Nabla inequalities and permanence for a logistic integrodifferential equation on time scales

Abstract: In this paper, by using the theory of calculus on time scales and some mathematical methods, several nabla dynamic inequalities on time scales are established. As an application, we apply the obtained results to a logistic integrodifferential equation on time scales and sufficient conditions for the permanence of the equation are derived. Finally, numerical examples together with their simulations are presented to illustrate the feasibility and effectiveness of the results.

Keywords: Nabla inequality, Permanence, Integrodifferential equation, Time scale

MSC: 34N05, 34k38, 92B05

1 Introduction

The theory of calculus on time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph.D. thesis in 1988 [1] in order to unify continuous and discrete analysis. The time scales approach not only unifies differential and difference equations, but also solves some other problems such as a mix of stop-start and continuous behaviors powerfully. For example, it can model insect populations that are continuous while in season (and may follow a difference scheme with variable step-size), die out in winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population. The study of population dynamic systems on time scales can reveal new qualitative phenomenon, see, for example, [2-8].

It is well known that permanence (or persistence) is an important property of dynamical systems and of the systems arising in ecology, epidemics etc, since permanence addresses the limits of growth for some or all components of a system, while persistence also deals with the long-term survival of some or all components of the system. To keep the biology’s variety of the nature, the permanence of biotic population is a significant and comprehensive problem in biomathematics. In the last decade, permanence of different types of continuous or discrete population dynamic systems in ecology have been studied wildly both in theories and applications, see [9-16] and the references therein.

On the other hand, the growth rate of a natural species population will often not respond immediately to changes in its own population or that of an interacting species, but rather will do so after a time lag. Research has shown that time delays have a great destabilizing influence on species populations. The dependence
of the rate of change of current population on the population at a particular point of past time is usually a simplified assumption, and a more reasonable dependence would be on the whole historical information of the population. A distributed delay has been proposed to describe the population growth of some species, which can date back to the works of Volterra [17,18,19]. Up to now, literature has focused on permanence of ecosystems on time scales, see [20,21]. However, to the best of the authors’ knowledge, there are few papers studying the permanence of ecosystem with distributed delays on time scales.

Motivated by the above statements, in the present paper we focus our attention on the following logistic integrodifferential equation on time scales:

\[ x^\nabla(t) = x(t) \left[ r(t) - a(t) x(t) - b(t) \int_{t_0}^{+\infty} k_\alpha(s) x(t-s) \nabla s \right], \quad t \in \mathbb{T} \]  \hfill (1)

where \( \mathbb{T} \) is a time scale, \( t_0 \in \mathbb{T} \); \( r(t), a(t), b(t) \) are continuous, positive and bounded functions; \( k_\alpha : [t_0, +\infty)_\mathbb{T} \to [0, +\infty) \) is piecewise continuous and integrable on \([t_0, +\infty)_\mathbb{T} \) for each \( \alpha \in (0, +\infty) \), and \( \int_{t_0}^{+\infty} k_\alpha(s) \nabla s = 1 \).

The initial conditions of (1) are of the form

\[ x(s) = \varphi(s) > 0, \quad s \in (-\infty, t_0]_\mathbb{T}, \quad \varphi(t_0) > 0, \]  \hfill (2)

where \( \varphi \) is bounded and continuous on \((-\infty, t_0]_\mathbb{T}\).

In this paper, we further study some dynamic inequalities on time scales by using the theory of calculus on time scales and some mathematical methods. Then, based on the obtained results, we shall study the permanence of equation (1).

For convenience, we introduce the notation

\[ f^u = \sup_{t \in \mathbb{T}} \{ f(t) \}, \quad f^l = \inf_{t \in \mathbb{T}} \{ f(t) \}, \]

where \( f \) is a positive and bounded function.

## 2 Nabla inequalities

Let \( \mathbb{T} \) be a nonempty closed subset (time scale) of \( \mathbb{R} \). The forward jump operator \( \sigma : \mathbb{T} \to \mathbb{T} \) is defined by \( \sigma(t) = \inf \{ s \in \mathbb{T} : s > t \} \) for all \( t \in \mathbb{T} \), while the backward jump operator \( \rho : \mathbb{T} \to \mathbb{T} \) is defined by \( \rho(t) = \sup \{ s \in \mathbb{T} : s < t \} \) for all \( t \in \mathbb{T} \).

