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Research Article

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Algebraic proofs for shallow water bi–Hamiltonian systems for three cocycle of the semi-direct product of Kac–Moody and Virasoro Lie algebras

https://doi.org/10.1515/math-2018-0002
Received November 30, 2016; accepted January 4, 2018.

Abstract: We prove new theorems related to the construction of the shallow water bi-Hamiltonian systems associated to the semi-direct product of Virasoro and affine Kac–Moody Lie algebras. We discuss associated Verma modules, coadjoint orbits, Casimir functions, and bi-Hamiltonian systems.

Keywords: Affine Kac–Moody Lie algebras, Bi-Hamiltonian systems, Verma modules, Coadjoint orbits

MSC: 17B69, 17B08, 70G60, 82C23

1 Introduction: The semi-direct product of Virasoro algebra with the Kac–Moody algebra

This paper is a continuation of the paper [1] where we studied bi-Hamiltonian systems associated to the three-cocycle extension of the algebra of diffeomorphisms on a circle. In this note we show that certain natural problems (classification of Verma modules, classification of coadjoint orbits, determination of Casimir functions) [2–5] for the central extensions of the Lie algebra $\text{Vect}(S^1) \ltimes LG$ reduce to the equivalent problems for Virasoro and affine Kac–Moody algebras (which are central extensions of $\text{Vect}(S^1)$ and $LG$ respectively). Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra. The group $\text{Diff}(S^1)$ of diffeomorphisms of the circle is included in the group of automorphisms of the Loop group $LG$ of smooth maps from $S^1$ to $G$. For any pairs $(\phi, \psi) \in \text{Diff}(S^1)^2$ and $(g, h) \in LG^2$ the composition law of the group $\text{Diff}(S^1) \ltimes LG$ is

$$(\phi, a) \cdot (\psi, b) = (\phi \circ \psi, a \cdot b \circ \phi^{-1}).$$

The Lie algebra of $\text{Diff}(S^1) \ltimes LG$ is the semi-direct product $\text{Vect}(S^1) \ltimes LG$ of the Lie algebras $\text{Vect}(S^1)$ and $LG$. Let $\mathfrak{g}$ be a Lie algebra and $(.,.)$ a non-degenerated invariant bilinear form. $\text{Vect}(S^1)$ is the Lie algebra of vector fields on the circle and $LG$ the loop algebra (i.e., the Lie algebra of smooth maps from $S^1$ to $\mathfrak{g}$), $\text{Vect}(S^1)_C$ is the Lie algebra over $C$ generated by the elements $L_n, n \in \mathbb{Z}$ with the relations

$$[L_m, L_n] = (n - m)L_{n+m}.$$

We denote by $LG_C$ the Lie algebra over $C$ generated by the elements $g_n, n \in \mathbb{Z}, g \in \mathfrak{g}$ where $(\lambda g + \mu h)_n$ is identified with $\lambda g_n + \mu h_n$ with the relations

$$[g_n, h_m] = [g, h]_{n+m}.$$

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The semi-direct product of \( \text{Vect}(S^1) \) with \( \mathcal{L}G \) is as a vector space isomorphic to \( C^\infty(S^1, \mathbb{R}) \oplus C^\infty(S^1, \mathcal{G}) \) [6]. The Lie bracket of \( \mathcal{SU}(\mathcal{G}) \) has the form

\[
[(u, a), (v, b)] = ([., \partial], u \otimes v, va' - ub' + [a, b]),
\]

for any \((u, v) \in C^\infty(S^1, \mathbb{R})^2\) and any \((a, b) \in C^\infty(S^1, \mathcal{G})^2\), where prime denote derivative with respect to a coordinate on \(S^1\). The Lie algebra \( \text{Vect}(S^1) \times \mathcal{L}G \) can be extended with a universal central extension \( \mathcal{SU}(\mathcal{G}) \) by a two-dimensional vector space. Let us denote by \( \mathcal{J}(u) = \int u \). Two independent cocycles are given by

\[
\omega_{\text{Vir}}((u, a), (v, b)) = \mathcal{J}(u''v), \quad \omega_{\text{K} - \text{M}}((u, a), (v, b)) = \mathcal{J}(a', b)).
\]

We denote by \((u, a, \chi, \alpha)\) the elements of \( \mathcal{SU}(\mathcal{G}) \) with \( u \in C^\infty(S^1, \mathbb{R}), a \in C^\infty(S^1, \mathcal{G}) \) and \((\chi, \alpha) \in \mathbb{R}^2\). The algebra \( \mathcal{SU}(\mathcal{G}) \) can be also represented as the semi-direct product of Virasoro algebra on the affine Kac–Moody algebra. We denote by \( c_{\text{Vir}} \) and \( c_{\text{K} - \text{M}} \) the elements \((0, 0, 1, 0)\) and \((0, 0, 0, 1)\) respectively. If \( \mathcal{G} = \mathbb{R} \), then the Lie algebra \( \text{Vect}(S^1) \times \mathcal{L}R \) has a universal central extension \( \mathcal{SU}(\mathbb{R}) \) by a three-dimensional vector space. The third independent cocycle is given by

\[
\omega_{\text{SP}}((u, a), (v, b)) = \mathcal{J}(ub'' - va'').
\]

