have

\[
|G(2\alpha_1)| \leq |G(\alpha_1)| + 2 \leq \lambda L + 2 \leq (1 + o(1))\lambda L.
\]

Then using the definition of \( \mathcal{E}_A \), the trivial bound of \( G(\alpha_i) \), Lemmas 2.1, 2.5 and 2.6, we have

\[
R_{12}(N_1, N_2) = \int_{\mathcal{C}(M_2) \cap \mathcal{E}_A} f(\alpha_1, N_1) g(\alpha_1, N_1) S(\alpha_1, U_1) T(\alpha_1, V_1) \\
\times f(\alpha_2, N_2) g(\alpha_2, N_2) S(\alpha_2, U_2) T(\alpha_2, V_2) G^k(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) d\alpha_1 d\alpha_2 \\
\ll \int_{\mathcal{C}(M_2) \cap \mathcal{E}_A} |f(\alpha_1, N_1) g(\alpha_1, N_1) S(\alpha_1, U_1) T(\alpha_1, V_1) G^{k/2}(2\alpha_1)| d\alpha_1 \\
\times \int_{\mathcal{C}(M_2) \cap \mathcal{E}_A} |f(\alpha_2, N_2) g(\alpha_2, N_2) S(\alpha_2, U_2) T(\alpha_2, V_2) G^{k/2}(2\alpha_2)| d\alpha_2 \\
\ll N_1^{10/9} L^{k/2} \max_{\alpha \in \mathcal{C}(M_2)} |f(\alpha_2, N_2) g(\alpha_2, N_2) S(\alpha_2, U_2) T(\alpha_2, V_2)| \left( \int_{\mathcal{E}_A} 1 d\alpha_2 \right) \\
\ll N_1^{10/9} L^{k} N_2^{10/9} L^{k - 1} \left( \text{meas}(\mathcal{E}_A) \right) \\
\ll N_1^{10/9} L^{k} N_2^{10/9} L^{k - 1} \ll N_1^{10/9} N_2^{10/9} L^{k - 1}.
\]

Similarly, we can get

\[
R_{21}(N_1, N_2) \ll N_1^{10/9} N_2^{10/9} L^{k - 1}.
\]

Next we give an upper bound for \( R_{13}(N_1, N_2) \). By Lemma 2.6, using the trivial bound \( |G(2\alpha)| \leq L \) when \( \alpha \in \mathcal{M}_1 \) and the bound \( |G(2\alpha)| \leq (1 + o(1))\lambda L \) when \( \alpha \in \mathcal{C}(M_2) \backslash \mathcal{E}_A \), we have

\[
|R_{13}(N_1, N_2)| = \int_{\mathcal{C}(M_2) \backslash \mathcal{E}_A} f(\alpha_1, N_1) g(\alpha_1, N_1) S(\alpha_1, U_1) T(\alpha_1, V_1) \\
\times f(\alpha_2, N_2) g(\alpha_2, N_2) S(\alpha_2, U_2) T(\alpha_2, V_2) G^2(\alpha_1 + \alpha_2) e(-\alpha_1 N_1 - \alpha_2 N_2) d\alpha_1 d\alpha_2|
\]
\[
\begin{align*}
&\leq \int_{\mathbb{N}_1} |f(a_1, N_1)g(a_1, N_1)S(a_1, U_1)T(a_1, V_1)G^{k/2}(a_1)| \, da_1 \\
&\quad \times \int_{C(M_2) \setminus E_4} |f(a_2, N_2)g(a_2, N_2)S(a_2, U_2)T(a_2, V_2)G^{k/2}(a_2)| \, da_2 \\
&\leq L^{k/2-2} \int_{\mathbb{N}_1} |f(a_1, N_1)g(a_1, N_1)S(a_1, U_1)T(a_1, V_1)G^2(a_1)| \, da_1 \\
&\quad \times (\lambda L)^{k/2-2} \int_{C(M_2) \setminus E_4} |f(a_2, N_2)g(a_2, N_2)S(a_2, U_2)T(a_2, V_2)G^2(a_2)| \, da_2 \\
&\leq (170.1881)^2 \lambda^{10/9} N_1^{10/9} N_2^{10/9} L^k.
\end{align*}
\]

We can obtain the estimate for \(R_{31}(N_1, N_2)\) analogously,

\[
|R_{31}(N_1, N_2)| \leq (170.1881)^2 \lambda^{10/9} N_1^{10/9} N_2^{10/9} L^k. \tag{18}
\]