A point \( t \in \mathbb{T} \) is called left-dense if \( t > \inf \mathbb{T} \) and \( \rho(t) = t \), left-scattered if \( \rho(t) < t \), right-dense if \( t < \sup \mathbb{T} \) and \( \sigma(t) = t \), and right-scattered if \( \sigma(t) > t \). If \( \mathbb{T} \) has a left-scattered maximum \( m \), then \( \mathbb{T}^k = \mathbb{T} \setminus \{ m \} \); otherwise \( \mathbb{T}^k = \mathbb{T} \). If \( \mathbb{T} \) has a right-scattered minimum \( m \), then \( \mathbb{T}^k = \mathbb{T} \setminus \{ m \} \); otherwise \( \mathbb{T}^k = \mathbb{T} \). The backwards graininess function \( \nu : \mathbb{T}^k \to [0, +\infty) \) is defined by \( \nu(t) = t - \rho(t) \).

A function \( f : \mathbb{T} \to \mathbb{R} \) is ld-continuous provided it is continuous at left-dense point in \( \mathbb{T} \) and its right-side limits exist at right-dense points in \( \mathbb{T} \).

For the basic theory of calculus on time scales, see [22].

A function \( p : \mathbb{T} \to \mathbb{R} \) is called \( \nu \)-regressive if \( 1 - \nu(t) p(t) \neq 0 \) for all \( t \in \mathbb{T}^k \). The set of all \( \nu \)-regressive and ld-continuous functions \( p : \mathbb{T} \to \mathbb{R} \) will be denoted by \( \mathcal{R}_\nu = \mathcal{R}_\nu(\mathbb{T}, \mathbb{R}) \). Define the set \( \mathcal{R}_\nu^k = \mathbb{R}_\nu(\mathbb{T}^k, \mathbb{R}) = \{ p \in \mathcal{R}_\nu : 1 - \nu(t) p(t) > 0, \forall t \in \mathbb{T} \} \).

If \( p \in \mathcal{R}_\nu \), then the nabla exponential function \( \hat{e}_p \) is defined by

\[ \hat{e}_p(t,s) = \exp \left\{ \int_s^t \hat{\xi}_\nu(p(\tau)) \nabla \tau \right\}, \quad \forall s, t \in \mathbb{T} \]

with the \( \nu \)-cylinder transformation

\[ \hat{\xi}_h(z) = \begin{cases} \frac{-\log(1-hz)}{h}, & \text{if } h \neq 0, \\ z, & \text{if } h = 0. \end{cases} \]
Lemma 2.2. If $p, q \in \mathcal{R}_v$, define
\[ p \oplus v q = p + q - v p q, \ominus_v p = \frac{-p}{1 - v p}. \]

Lemma 2.1 (see [22]). If $p, q \in \mathcal{R}_v$, and $s, t, r \in \mathbb{T}$, then
(i) $\hat{e}_0(t, s) \equiv 1$ and $\hat{e}_p(t, t) \equiv 1$;
(ii) $\hat{e}_p(p(t), s) = (1 - v(t)p(t)) \hat{e}_p(t, s)$;
(iii) $\hat{e}_p(t, s) = \frac{1}{e^{\frac{-t}{s}}(s)} = \hat{e}_{\ominus p}(s, t)$;
(iv) $\hat{e}_p(t, s) \hat{e}_p(s, r) = \hat{e}_p(t, r)$;
(v) $(\hat{e}_p(t, s))^\nabla = p(t) \hat{e}_p(t, s)$.

Lemma 2.2. If $p \in \mathcal{R}_v$, and $a, b, c \in \mathbb{T}$, then
\[ [\hat{e}_p(c, \cdot)]^\nabla = -p[\hat{e}_p(c, \cdot)]^p \]
and
\[ \int_a^b p(t) \hat{e}_p(c, \rho(t)) \nabla t = \hat{e}_p(c, a) - \hat{e}_p(c, b). \]

Proof. By Lemma 2.1,
\[ p(t) \hat{e}_p(c, \rho(t)) = p(t) \hat{e}_{\ominus p}(\rho(t), c) \]
\[ = p(t) [1 - v(t)(\ominus_v p)(t)] \hat{e}_{\ominus p}(t, c) \]
\[ = p(t) [1 - v(t)(\ominus_v p)(t)] \hat{e}_{\ominus p}(t, c) \]
\[ = p(t) \left[ 1 + \frac{v(t)p(t)}{1 - v(t)p(t)} \right] \hat{e}_{\ominus p}(t, c) \]
\[ = p(t) \frac{1}{1 - v(t)p(t)} \hat{e}_{\ominus p}(t, c) \]
\[ = -(\ominus_v p)(t) \hat{e}_{\ominus p}(t, c) \]
\[ = -\hat{e}_{\ominus p}^\nabla(t, c) \]
\[ = -[\hat{e}_p(c, t)]^\nabla, \]
where $\nabla$ denotes differentiation with respect to $t$. Thus
\[ \int_a^b p(t) \hat{e}_p(c, \rho(t)) \nabla t = - \int_a^b [\hat{e}_p(c, t)]^\nabla \nabla t = \hat{e}_p(c, a) - \hat{e}_p(c, b). \]