We denote by \((u, a, \chi, \alpha, \gamma, \beta)\) elements of \( \mathcal{SU}(\mathbb{R}) \) with \( u \in C^\infty(S^1, \mathbb{R}), a \in C^\infty(S^1, \mathcal{G}), \) and \((\chi, \alpha, \gamma) \in \mathbb{R}^3\). The Lie bracket of \( \mathcal{SU}(\mathbb{R}) \) is given by

\[
[(u, a, \phi, \chi, \alpha, \gamma), (v, b, \xi, \beta, \delta)] = (uv' - u'v, [a, b] - ub' + va', \mathcal{J}(u''v), \mathcal{J}((a', b)), \mathcal{J}(ub'' - va'')).
\]

In this paper we discuss a few questions. Let us mention the main results. First, in Section 2 we consider Kirillov-Kostant Poisson brackets [7] of the regular dual of the semi-direct product of Virasoro Lie algebra with the affine Kac–Moody Lie algebra. Let us denote by \( \mathcal{SU}(\mathcal{G})' \) the subset of \( \mathcal{SU}(\mathcal{G}) \) of elements \((u, a, \xi, \beta)\) with non-vanishing \( \beta \). We denote by \( (\text{Vect}(S^1) \oplus \mathcal{L}G)' \) the subset of \( \text{Vect}(S^1) \oplus \mathcal{L}G \) composed of elements \((u, a, \xi, \beta)\) with \( \beta \neq 0 \). Then introduce two new maps \( \mathcal{I}(u, a, \xi, \beta) \) from \( \mathcal{SU}(\mathcal{G})' \) to \( (\text{Vect}(S^1) \oplus \mathcal{L}G)' \), and \( \mathcal{I}(u, a, \xi, \beta, \gamma) \) from \( \mathcal{SU}(\mathcal{G}) \) to \( \text{Vect}(S^1) \oplus \mathcal{L}G \). We prove that \( \mathcal{I}(u, a, \xi, \beta) \) and \( \mathcal{T}(u, a, \xi, \beta, \gamma) \) are Poisson maps. In Section 3 we discuss coadjoint orbits and Casimir functions for \( \mathcal{SU}(\mathcal{G}) \). Let \( \mathcal{H} \) be a central extension of a Lie algebra \( \mathcal{H} \) and \( H \) be a Lie group with Lie algebra is \( \mathcal{H} \). We find explicit form for the the coadjoint actions of the groups \( \text{Diff}(S^1) \times LG \) and \( \text{Diff}(S^1) \times LR^* \). As a result we obtain the following new theorem. We prove that a coadjoint orbit of \( \mathcal{SU}(\mathcal{G}) \) is mapped by \( \mathcal{I} \) to a coadjoint orbit of \( \text{Vect}(S^1) \oplus \mathcal{L}G \) to a coadjoint orbits of \( \text{Vect}(S^1) \). We prove that the map \( \mathcal{T} \) sends the coadjoint orbits of \( \mathcal{SU}(\mathcal{G}) \) to coadjoint orbits of \( \text{Vect}(S^1) \oplus \mathcal{L}G \). Previously, we determined Casimir functions on \( \mathcal{SU}(\mathcal{G})' \) and \( \mathcal{SU}(\mathbb{R}) \). We then prove new propositions concerning the explicit form of Casimir functions on \( \mathcal{SU}(\mathcal{G}) \), and in particular on on \( \mathcal{SU}(\mathbb{R}) \). This paper was partially inspired by the construction of bi-Hamiltonian systems as natural generalization of the classical Korteweg-de Vries equation. [1, 8–11]. It has been showed in [1], that the dispersive water waves system equation [9, 10, 12] is a bi–Hamiltonian system related to the semi-direct product of a Kac–Moody and Virasoro Lie algebras, and the hierarchy for this system was found. In Section 4 some results of [1] are obtained from another point of view. We prove new proposition for pairwise commuting functions under certain brackets. In section 5 we discuss properties of the universal enveloping algebra of \( \mathcal{SU}(\mathcal{G}) \). In subsection 5.1 we consider a decomposition of the enveloping algebra of a semi-direct product. We introduce the notion of realizability of the action of \( \mathcal{K} \) on \( \mathcal{H} \) in \( \mathcal{U}_\lambda(\mathcal{H}) \). Then we show (Theorem 5.1) that the realizability of the action of \( \mathcal{K} \) in \( \mathcal{U}_\lambda(\mathcal{H}) \) leads to the isomorphism

\[
\mathcal{U}_{\omega_\mathcal{K} - \lambda}(\mathcal{K} \ltimes \mathcal{H}) \cong \mathcal{U}_{\omega_\mathcal{K} - \lambda}(\mathcal{K}) \otimes \mathcal{U}_\lambda(\mathcal{H}).
\]

In subsection 5.2 the case of \( \mathcal{SU}(\mathcal{G}) \) is considered. In subsection 5.3 we discuss representations of \( \mathcal{SU}(\mathcal{G}) \). We prove that positive energy representation \( V \) of \( \mathcal{SU}(\mathcal{G}) \) with non-vanishing \( \beta Id \)-action of the cocyle \( c_{\text{K} - \text{M}} \) delivers a pair of commuting representations of Virasoro and affine Kac–Moody Lie algebras. This proposition determines whether a \( \mathcal{SU}(\mathcal{G}) \) Verma module is a sub-module of another Verma module of \( \mathcal{SU}(\mathcal{G}) \). We also prove a proposition regarding a linear form over \( \mathfrak{h} \) with non-vanishing \( \lambda(c_{\text{K} - \text{M}}) \). In this paper we present proofs for corresponding theorems and lemmas.
2 The Kirillov-Kostant structure of $SU(\mathcal{G})$