We give the estimate for \(R_{22}(N_1, N_2)\) by the trivial bound for \(G(a)\), Lemma 2.5 and the definition of \(E_\lambda\),

\[
R_{22}(N_1, N_2) = \int_{C(M_1) \cap E_4} \int_{C(M_2) \cap E_4} f(a_1, N_1)g(a_1, N_1)S(a_1, U_1)T(a_1, V_1) \\
\times f(a_2, N_2)g(a_2, N_2)S(a_2, U_2)T(a_2, V_2)G^k(a_1 + a_2)e(-a_1 N_1 - a_2 N_2) \, da_1 \, da_2 \\
\ll \int_{C(M_1) \cap E_4} |f(a_1, N_1)g(a_1, N_1)S(a_1, U_1)T(a_1, V_1)G^{k/2}(a_1)| \, da_1 \\
\times \int_{C(M_2) \cap E_4} |f(a_2, N_2)g(a_2, N_2)S(a_2, U_2)T(a_2, V_2)G^{k/2}(a_2)| \, da_2 \\
\ll N_1^{10/9} L^{k/2-1} N_2^{10/9} L^{k/2-1} \ll N_1^{10/9} N_2^{10/9} L^{k-1}.
\]

For \(R_{23}(N_1, N_2)\), we can easily get

\[
R_{23}(N_1, N_2) = \int_{C(M_1) \cap E_4} \int_{C(M_2) \setminus E_4} f(a_1, N_1)g(a_1, N_1)S(a_1, U_1)T(a_1, V_1) \\
\times f(a_2, N_2)g(a_2, N_2)S(a_2, U_2)T(a_2, V_2)G^k(a_1 + a_2)e(-a_1 N_1 - a_2 N_2) \, da_1 \, da_2 \\
\ll \int_{C(M_1) \cap E_4} |f(a_1, N_1)g(a_1, N_1)S(a_1, U_1)T(a_1, V_1)G^{k/2}(a_1)| \, da_1 \\
\times \int_{C(M_2) \setminus E_4} |f(a_2, N_2)g(a_2, N_2)S(a_2, U_2)T(a_2, V_2)G^{k/2}(a_2)| \, da_2 \\
\ll N_1^{10/9} L^{k/2-1} N_2^{10/9} L^{k/2} \ll N_1^{10/9} N_2^{10/9} L^{k-1}.
\]

Similarly, we have

\[
R_{32}(N_1, N_2) \ll N_1^{10/9} N_2^{10/9} L^{k-1}. \tag{21}
\]

In the end, we provide the upper bound for \(R_{33}(N_1, N_2)\).

\[
|R_{33}(N_1, N_2)| = \int_{C(M_2) \setminus E_4} \int_{C(M_2) \setminus E_4} f(a_1, N_1)g(a_1, N_1)S(a_1, U_1)T(a_1, V_1)
\]
\[
\times f(a_2, N_2)g(a_2, N_2)T(a_2, U_2)V_2G^{k/2}(2a_2)d_2| \leq \int_{C(N_1), \xi_1} |f(a_1, N_1)g(a_1, N_1)T(a_1, U_1)V_1G^{k/2}(2a_1)|d_1 \times \int_{C(M_2), \xi_2} |f(a_2, N_2)g(a_2, N_2)T(a_2, U_2)V_2G^{k/2}(2a_2)|d_2 \leq (AL)^{k/2-2} \times 170.1881N_1^{10/9}L^2 \times (AL)^{k/2-2} \times 170.1881N_2^{10/9}L^2 \leq \lambda^{k-q}(170.1881)^2N_1^{10/9}N_2^{10/9}L^k.
\]

Combining (3.1)-(3.9), we can obtain

\[
R(N_1, N_2) > R_{11}(N_1, N_2) - R_{13}(N_1, N_2) - R_{31}(N_1, N_2) - R_{33}(N_1, N_2) + O(N_1^{10/9}N_2^{10/9}L^{k-1}) + O(N_1^{10/9}N_2^{10/9}L^{k-1})
\]

\[
> \frac{\pi^2}{16} \cdot (0.2448) \cdot 2 \cdot 5.4671N_1^{10/9}N_2^{10/9}L^k - 2 \times (170.1881)^2\lambda^{k/2-2}N_1^{10/9}N_2^{10/9}L^k - \lambda^{k-q}(170.1881)^2N_1^{10/9}N_2^{10/9}L^k + O(N_1^{10/9}N_2^{10/9}L^{k-1}).
\]

Therefore, we solve the inequality

\[
R(N_1, N_2) > 0
\]

and get \( k \geq 455 \). Consequently, we deduce that every pair of large odd integers \( N_1, N_2 \) satisfying \( N_2 \gg N_1 > N_2 \) and \( N_1 \equiv N_2 \equiv 0(\text{mod} \ 2) \) can be written in the form of (1.3) for \( k \geq 455 \). Thus Theorem 1.1 follows.

**Acknowledgement:** This work is supported by Natural Science Foundation of China (Grant Nos. 11761048). The authors would like to express their thanks to the referee for many useful suggestions and comments on the manuscript.

**References**