This completes the proof. \qed

Lemma 2.3. Assume that $a > 0, b > 0$. Then
\[ y^\nabla(t) \geq (\leq) b - ay(t), y(t) > 0, t \in [t_0, +\infty) \mathbb{T} \]
implies
\[ y(t) \geq (\leq) \frac{b}{a} \left[ 1 + \left( \frac{ay(t_0)}{b} - 1 \right) \hat{e}_{(-a)}(t, t_0) \right], t \in [t_0, +\infty) \mathbb{T}. \]

Proof. We only prove the "$\geq$" case, the proof of the "$\leq$" case is similar. For
\[ [y \hat{e}_{\ominus (-a)}(\cdot, t_0)]^\nabla(t) = y^\nabla(t) \hat{e}_{\ominus (-a)}(\rho(t), t_0) + y(t)(\ominus_v (-a)) \hat{e}_{\ominus (-a)}(t, t_0) \]
\[ = y^\nabla(t) \hat{e}_{\ominus (-a)}(\rho(t), t_0) + y(t) \frac{(\ominus_v (-a)) \hat{e}_{\ominus (-a)}(t, t_0)}{1 - v(t)(\ominus_v (-a))} \hat{e}_{\ominus (-a)}(\rho(t), t_0) \]
Lemma 2.4. Assume that 

\[ \nu = \nu(t) = (\nu(t) - (\nu(t) - a))y(t) \]

then, integrate both side from \( t_0 \) to \( t \) to conclude

\[
y(t)\hat{e}_{\nu,-a}(t, t_0) - y(t_0) = \int_{t_0}^{t} [y^\nu(s) - (-a)y(s)] \hat{e}_{\nu,-a}(s, t_0) \Delta s
\]

\[
\geq \int_{t_0}^{t} b \hat{e}_{\nu,-a}(s, t_0) \Delta s
\]

\[
= b \int_{t_0}^{t} \hat{e}_{-a}(s, t_0, \rho(s)) \Delta s,
\]

that is

\[
y(t) \geq y(t_0)\hat{e}_{-a}(t, t_0) + b \int_{t_0}^{t} \hat{e}_{-a}(s, t_0, \rho(s)) \Delta s.
\]

So,

\[
y(t) \geq y(t_0)\hat{e}_{-a}(t, t_0) + b \int_{t_0}^{t} \hat{e}_{-a}(s, t_0, \rho(s)) \Delta s
\]

\[
= y(t_0)\hat{e}_{-a}(t, t_0) - \frac{b}{a} \int_{t_0}^{t} \hat{e}_{-a}(s, t_0, \rho(s)) \Delta s \quad \text{(by Lemma 2.2)}
\]

\[
= y(t_0)\hat{e}_{-a}(t, t_0) - \frac{b}{a}[\hat{e}_{-a}(t, t_0) - 1]
\]

\[
= \hat{e}_{-a}(t, t_0)[y(t_0) - \frac{b}{a}] + \frac{b}{a},
\]

i.e. \( y(t) \geq \frac{b}{a}[1 + (\frac{ay(t_0)}{b} - 1)\hat{e}_{-a}(t, t_0)] \). This completes the proof. \( \square \)

**Lemma 2.4.** Assume that \( a > 0, b > 0 \) and \(-a \in \mathbb{R}^+\). Then

\[
y^\nu(t) \geq (\lessgtr) b - ay(\rho(t)), \quad y(t) > 0, \quad t \in [t_0, +\infty)_T
\]

implies

\[
y(t) \geq (\lessgtr) \frac{b}{a}[1 + (\frac{ay(t_0)}{b} - 1)\hat{e}_{\nu,a}(t, t_0)], \quad t \in [t_0, +\infty)_T.
\]