Now we consider Kirillov-Kostant Poisson brackets of the regular dual of the semi-direct product of Virasoro Lie algebra with the Affine Kac-Moody Lie algebra. Let $\mathcal{K}$ be a Lie algebra with a non-degenerated bilinear form $(\cdot, \cdot)$. A function $f : \mathcal{K} \to \mathbb{R}$ is called regular at $x \in \mathcal{K}$ if there exists an element $\nabla f(x)$ such that

$$f(x + \epsilon a) = f(x) + \epsilon (\nabla f(x), a) + o(\epsilon),$$

for any $a \in \mathcal{K}$. For two regular functions $f, g : \mathcal{K} \to \mathbb{R}$, we define the Kirillov-Kostant structure as a Poisson structure on $\mathcal{K}$ with

$$\{f, g\} (x) = \{x, [\nabla f(x), \nabla g(x)]\}.$$

Then for any $e \in \mathcal{G}$, the second Poisson structure $\{f, g\}_e (x)$ compatible with the Kirillov-Kostant Poisson structure is defined by

$$\{f, g\}_e (x) = \{e, [\nabla f(x), \nabla g(x)]\}.$$

A non-degenerated bilinear form on $SU(\mathcal{G})$ and $\text{Vect}(\mathcal{S}^1) \oplus \mathbb{C} \mathcal{G}$ is defined by

$$\langle (u_1, a_1, \beta_1, \xi_1), (u_2, a_2, \beta_2, \xi_2) \rangle = \int \sum_{\mathcal{S}} u_1 u_2 + \int \sum_{\mathcal{S}} a_1 a_2 + \xi_1 \xi_2 + \beta_1 \beta_2.$$

We denote by $SU(\mathcal{G})'$ the subset of $SU(\mathcal{G})$ of elements $(u, a, \xi, \beta)$ with non-vanishing $\beta$. Let $u' = u - \frac{1}{2} \frac{d}{d\beta}$. We denote by $(\text{Vect}(\mathcal{S}^1) \oplus \mathbb{C} \mathcal{G})'$ the subset of $\text{Vect}(\mathcal{S}^1) \oplus \mathbb{C} \mathcal{G}$ composed of elements $(u, a, \xi, \beta)$ with $\beta \neq 0$. Let us introduce a new map $\mathcal{I}(u, a, \xi, \beta) = (u', a, \xi, \beta)$ from $SU(\mathcal{G})'$ to $(\text{Vect}(\mathcal{S}^1) \oplus \mathbb{C} \mathcal{G})'$. Then for non-vanishing $\beta$, let us introduce another new map $\mathcal{I}(u, a, \xi, \beta, \gamma) = (u' - \frac{\beta}{\gamma}, a, \xi - \frac{1}{\beta})$ from $SU(\mathcal{G})$ to $\text{Vect}(\mathcal{S}^1) \oplus \mathbb{C} \mathcal{R}$. Here we give a proof for the following new theorem:

**Theorem 2.1.** $\mathcal{I}$ and $\mathcal{I}$ are Poisson maps.

**Proof.** For any regular function $f(u, a, \xi, \beta)$ from $\text{Vect}(\mathcal{S}^1) \oplus \mathbb{C} \mathcal{G}$ to $\mathbb{R}$ let us define a regular function $\mathcal{F}$ from $SU(\mathcal{G})'$ to $\mathbb{R}$ by $\mathcal{F}(u, a, \xi, \beta) = f(u', a, \xi, \beta)$. For $f(u, a, \xi, \beta)$ a function on $SU(\mathcal{G})$, or $(\text{Vect}(\mathcal{S}^1) \oplus \mathbb{C} \mathcal{G})$, let us denote $f_a$ the function of the variables $a$ and $\beta$ that we get when we fix $u$ and $\xi$. Let us denote $f_u$ the function of the variables $u$ and $\xi$ that we get when we fix $a$ and $\beta$. With the previous notations, one has for $\beta \neq 0$ for the bracket $\langle \cdot, \cdot \rangle^u_{SU(\mathcal{G})}$

$$\{f, g\}^u_{SU(\mathcal{G})}(u, a, \xi, \beta) = \{f'_u, g_u\}^u + \{f_u, g'_u\}^u + \{f_a, g_a\}^u + \{f_a, g_\xi\}^u (u, a, \xi, \beta),$$

and for the bracket $\langle \cdot, \cdot \rangle^u_{\text{Vect}(\mathcal{S}^1) \oplus \mathbb{C} \mathcal{G}}$ we have

$$\{f, g\}^u_{\text{Vect}(\mathcal{S}^1) \oplus \mathbb{C} \mathcal{G}} (u, a, \xi, \beta) = \{f'_u, g_u\}^u_{\text{Vect}(\mathcal{S}^1) \oplus \mathbb{C} \mathcal{G}} + \{f_u, g_\xi\}^u_{\text{Vect}(\mathcal{S}^1) \oplus \mathbb{C} \mathcal{G}}.$$

Then the map $\pi_1$ from $SU(\mathcal{G})$ onto $\text{Vect}(\mathcal{S}^1)$ which sends $(u, a, \xi, \beta)$ onto $(u', \xi)$ is a Poisson morphism. The map $\pi_2$ from $SU(\mathcal{G})$ onto $\mathbb{C} \mathcal{G}$ which sends $(u, a, \xi, \beta)$ to $(a, \beta)$ is a Poisson morphism. For any regular function $f$ on $\text{Vect}(\mathcal{S}^1)$ and any regular function $g$ on $\mathbb{C} \mathcal{G}$ we have

$$\pi_1^* f, \pi_2^* g = 0.$$

Indeed, for $i = 1, 2$, $(\delta_a - \frac{a}{\beta} \delta_u) f_i(\tilde{u}, \xi) = 0$. We have:

$$\{f_1(\tilde{u}, \xi), f_2(\tilde{u}, \xi)\}^u_{\xi, \beta}(u, a, \xi, \beta) = \mathcal{F}((\xi, \delta f_{1,u}(\tilde{u}), \xi, \delta f_{2,u}(\tilde{u}), \xi) + 2(\delta f_{1,u}(\tilde{u}), \xi) u + \delta f_{1,u}(\tilde{u}, \xi) a_2, \xi, \delta f_{2,u}(\tilde{u}, \xi)) + \beta^{-1}(\delta f_{1,u}(\tilde{u}, \xi) a, \xi) + \beta^{-1}(\delta f_{2,u}(\tilde{u}, \xi) a, \xi) a.$$
Proposition 3.1. The coadjoint actions of the groups $H \times \tilde{G}$ are given by

$$\{f(\tilde{u}, \xi), g(\tilde{a}, \beta)\}^U \cdot (u, a, \xi, \beta) = \{f_1(\tilde{u}, \xi), f_2(\tilde{a}, \beta)\}^U \cdot \{f_1(\tilde{u}, \xi), f_2(\tilde{a}, \beta)\}.$$

Let $g_i(\alpha, \beta), i = 1, 2$ be two regular functions on the affine Kac–Moody algebra. One notes that $\delta g_1, u = \delta g_2, u = 0$. Therefore,

$$\{g_1, g_2\}^U \cdot (u, a, \xi, \beta) = \beta (dx(\delta g_1, a, \beta), a, \xi, \beta) + \{a, \beta(\delta g_1, a, \beta), \delta g_2, a, \beta\}.$$

Then,

$$\{g_1, g_2\}^U \cdot (u, a, \xi, \beta) = \{f, g\} \tilde{G}(a, \beta).$$

We have:

$$\{f(\tilde{u}, \xi), g(a, \beta)\}^U = \mathcal{J}(\delta f_1, a, \beta), a, \xi, \beta) = -\beta dx(\delta f_1, a, \beta), a, \xi, \beta) + \{a, \delta f_1, a, \beta, \delta g(a, \beta)\}.$$

The sum of the first two terms is equal to 0. The last term is $\mathcal{J}(\delta f_1, a, \beta, \delta g(a)), \beta, \gamma)$, and is equal to zero. One can proceed similarly for $\tilde{T}$.

\section{Coadjoint orbits Casimir functions and for $SU(\mathcal{G})$}

Let $\tilde{H}$ be a central extension of a Lie algebra $H$, and $H$ be a Lie group with Lie algebra is $H$. Then $H$ acts on $\tilde{H}^*$ by the coadjoint action along coadjoint orbits.

\textbf{Proposition 3.1.} The coadjoint actions of the groups $Diff(S^1) \times LG$ and $Diff(S^1) \times L\mathbb{R}_+$ are given by

$$Ad^*(\phi, g)^{-1}(u, a, \xi, \beta) = \left(\frac{1}{2} \beta \parallel g^{-1}g'\parallel^2, \phi'Ad(g^{-1})a \circ \phi + \beta g^{-1}g', \xi, \beta\right).$$

The classification of coadjoint orbits of $\text{Vect}(S^1) \times LG$ can be known from the classification of coadjoint orbits of the Virasoro and affine Kac-moody algebra. Here we obtain the following new

\textbf{Theorem 3.2.} A coadjoint orbit of $SU(\mathcal{G})$ is mapped by $\mathcal{I}$ to a coadjoint orbit of $\text{Vect}(S^1) \otimes \tilde{G}$ to a coadjoint orbits of $\text{Vect}(S^1)$.

In other words, this means that if $\beta_1 \neq 0$, the elements $(u_1, a_1, \xi_1, \beta_1)$ and $(u_2, a_2, \xi_2, \beta_2)$ are in the same coadjoint orbit if and only if: $\xi_1 = \xi_2, \beta_1 = \beta_2, (a_1, \beta_1)$ and $(a_2, \beta_2)$ are on the same coadjoint orbit of $\tilde{G}$, $(u_1 - \frac{a_1}{2\beta_1}, \xi_1)$ and $(u_2 - \frac{a_2}{2\beta_2}, \xi_2)$ are elements of the same coadjoint orbit of $\text{Vect}(S^1)$.

\textbf{Proof.} For any $\phi \in Diff(S^1)$, there exists $h \in LG$ such that

$$hah^{-1} + \beta \frac{\partial h(x)}{\partial x} h^{-1} = a \circ \phi(x').$$

By direct computation we check that

$$\mathcal{I}(Ad^*(\phi, g)(u, a, \xi, \beta) = (Ad^*(\phi, g, h)\mathcal{I}(u, a, \xi, \beta).$$

This implies Theorem 3.2.
Proposition 3.3. The map $\mathcal{F}$ sends the coadjoint orbits of $\tilde{SU}(\mathcal{G})$ to coadjoint orbits of $\text{Vect}(S^1) \otimes \tilde{\mathcal{G}}$.