**Proof.** We only prove the “\( \geq \)” case, the proof of the “\( \leq \)” case is similar. For

\[
[y\hat{e}_a(\cdot, t_0)]^\nu(t) = y^\nu(t)\hat{e}_a(t, t_0) + a\hat{e}_a(t_0, t_0)y(\rho(t))
\]

\[
= \hat{e}_a(t, t_0)[y^\nu(t) + ay(\rho(t))],
\]

then, integrate both side from \( t_0 \) to \( t \) to conclude

\[
y(t)\hat{e}_a(t, t_0) - y(t_0) = \int_{t_0}^{t} \hat{e}_a(s, t_0)[y^\nu(s) + ay(\rho(s))] \Delta s
\]

\[
\geq b \int_{t_0}^{t} \hat{e}_a(s, t_0) \Delta s,
\]
then
\[ y(t) \geq \hat{e}_{\Theta,t}(t,t_0)y(t_0) + \int_{t_0}^{t} \hat{e}_{\Theta,t}(t,s) \nabla s \]

\[ = \hat{e}_{\Theta,t}(t,t_0)y(t_0) + \int_{t_0}^{t} (1 - \nu(s)a)\hat{e}_a(s,t) \frac{1}{1 - \nu(s)a} \nabla s \]

\[ = \hat{e}_{\Theta,t}(t,t_0)y(t_0) + \int_{t_0}^{t} \hat{e}_a(s,t) \frac{1}{1 - \nu(s)a} \nabla s \]

\[ = \hat{e}_{\Theta,t}(t,t_0)y(t_0) + \int_{t_0}^{t} \hat{e}_{\Theta,t}(t,s) \frac{1}{1 - \nu(s,a)} \nabla s \]

\[ = \hat{e}_{\Theta,t}(t,t_0)y(t_0) - \frac{b}{a} \int_{t_0}^{t} \hat{e}_{\Theta,t}(t,s)(\Theta_{\nu})a \nabla s \quad \text{(by Lemma 2.2)} \]

\[ = \hat{e}_{\Theta,t}(t,t_0)y(t_0) - \frac{b}{a}(\hat{e}_{\Theta,t}(t,t_0) - 1) \]

\[ = \hat{e}_{\Theta,t}(t,t_0)[y(t_0) - \frac{b}{a} + \frac{b}{a}] \]

i.e. \( y(t) \geq \frac{b}{a}[1 + (\frac{ay(t_0)}{b} - 1)e_{\Theta,a}(t,t_0)] \). This completes the proof. \( \square \)

**Lemma 2.5.** Assume that \( a > 0, b > 0 \). Then

\[ y^\nu(t) \leq (\geq)y(\nu(t))(b - ay(t)), \quad y(t) > 0, \quad t \in [t_0, +\infty) \]

implies

\[ y(t) \leq (\geq)\frac{b}{a}[1 + (\frac{b}{ay(t)} - 1)e_{(-b)}(t,t_0)]^{-1}, \quad t \in [t_0, +\infty) \]

**Proof.** We only prove the "\( \leq \)" case, the proof of the "\( \geq \)" case is similar.

Let \( y(t) = \frac{1}{x(t)} \), then

\[ (\frac{1}{x(t)})^\nu \leq \frac{1}{x(\nu(t))}(b - \frac{a}{x(t)}), \]

that is

\[ -\frac{x^\nu(t)}{x(t)x(\nu(t))} \leq \frac{1}{x(\nu(t))}(b - \frac{a}{x(t)}), \]

so, \( x^\nu(t) \geq a - bx(t) \). By Lemma 2.2, we have

\[ x(t) \geq \frac{a}{b}[1 + (\frac{bx(t_0)}{a} - 1)e_{(-b)}(t,t_0)]. \]

Therefore,

\[ y(t) \leq \frac{b}{a}[1 + (\frac{b}{ay(t_0)} - 1)e_{(-b)}(t,t_0)]^{-1}. \]

This completes the proof. \( \square \)

**Lemma 2.6.** Assume that \( a > 0, b > 0 \) and \( -b \in R^+_y \). Then

\[ y^\nu(t) \leq (\geq)y(t)(b - ay(\nu(t))), \quad y(t) > 0, \quad t \in [t_0, +\infty) \]

implies

\[ y(t) \leq (\geq)\frac{b}{a}[1 + (\frac{b}{ay(t_0)} - 1)e_{\Theta,b}(t,t_0)], \quad t \in [t_0, +\infty) \]


Proof. We only prove the ”≤” case, the proof of the ”≥” case is similar.