In other words, this means that if $\beta_1 \neq 0$ the elements $(u_1, a_1, \xi_1, \beta_1, \gamma_1)$ and $(u_1, a_1, \xi_2, \beta_2, \gamma_2)$ are in the same coadjoint orbit if and only if $\gamma_1 = \gamma_2, \xi_1 = \xi_2, \beta_1 = \beta_2, (a_1, \beta_1)$ and $(a_2, \beta_2)$ are on the same coadjoint orbit of $\tilde{\mathcal{G}}$, $(u_1 - \frac{a_1}{\beta_1}, \xi_1 - \frac{\gamma_1}{\beta_1})$ and $(u_2 - \frac{a_1}{\beta_2}, \xi_2 - \frac{\gamma_2}{\beta_2})$ are elements of the same coadjoint orbit of $\text{Vect}(S^1)$. In a particular case, if $\beta_1 = \beta_2 = 0$, then:

Proposition 3.4. If the elements $(u_1, a_1, \xi_1, \beta_1, \gamma_1)$ and $(u_1, a_1, \xi_2, \beta_2, \gamma_2)$ are in the same coadjoint orbit then $\gamma_1 = \gamma_2, (a_1^2 + \gamma_1 a_1')$ and $(a_2^2 + \gamma_2 a_2')$ are in the same coadjoint orbit of the Virasoro Lie algebra.

Proof. We have: $\text{Ad}(\phi, g)(a_1^2 + \gamma_1 a_1') = (a_2^2 + \gamma_2 a_2') \circ \phi + \gamma_1 S(\phi)$.

Previously, we determined Casimir functions on $\tilde{SU}(\mathcal{G})'$ and $\tilde{SU}(\tilde{\mathcal{G}})$. We gave the following proposition:

Proposition 3.5. Let $C_{\text{Vir}}, C_{K-M} C_A$ be Casimir functions for Virasoro, affine Kac–Moody, and the Heisenberg Lie algebras $A$ correspondingly. Let $\mathcal{S}_{\text{SU}}(\mathcal{G}), \mathcal{S}_{\tilde{\mathcal{G}}}(\tilde{\mathcal{G}})$ be Poisson submanifolds of $\mathcal{SU}(\mathcal{G})$ and $\tilde{\mathcal{SU}}(\tilde{\mathcal{G}})$ defined by $\xi = 0$. Then the functions $C_{\text{Vir}}(u', \xi), C(u, a, \beta, \xi) = C_{K-M}(a, \beta)$, and $f_{\mathcal{S}}(u')^{1/2}$, are Casimir functions on $\tilde{SU}(\mathcal{G})'$. In particular, the functions $C_A(u, a, \beta, \xi) = C_A(a, \beta), C_{\text{Vir}}(u' - \frac{\gamma}{\beta} a', \xi)$, and $f_{\mathcal{S}}(u' - \frac{\gamma}{\beta} a')^{1/2}$, are Casimir functions on $\tilde{SU}(\tilde{\mathcal{G}})$.

4 Bi-hamiltonian dispersive water waves systems associated to $\mathcal{SU}(\mathcal{G})$

It has been showed in [1], that the dispersive water waves system equation [9, 10, 12] is a bi–Hamiltonian system related to the semi-direct product of a Kac–Moody and Virasoro Lie algebras, and the hierarchy for this system was found. In this section some results of [1] are obtained from another point of view. We obtain new propositions.

Proposition 4.1. The functions $\phi_1(A(u + B \frac{da}{dx} + C))|\lambda \in \mathbb{R}|$ commute pairwise for the Sugawara $\{\ldots\}^{\text{Sug}}$ and e-braket $\{\ldots\}_{e}$ with $e = (1, 0, 0, 2, 0)$, and $A = \left(\frac{\xi - \frac{\gamma}{\beta - 2\gamma}}{\beta - 2\gamma}\right)^{-2}, B = \left(\frac{\gamma}{\beta - 2\gamma}\right)^{-1}, C = \frac{||a||^2}{\beta - 2\gamma} - \lambda$.

The function $\lambda \mapsto \phi_1(A(u + B \frac{da}{dx} + C))$ has an asymptotic development. The coefficients of this development form a hierarchy. The first term of this development is $f_{\mathcal{S}}(u)$, and the second one is $f_{\mathcal{S}}(u^2 + \gamma u + \parallel a \parallel^2)$. A linear combination of these two terms gives the Hamiltonian of equations $H(u, a) = f_{\mathcal{S}}(u^2 + \parallel a \parallel^2)$.

Let $\{\phi_i, i \in I\}$ be a set of Casimir functions and $e \in \mathcal{G}$. Define $x_e = x - \chi e$, for some $\chi \in \mathbb{R}$.

Lemma 4.2. For any $(i, j) \in I^2$ and any $(\lambda, \mu) \in \mathbb{R}^2$ we have $\{\phi_i(x_\lambda), \phi_j(x_\mu)\} = (\phi_i(x_\lambda), \phi_j(x_\mu))_e = 0$.

Lemma 4.3. Suppose $\phi_i(x_\lambda)$ can be expanded in terms of inverse powers of $\lambda$ with some extra function $f(\lambda)$, and modes $F_{i,k}(x)$, i.e.,

$$\phi_i(x_\lambda) = f(\lambda) \sum_{k \in \mathbb{R}} \lambda^{-k} F_{i,k}(x),$$

then $\{F_{i,k+1}, f\}_e = \{F_{i,k}, f\}_0$. We can choose $e$ so that the Hamiltonian $H(x) = \frac{1}{2}(x, x)$ commute with these functions.

Lemma 4.4. If an element $e \in \mathcal{G}$ satisfies two conditions: (i) $ad^*(e)e = 0$; (ii) for any $u \in \mathcal{G}$, $ad^*(u)e$ belongs to the tangent space to the coadjoint orbit of $u$ (i.e., for any $u \in \mathcal{G}$ there exists $v \in \mathcal{G}$ such that $ad^*(u)e = ad^*(v)u$), then the functions $\phi(a - \lambda e)$ commute with the Hamiltonian of the geodesics $H(a) = \frac{1}{2} || a ||^2$ with respect to the brackets $\{\ldots\}_0$ and $\{\ldots\}_e$. 
5 The universal enveloping algebra of \( \mathcal{SU}(G) \)

When \( \mathcal{H} = \sum_{k \in \mathbb{Z}} \mathcal{H}_k \) has a structure of graded algebra, its universal enveloping algebra \( \mathcal{U}(\mathcal{H}) \) is also naturally endowed with a structure of a graded Lie algebra. Indeed, the weight of a product \( h_1, \ldots, h_n \in \mathcal{U}(\mathcal{H}) \) of homogeneous elements is defined to be the sum of the weights of the elements \( h_i, \ i = 1, \ldots, n \). The universal enveloping algebra \( \mathcal{U}(\mathcal{H}) \) admits a filtration \( \mathcal{U}(\mathcal{H}) = \bigcup_{n=0}^{\infty} F_n \) where \( F_n \) is the vector space generated by the products of at most \( k \) elements of \( \mathcal{H} \). The generalized enveloping algebra is the algebra of the elements of the form \( \sum_{k \in \mathbb{Z}} u_k \) where \( u_k \) is an element of weight \( k \) of \( \mathcal{U}(\mathcal{H}) \). The product of two such elements is defined by:

\[
\sum_{k_1 \leq n} u_{k_1} \cdot \sum_{k_2 \leq m} v_{k_2} = \sum_{k_3 \leq m} w_{k_3},
\]

where \( w_k = \sum_{i \in \mathbb{Z}} u_i v_{k-i} \), which is a finite sum. Let \( \omega_1, \ldots, \omega_n \) be two-cocycles on the Lie algebra \( \mathcal{H} \), let \( \tilde{\mathcal{H}} \) be the central extension associated with and let \( e_1, \ldots, e_n \) be the central elements associated with these cocycles.

The modified generalized enveloping algebra \( \mathcal{U}_{\omega_1, \ldots, \omega_n}^{\mathcal{H}} \) is defined to be the quotient of the generalized enveloping algebra of \( \tilde{\mathcal{H}} \) by the ideal generated by the elements \( \{ e_1 - 1, \ldots, e_n - 1 \} \). We denote again by \( 1 \) the neutral element of \( \mathcal{U}_{\omega_1, \ldots, \omega_n}^{\mathcal{H}} \). The algebra \( \mathcal{U}_{\omega_1, \ldots, \omega_n}^{\mathcal{H}} \) is by construction a graded algebra and a filtered algebra. We denote by \( F_n, \ n \in \mathbb{N} \) its filtration. Let us recall shortly the main properties of the modified generalized enveloping algebra. Let \( V \) be a module over \( \tilde{\mathcal{H}} \) such that for any \( v \in V \), there exists \( n_0 \in \mathbb{Z} \) such that for any \( n > n_0 \) and any \( h \) in \( \tilde{\mathcal{H}}_n \) we have \( h.v = 0 \). Such modules are called representations of positive energy, and \( e_1 \) acts on \( V \) by \( \lambda_1 \cdot 1 \). Then \( V \) is a module over \( \mathcal{U}_{\omega_1, \ldots, \omega_n}^{\mathcal{H}} \). Such modules are named modules of positive energy. The anticommutator provides a structure of Lie algebra on \( \mathcal{U}_{\omega_1, \ldots, \omega_n}^{\mathcal{H}} \). For this bracket \( F_1 \) is a Lie sub-algebra isomorphic to the central extension of \( \mathcal{H} \) by the cocycle \( \omega = \sum_{i=1}^{n} \omega_i \). We denote by \( i \) be the natural inclusion of \( \tilde{\mathcal{H}} \) into \( \mathcal{U}_{\omega_1, \ldots, \omega_n}^{\mathcal{H}} \) given by this identification.

5.1 Decomposition of the enveloping algebra of a semi-direct product

In some very particular cases, the modified generalized enveloping algebra of a semi-direct product \( \mathcal{K} \ltimes \mathcal{H} \) of two Lie algebras is isomorphic to the tensor product of some modified generalized enveloping algebras of \( \mathcal{K} \) and of \( \mathcal{H} \). Let \( \tilde{\mathcal{H}} \) be the central extension of \( \mathcal{H} \) with the two-cocycle \( \omega_\mathcal{H} \). Denote by \( \cdot \) the action of the Lie algebra \( \mathcal{K} \) on the Lie algebra \( \mathcal{H} \). Let us introduce the semi-direct product \( \mathcal{K} \ltimes \mathcal{H} \) which is a central extension of \( \mathcal{K} \ltimes \mathcal{H} \) by a two-cocycle \( \omega_{\mathcal{K}_\mathcal{H}} \) with

\[
\omega_{\mathcal{K}_\mathcal{H}}((0, h_1), (0, h_2)) = \omega_\mathcal{H}(h_1, h_2).
\]