Let \( y(t) = \frac{1}{x(t)} \), then

\[
\left( \frac{1}{x(t)} \right)^\nabla \leq \frac{1}{x(t)} \left( b - \frac{a}{x(\rho(t))} \right),
\]

that is

\[
-\frac{x^\nabla(t)}{x(t)x(\rho(t))} \leq \frac{1}{x(t)} \left( b - \frac{a}{x(\rho(t))} \right),
\]

so, \( x^\nabla(t) \geq a - bx(\rho(t)) \). By Lemma 2.6, we have

\[
x(t) \geq \frac{a}{b} [1 + (\frac{bx(t_0)}{a} - 1)\hat{e}_{\rho,b}(t,t_0)].
\]

Therefore,

\[
y(t) \leq \frac{b}{a} [1 + (\frac{b}{ay(t_0)} - 1)\hat{e}_{\rho,b}(t,t_0)]^{-1}.
\]

This completes the proof. \( \square \)

Theorem 2.7. Assume that \( a > 0, b > 0, -b \in \mathbb{R}_+^\ast \), and \( y(t) \geq 0, t \in [t_0, +\infty)_{\mathbb{T}} \).

(i) If \( y^\nabla(t) \geq y(t)(b - ay(t)) \), then \( \liminf_{t \to +\infty} y(t) = \frac{b}{a} \).

(ii) If \( y^\nabla(t) \leq y(t)(b - ay(t)) \), then \( \limsup_{t \to +\infty} y(t) = \frac{b}{a} \).

Proof. We only need to prove (i), the proof of (ii) is similar.

If \( y(\rho(t)) \geq y(t), t \in [t_0, +\infty)_{\mathbb{T}} \), then

\[
y^\nabla(t) \geq y(t)(b - ay(t)) \geq y(t)(b - ay(\rho(t))),
\]

by Lemma 2.6, we can get

\[
y(t) \geq \frac{b}{a} [1 + (\frac{b}{ay(t_0)} - 1)\hat{e}_{\rho,b}(t,t_0)], \quad t \in [t_0, +\infty)_{\mathbb{T}}.
\] (3)

If \( y(\rho(t)) \leq y(t), t \in [t_0, +\infty)_{\mathbb{T}} \), then

\[
y^\nabla(t) \geq y(t)(b - ay(t)) \geq y(\rho(t))(b - ay(t)),
\]

by Lemma 2.5, we can get

\[
y(t) \geq \frac{b}{a} [1 + (\frac{b}{ay(t_0)} - 1)\hat{e}_{-(\rho)}(t,t_0)]^{-1}, \quad t \in [t_0, +\infty)_{\mathbb{T}}.
\] (4)

It follows from (3) and (4) that

\[
\liminf_{t \to +\infty} y(t) = \frac{b}{a}.
\]

This completes the proof. \( \square \)

Theorem 2.8. Assume that \( y(t) > 0, t \in \mathbb{T} \). Let \( t \in \mathbb{T}_k \), if \( y(t) \) is differentiable at \( t \), then

\[
\frac{y^\nabla(t)}{y(t)} \leq [\ln(y(t))]^\nabla.
\]

Proof. Let \( t \in \mathbb{T}_k \), if \( y(t) \) is differentiable at \( t \), then

\[
y(\rho(t)) = y(t) - \nu(t)y^\nabla(t).
\]
Since \( y(t) > 0 \), \( t \in \mathbb{T} \), we have \( y(t) - v(t)y^\nabla(t) > 0 \), \( t \in \mathbb{T}_k \).

Now, we consider \( \left[ \ln(y(t)) \right]^\nabla, t \in \mathbb{T}_k \).

**Case I.** If \( v(t) = 0 \), then \( \left[ \ln(y(t)) \right]^\nabla = \frac{y^\nabla(t)}{y(t)} \).

**Case II.** If \( v(t) > 0 \), by the Chain Rule,

\[
\left[ \ln(y(t)) \right]^\nabla = \left( \int_0^1 \frac{d\tau}{y(t) - \tau v(t)y^\nabla(t)} \right)y^\nabla(t)
\]

\[
= -\frac{1}{v(t)y^\nabla(t)} \int_y^t \frac{dy(s)}{s}y^\nabla(t)
\]

\[
= -\frac{1}{v(t)} \ln \left( \frac{y(t) - v(t)y^\nabla(t)}{y(t)} \right).
\]

By using the inequality \( x \geq \ln(1 + x), \forall x > -1 \), we have

\[
y^\nabla(t) = \frac{y^\nabla(t)}{y(t)} - 1 \leq \frac{1}{v(t)} \ln \left( \frac{y(t) - v(t)y^\nabla(t)}{y(t)} \right) = \left[ \ln(y(t)) \right]^\nabla.
\]

This completes the proof. \(\square\)

**Remark 2.9.** If the time scale \( \mathbb{T} \) is unbounded above, then \( \mathbb{T} = \mathbb{T}_k \); therefore, in Theorem 2.8, if \( \sup \mathbb{T} = +\infty \), then

\[
\frac{y^\nabla(t)}{y(t)} \leq \left[ \ln(y(t)) \right]^\nabla, \forall t \in \mathbb{T}.
\]

## 3 Permanence

As an application, based on the results obtained in section 2, we shall establish a permanent result for equation (1).