A two-cocycle \( \omega_{\mathcal{K}} \) on \( \mathcal{K} \) defines also a two-cocycle \( \omega_{\mathcal{K}_\mathcal{H}}' \) by

\[
\omega_{\mathcal{K}_\mathcal{H}}'(g_1 h_1, g_2 h_2) = \omega_{\mathcal{K}}(g_1, g_2),
\]

of \( \mathcal{K} \ltimes \mathcal{H} \). Let \( I \) be the natural inclusion of \( \tilde{\mathcal{H}} \) into \( \mathcal{U}_{\omega_\mathcal{H}}(\mathcal{H}) \) and \( J \) be the natural inclusion of \( \tilde{\mathcal{H}} \) into \( \mathcal{U}_{\omega_{\mathcal{K}_\mathcal{H}}}(\mathcal{K} \ltimes \mathcal{H}) \).

We call the action of \( \mathcal{K} \) on \( \mathcal{H} \) realizable in \( \mathcal{U}_{\omega_\mathcal{H}}(\mathcal{H}) \) when there exists a map \( F : \mathcal{K} \rightarrow \mathcal{U}_{\omega_\mathcal{H}}(\mathcal{H}) \) and a two-cocycle \( \alpha \) on \( \mathcal{K} \) such that for any pair \( (g_1, g_2) \) in \( \mathcal{K}^2 \)

\[
F ([g_1, g_2]) = [F(g_1), F(g_2)] + \alpha(g_1, g_2) 1,
\]

and the map \( F \) satisfies the compatibility condition, i.e., for any \( g \in \mathcal{K} \) and \( h \in \tilde{\mathcal{H}} \) with the anti-commutator \([F(g), I(h)] = I(g \cdot h)\), of the algebra \( \mathcal{U}_{\omega_\mathcal{H}}(\mathcal{H}) \).

Theorem 5.1. If the action of \( \mathcal{K} \) is realizable in \( \mathcal{U}_{\omega_\mathcal{H}}(\mathcal{H}) \) then

\[
\mathcal{U}_{\omega_{\mathcal{K}_\mathcal{H}}}(\mathcal{K} \ltimes \mathcal{H}) = \mathcal{U}_{\omega_{\mathcal{K}} - \alpha}(\mathcal{K}) \odot \mathcal{U}_{\omega_\mathcal{H}}(\mathcal{H}).
\]
Proof. Let $U_\theta = \{ g : g \in K \}$ with be the unitary subalgebra of $U_{\omega_K \omega_M}(K \ltimes H)$ generated by the elements $g = F(g)$, and $U_\theta = \{ j(h), h \in H \}$ be the unitary subalgebra of $U_{\omega_K \omega_M}(K \ltimes H)$. For any $(g, h)$ this implies that the generators of $U_\theta$ and $U_\theta$ commute, i.e., $[\overline{g}, j(h)] = 0$. The subalgebras $U_\theta$ and $U_\theta$ therefore commute. The subalgebra $U_\theta$ is isomorphic to $U_{\omega_K \omega_M}(K \ltimes H)$. Let us check that the generators $g$ of this algebra satisfy the relations of the generators of $U_{\omega_K \omega_M}(K \ltimes H)$:

$$[\overline{g}, \overline{g}] = [g_1, g_2] + \omega_K(g_1, g_2)1 + [F(g_1), F(g_2)] - [F(g_1), g_2] - [g_1, F(g_2)].$$

Since $F(g_1)$ is an element of $U_\theta$ and since the algebras $U_\theta$ and $U_\theta$ commute $[F(g_1), g_2] = [F(g_1), F(g_2)]$ and $[g_1, F(g_2)] = [F(g_1), F(g_2)]$. Therefore:

$$[\overline{g}, \overline{g}] = [g_1, g_2] + \omega_K(g_1, g_2)1 - [F(g_1), F(g_2)],$$

and finally

$$[\overline{g}, \overline{g}] = [g_1, g_2] - F([g_1, g_2]) + \left(\omega_K(g_1, g_2) - \alpha(g_1, g_2)\right)1.$$

The subalgebra $U_\theta$ is obviously isomorphic to $U_{\omega_K}(H)$. The generalized modified enveloping algebra $U_{\omega_K + \omega_M}(K \ltimes H)$ is therefore isomorphic to the tensor product over $C$ of $U_{\omega_K}(K \ltimes H)$ with $U_{\omega_K}(H)$. □

5.2 The case of $SU(C)$

Let $G$ be a simple complex Lie algebra and $C_\varphi$ its dual Coxeter number. Introduce the $\{ K_1, \ldots, K_n \}$ a basis of $G$, and the dual basis $\{ K^1, \ldots, K^n \}$ with respect to the Killing form $\langle, \rangle$. We apply Theorem 5.1 for $K = Vect(S^1)$, $H = L_G$, $\omega_K = \epsilon \omega_V$, and $\omega_H = \beta \omega_M$. In this case, $\omega^*_H = \beta \omega_M$. For $\eta = \beta + C_\varphi \neq 0$, the Sugawara construction, delivers a map $F : Vect(S^1) \to U_{\omega_K}(L_G \otimes C)$ defined by

$$(\beta + \eta)F(L_n) = K \cdot K^*,$$

where

$$K \cdot K^* = \sum_{i \in Z, j = 1, \ldots, n} : (K_i)(K^j)_{n-i} ;,$$

(here dots denote the normal ordering), i.e., the action of $Vect(S^1)$ is realizable in $U_{\beta \omega_K \omega_M}(L_G)$, with $\alpha = \beta \omega_V / 12\eta$. Thus we obtain