**Definition 3.1.** Equation (1) is said to be permanent if there exist a compact region \( D \subseteq \text{Int}\mathbb{R}_+ \), such that for any positive solution \( x(t) \) of equation (1) with initial condition (2) eventually enters and remains in region \( D \).

Hereafter, we assume that

\[
(H_1) \int_{t_0}^{+\infty} k_a(s)e^{-[\gamma-(a+b)\gamma]s}ds < +\infty.
\]

**Proposition 3.2.** Assume that \( x(t) \) is any positive solution of equation (1) with initial condition (2). Then

\[
\limsup_{t \to +\infty} x(t) = M_1 := \frac{r^\mu}{a^\ell}.
\]

**Proof.** By the positivity of \( x(t) \) and (1), we have

\[
x^\nabla(t) = x(t) \left[ r(t) - a(t)x(t) - b(t) \int_{t_0}^{+\infty} k_a(s)x(t-s)\nabla s \right]
\]

\[
\leq x(t)[r(t) - a(t)x(t)]
\]

\[
\leq x(t)[r^\mu - a^\ell x(t)].
\]

(5)
By Theorem 2.7, we get
\[ \limsup_{t \to +\infty} x(t) = \frac{r'}{a'} := M_1. \]
This completes the proof. \(\square\)

**Proposition 3.3.** Assume that \(x(t)\) is any positive solution of equation (1) with initial condition (2). If \((H_1)\) holds, then
\[
\liminf_{t \to +\infty} x(t) = m_1 := \frac{r'}{a' + b' \int_{t_0}^{+\infty} k_a(s) e^{-[a'' + b']M_1]s}}.
\]

**Proof.** Assume that \(x(t)\) is any positive solution of equation (1) with initial condition (2). From Proposition 3.2, for arbitrarily small positive constant \(\epsilon\), there exists a \(T_1 \in \mathbb{T} \), \(T_1 > t_0\) such that
\[
x(t) < M_1 + \epsilon, \ t \in [T_1, +\infty)_\mathbb{T}.
\]
By the positivity of \(x(t)\) and (1), we have for \(t \in [T_1, +\infty)_\mathbb{T},\)
\[
x^\nabla(t) \geq x(t) \left[ r' - a' x(t) - b' \int_{t_0}^{+\infty} k_a(s) x(t - s) \nabla s \right]
\geq x(t) \left[ r' - a' x(t) - b' \int_{t_0}^{T_1 - t} k_a(s) x(t - s) \nabla s \right. \\
- b' \int_{T_1 - t}^{+\infty} k_a(s) x(t - s) \nabla s \right],
\]
which together with (6) implies
\[
x^\nabla(t) > x(t) \left[ r' - a' (M_1 + \epsilon) - b' \int_{t_0}^{T_1 - t} k_a(s) (M_1 + \epsilon) \nabla s \right. \\
- b' \int_{T_1 - t}^{+\infty} k_a(s) x(t - s) \nabla s \right], \ t \in [T_1, +\infty)_\mathbb{T}.
\]
Let \(\xi(t)\) be defined by
\[
\xi(t) = r' - a' (M_1 + \epsilon) - b' \int_{t_0}^{T_1 - t} k_a(s) (M_1 + \epsilon) \nabla s \\
- b' \int_{T_1 - t}^{+\infty} k_a(s) x(t - s) \nabla s.
\]
Then, by the boundedness of \(x(t)\) and the property of the kernel \(k_a(s)\),
\[
\lim_{t \to +\infty} \xi(t) = r' - (a'' + b'')(M_1 + \epsilon),
\]
and also from (8),
\[
x^\nabla(t) > \xi(t) x(t), \ t \in [T_1, +\infty)_\mathbb{T}.
\]
By Theorem 2.8 and Remark 2.9,
\[
(\ln x(t))^\nabla \geq \frac{x^\nabla(t)}{x(t)} > \xi(t), \ t \in [T_1, +\infty)_\mathbb{T}.
\]