Proposition 5.2. If $\eta \neq 0$, then $\xi_{\omega_V, \omega_K \omega_M}(SU_C) \simeq U_{\beta \omega_K \omega_M}(Vect(S^1)) \otimes U(L_G)$. The Lie algebra $Vect(C)$ acts on the Heisenberg algebra by

$$L_n a_m = m a_{n+m} + \delta_{n-m} m^2 c_{K-M}.$$ 

In this case, on has $\omega^*_H = \beta \omega_H + \gamma \omega_M$. The map $F : Vect(S^1) \to SU(C)$ defined by

$$\beta F(L_n) = \frac{1}{2} \sum_{i \in Z} : a_i a_{n-i} ; + \gamma a_n,$$

for a cocycle $\alpha = (\alpha + \gamma \beta^{-1}) \omega_V$. For $SU(C)$ we obtain

Proposition 5.3. For $\beta \neq 0$, we have

$$U_{\xi_{\omega_V, \beta \omega_K \omega_M, \gamma \omega_M}}(SU(C)) \simeq U_{\beta \omega_V}(Vect(S^1)) \otimes U_{\omega_K \omega_M}(L_G),$$

with $\theta = \xi - \gamma^2 / \beta - 1/12$. 
5.3 Representations of $\mathcal{SU}(G)$

Proposition 5.4. A positive energy representation $V$ of $\mathcal{SU}_c(G)$ with non-vanishing $\beta\text{Id}$-action of the cocycle $c_{k-M}$ brings about a pair of commuting representations of Virasoro and affine Kac–Moody Lie algebras.

This proposition determines whether a $\mathcal{SU}_c(G)$ Verma module is a sub-module of another Verma module of $\mathcal{SU}_c(G)$. Let $\mathfrak{h}$ be a Cartan algebra of $G$ with a basis $\{h_1, \ldots, h_k\}$. The Lie subalgebra $\mathfrak{t}$ of $\mathcal{SU}_c(G)$ is generated by the elements $\{c_{\text{Vir}}, c_{K-M}, u_0, (h_1)_0, \ldots, (h_k)_0\}$. A Verma module $V_\lambda(\mathcal{SU}_c(G))$ of $\mathcal{SU}_c(G)$ is associated to any linear form $\lambda \in \mathfrak{h}^*$. Verma modules $V_{\nu}^{\text{Vir}}$, $V^{K-M}_\mu$, are associated to linear forms $\nu$, $\mu$ over the spaces generated by $c_{\text{Vir}}$ and $u_0$, $c_{K-M}$ and $\{(h_1)_0, \ldots, (h_k)_0\}$ correspondingly. For any $\lambda \in \mathfrak{t}^*$, the Verma module $V_\lambda(\mathcal{SU}_c(G))$ is a positive energy representation. Thus, $V_\lambda(\mathcal{SU}_c(G))$ is Virasoro and affine Kac–Moody algebra module. The generator $e$ of $V_\lambda(\mathcal{SU}_c(G))$ brings about a Verma module $V_{\nu}^{\text{Vir}}$ for Virasoro algebra. It generates also a Verma module $V_{\nu}^{\text{Vir}}$ for the affine Kac–Moody algebra. The linear form $\nu$ satisfies $\nu(u_0) = \lambda(u_0 - F(u_0)) e$, i.e.,

$$ (u_0 - (\beta + \eta)^{-1} K \cdot K^* e = \nu(u_0) e. $$

Suppose the action of a Casimir element of $G$ is given by acts by $D(\lambda)\text{Id}$ for $D(\lambda) \in \mathbb{C}$. We then have

$$ (u_0 - (\beta + \eta)^{-1} K \cdot K^* e = (u_0 - (\beta + \eta)^{-1} \sum_{j=1}^{n} : (K_j)_0 (K_j^*)_0 : ). e, $$

$$ (\lambda(u_0) - \frac{D(\lambda)}{2\eta}). e. $$

This implies $\nu(u_0) = \lambda(u_0) - \frac{D(\lambda)}{2\eta}$. The other values of $\mu$ and $\nu$ can be computed by the same method.

Proposition 5.5. Let $\lambda$ be a linear form over $\mathfrak{h}$ with non-vanishing $\lambda(c_{K-M})$. Then

$$ V_\lambda(\mathcal{SU}_c(G)) = V^{\text{Vir}}_\nu \otimes V^{K-M}_\mu, $$

where $\mu(e_i) = \lambda(e_i)$, $i = 1, \ldots, n$, defines $\mu$, $\mu(c_{K-M}) = \lambda(c_{K-M})$, and $\nu(c_{\text{Vir}}) = \lambda(c_{\text{Vir}}) - \frac{\beta}{12\eta}$ defines $\nu$,

$$ \nu(u_0) = \lambda(u_0) - \frac{D(\lambda)}{2\eta}. $$

References

[1] Zuevsky A., Hamiltonian structures on coadjoint orbits of semidirect product $G = \text{Diff}_s(S^1) \ltimes C^\infty(S^1, \mathbb{R})$. Czechoslovak J. Phys., 2004, 54, no. 11, 1399-1406