Integrating (11) on \([t-s, t]_\mathbb{T}\), \(t-s \in [T_1, +\infty)\), we derive

\[ x(t) > x(t-s)e^{\int_{t-s}^{t} \xi(r) \, dr}, \]

which leads to

\[ x(t-s) < x(t)e^{-\int_{t-s}^{t} \xi(r) \, dr}. \tag{12} \]

It follows from (7) and (12) that

\[ x^\uparrow(t) > x(t) \left[ t^t \left( a^u + b^u \int_{t-s}^{t} k_a(s) e^{-\int_{t-s}^{t} \xi(r) \, dr} \, ds \right) x(t) - b^u \int_{t-s}^{t} k_a(s) x(t-s) \, ds \right]. \tag{13} \]

Noting that,

\[ \lim_{t \to +\infty} b^u \int_{t-T_1}^{+\infty} k_a(s) x(t-s) \, ds = 0, \]

for the above \(\varepsilon > 0\), there exists a \(T_2 \in \mathbb{T}\), \(T_2 > T_1\) large enough such that

\[ b^u \int_{t-T_1}^{+\infty} k_a(s) x(t-s) \, ds < \varepsilon, \tag{14} \]

also from (9) and (10), we have

\[ t^t - (a^u + b^u) (M_1 + \varepsilon) - \varepsilon < \xi(t) < t^t - (a^u + b^u) (M_1 + \varepsilon) + \varepsilon, \quad t \in [T_2, +\infty)\).

By (15), for \(t \in [T_2, +\infty)\),

\[ b^u \int_{t-T_1}^{+\infty} k_a(s) e^{-\int_{t-s}^{t} \xi(r) \, dr} \, ds < b^u \int_{t-T_1}^{+\infty} k_a(s) e^{-\int_{t-s}^{t} (a^u + b^u) (M_1 + \varepsilon) \, dr} \, ds. \tag{16} \]

From (13)-(16), we derive that, for \(t \in [T_2, +\infty)\),

\[ x^\uparrow(t) > x(t) \left[ t^t \left( a^u + b^u \int_{t-T_1}^{+\infty} k_a(s) e^{-\int_{t-s}^{t} (a^u + b^u) (M_1 + \varepsilon) \, dr} \, ds \right) x(t) \right]. \tag{17} \]

Let \(\varepsilon \to 0\), then

\[ x^\uparrow(t) \geq x(t) \left[ t^t \left( a^u + b^u \int_{t-T_1}^{+\infty} k_a(s) e^{-\int_{t-s}^{t} (a^u + b^u) M_1 \, dr} \, ds \right) x(t) \right]. \tag{18} \]

By Theorem 2.7,

\[ \liminf_{t \to +\infty} x(t) = \frac{t^t}{a^u + b^u \int_{t-T_1}^{+\infty} k_a(s) e^{-\int_{t-s}^{t} (a^u + b^u) M_1 \, dr} \, ds} := m_1. \tag{19} \]

This completes the proof.

Together with Propositions 3.2 and 3.3, we can obtain the following theorem.

**Theorem 3.4.** Assume that \((H_1)\) holds, then equation (1) is permanent.
4 Examples and simulations

In this section, we illustrate the calculation of the asymptotic upper and lower estimates $M_1$ and $m_1$ under some fixed timescales. The simulations are based on the technique of converting the scalar integrodifferential equations into a system of ordinary differential (difference) equations and then numerically solving them using Matlab and its built in graphical output routine.

Case I. $T = \mathbb{R}$. Choose the following two kernels

\[ k_1^{(1)}(s) = a e^{-as}; \]
\[ k_1^{(2)}(s) = a^2 s e^{-as}. \]

Example 4.1. Consider the following equation

\[ \frac{dx(t)}{dt} = x(t) \left[ (2 + \sin \frac{\pi}{2} t) - x(t) - (2 - \cos \pi t) \int_0^{+\infty} a e^{-as} x(t-s) ds \right]. \tag{20} \]

then $M_1 = \frac{r}{u} = 3$. Let $a = 12.1$, we get

\[ \int_0^{+\infty} k_a(s) e^{-\left[ \frac{r}{a^2 + b^2} M_1 \right] s} s \, ds = \int_0^{+\infty} 12.1 e^{-12.1s} e^{11s} ds = 11 < +\infty, \]

that is, $(H_1)$ holds. Furthermore, we have

\[ m_1 = \frac{r}{a^2 + b^2} \int_0^{+\infty} k_a(s) e^{-\left[ \frac{r}{a^2 + b^2} M_1 \right] s} s \, ds = 0.0294. \]

This integrodifferential equation satisfies the assumptions of Theorem 3.4 and can be converted into a system of ordinary differential equations by the introduction of an auxiliary variable $y$, where

\[ y(t) = \int_0^{+\infty} a e^{-as} x(t-s) ds = \int_{-\infty}^t e^{-a(t-s)} x(s) ds. \]

The scalar integrodifferential equation (20) becomes

\[ \begin{cases} \frac{dx(t)}{dt} = x(t)[(2 + \sin \frac{\pi}{2} t) - x(t) - (2 - \cos \pi t) y(t)], \\ \frac{dy(t)}{dt} = -a[y(t) - x(t)]. \end{cases} \tag{21} \]

The convergence of solutions of (20) corresponding to $a = 10$ and three initial values are displayed in Figure 1.

Fig. 1. Graphs of the solution $x(t)$ of (21) with $a = 10$ and the initial values $(x(0), y(0)) = ((0.1, 0.1), (1, 1), (2, 2)).$
Example 4.2. Consider the following equation

\[ \frac{dx(t)}{dt} = x(t)\left[1 - (2 - \sin t)x(t) - \int_0^\infty a^2 se^{-as} x(t - s)\,ds\right], \tag{22} \]

then \( M_1 = \frac{r^\alpha}{a^\alpha} = 1 \). Let \( \alpha = 4 \), we get

\[ \int_0^\infty k_a(s)e^{-[r\alpha - (a^\alpha + b^\alpha)M_1]s}\,ds = \int_0^\infty 16se^{-as} e^{3s}\,ds = 16 < +\infty, \]

that is, \( (H_1) \) holds. Furthermore, we have

\[ m_1 = \frac{r^\alpha}{a^\alpha + b^\alpha \int_0^\infty k_a(s)e^{-[r\alpha - (a^\alpha + b^\alpha)M_1]s}\,ds} = 0.0526. \]

This integrodifferential equation satisfies the assumptions of Theorem 3.4 and can be converted into a system of ordinary differential equations by the introduction of two auxiliary variables \( y \) and \( z \), where

\( \begin{align*}
   y(t) &= \int_0^\infty a^2 se^{-as} x(t - s)\,ds = a^2 \int_{-\infty}^t (t - s) e^{-a(t-s)} x(s)\,ds, \\
   z(t) &= \int_0^\infty a^\alpha e^{-as} x(t - s)\,ds = \alpha \int_{-\infty}^t e^{-a(t-s)} x(s)\,ds.
\end{align*} \)

The scalar integrodifferential equation (22) becomes

\[ \begin{align*}
   \frac{dx(t)}{dt} &= x(t)\left[1 - (2 - \sin t)x(t) - y(t)\right], \\
   \frac{dy(t)}{dt} &= a[z(t) - y(t)], \\
   \frac{dz(t)}{dt} &= a[x(t) - z(t)].
\end{align*} \tag{23} \]

The convergence of solutions of (22) corresponding to \( \alpha = 4 \) and three initial values are displayed in Figure 2. Case II. \( T = \mathbb{Z} \). Choose the following kernel

\[ k_a^{(3)}(s) = (e^a - 1)e^{-as}. \]

Example 4.3. Consider the following equation

\[ \nabla x(t) = x(t)\left[1 - 0.2\sin t - 0.5x(t) - 0.1 \sum_{s=1}^{+\infty} (e^a - 1)e^{-a^\alpha s}x(t - s)\right], \tag{24} \]

where \( \nabla \) is the backward difference operator, then \( M_1 = \frac{\omega^\alpha}{a^\alpha} = 2.4 \). Let \( \alpha = 0.7 \), we get

\[ \int_0^\infty k_a(s)e^{-[r\alpha - (a^\alpha + b^\alpha)M_1]s}\,ds = (e^{0.7} - 1) \sum_{s=1}^{+\infty} e^{-0.7s} e^{0.64s} = 16.3941 < +\infty, \]
that is, \((H_1)\) holds. Furthermore, we have

\[
m_1 = \frac{r}{a u + b u \int_0^{+\infty} k_a(s) e^{-(r+(a+b)M_1)} \sqrt{s}} = 0.3739.
\]

The convergence of solutions of (24) corresponding to \(\alpha = 0.7\) and three initial values are displayed in Figure 3.

**Fig. 3.** Graphs of the solution \(x(t)\) of (24) with \(\alpha = 0.7\) and the initial values \(x(0) = \{0.1, 1, 2\}\).

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**References**


