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Algebraic proofs for shallow water bi–Hamiltonian systems for three cocycle of the semi-direct product of Kac–Moody and Virasoro Lie algebras

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Abstract: We prove new theorems related to the construction of the shallow water bi-Hamiltonian systems associated to the semi-direct product of Virasoro and affine Kac–Moody Lie algebras. We discuss associated Verma modules, coadjoint orbits, Casimir functions, and bi-Hamiltonian systems.

Keywords: Affine Kac–Moody Lie algebras, Bi-Hamiltonian systems, Verma modules, Coadjoint orbits

MSC: 17B69, 17B08, 70G60, 82C23

1 Introduction: The semi-direct product of Virasoro algebra with the Kac–Moody algebra

This paper is a continuation of the paper [1] where we studied bi-Hamiltonian systems associated to the three-cocycle extension of the algebra of diffeomorphisms on a circle. In this note we show that certain natural problems (classification of Verma modules, classification of coadjoint orbits, determination of Casimir functions) [2–5] for the central extensions of the Lie algebra $\text{Vect}(S^1) \ltimes \mathcal{L}G$ reduce to the equivalent problems for Virasoro and affine Kac–Moody algebras (which are central extensions of $\text{Vect}(S^1)$ and $\mathcal{L}G$ respectively). Let $G$ be a Lie group and $\mathcal{G}$ its Lie algebra. The group $\text{Diff}(S^1)$ of diffeomorphisms of the circle is included in the group of automorphisms of the Loop group $\mathcal{L}G$ of smooth maps from $S^1$ to $G$. For any pairs $(\phi, \psi) \in \text{Diff}(S^1)^2$ and $(g, h) \in \mathcal{L}G^2$ the composition law of the group $\text{Diff}(S^1) \ltimes \mathcal{L}G$ is

$$(\phi, a) \cdot (\psi, b) = (\phi \circ \psi, a \cdot b \circ \phi^{-1}).$$

The Lie algebra of $\text{Diff}(S^1) \ltimes \mathcal{L}G$ is the semi-direct product $\text{Vect}(S^1) \ltimes \mathcal{L}G$ of the Lie algebras $\text{Vect}(S^1)$ and $\mathcal{L}G$.

Let $\mathcal{G}$ be a Lie algebra and $\langle \cdot, \cdot \rangle$ a non-degenerated invariant bilinear form. $\text{Vect}(S^1)$ is the Lie algebra of vector fields on the circle and $\mathcal{L}G$ the loop algebra (i.e., the Lie algebra of smooth maps from $S^1$ to $\mathcal{G}$). $\text{Vect}(S^1)_C$ is the Lie algebra over $\mathbb{C}$ generated by the elements $L_n, n \in \mathbb{Z}$ with the relations

$$[L_m, L_n] = (n - m)L_{n+m}.$$ 

We denote by $\mathcal{L}G_C$ the Lie algebra over $\mathbb{C}$ generated by the elements $g_n, n \in \mathbb{Z}, g \in \mathcal{G}$ where $(\lambda g + \mu h)_n$ is identified with $\lambda g_n + \mu h_n$ with the relations

$$[g_n, h_m] = [g, h]_{n+m}.$$ 

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The semi-direct product of $\text{Vect}(S^1)$ with $\mathcal{L}G$ is as a vector space isomorphic to $C^\infty(S^1, \mathbb{R}) \oplus C^\infty(S^1, \mathcal{G})$ [6].

The Lie bracket of $\mathcal{SU}(\mathcal{G})$ has the form

$$\left[(u, a), (v, b)\right] = ([.,\partial_\cdot].u \otimes v, va' - ub' + [a, b]),$$

for any $(u, v) \in C^\infty(S^1, \mathbb{R})^2$ and any $(a, b) \in C^\infty(S^1, \mathcal{G})^2$, where prime denote derivative with respect to a coordinate on $S^1$. The Lie algebra $\text{Vect}(S^1) \ltimes \mathcal{L}G$ can be extended with a universal central extension $\widetilde{\mathcal{SU}}(\mathcal{G})$ by a two-dimensional vector space. Let us denote by $\mathcal{SU}(\mathcal{G})$ the Lie algebra $\widetilde{\mathcal{SU}}(\mathcal{G})$.

We denote by $\mathcal{SU}(\mathcal{G})$ with the Affine Kac–Moody Lie algebra. Let us denote by $\mathcal{SU}(\mathcal{G})$ and $\mathcal{SU}(\mathcal{G}^\prime)$ the third independent cocycle is given by $\omega_{\mathcal{SU}}((u, a), (v, b)) = \mathcal{J}(u''v)$. We denote by $\omega_{\mathcal{SU}}((u, a), (v, b)) = \mathcal{J}((a', b))$.

We denote by $(u, a, \chi, \alpha)$ the elements of $\mathcal{SU}(\mathcal{G})$ with $u \in C^\infty(S^1, \mathbb{R})$, $a \in C^\infty(S^1, \mathcal{G})$, and $(\chi, \alpha) \in \mathbb{R}^2$. The algebra $\mathcal{SU}(\mathcal{G})$ can be also represented as the semi-direct product of Virasoro algebra on the affine Kac–Moody algebra. We denote by $c_{\mathcal{Vir}}$ and $c_{\mathcal{K-M}}$ the elements $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$ respectively, if $\mathcal{G} = \mathbb{R}$, then the Lie algebra $\text{Vect}(S^1) \ltimes \mathcal{L}R$ has a universal central extension $\widetilde{\mathcal{SU}}(\mathbb{R})$ by a three-dimensional vector space. The third independent cocycle is given by $\omega_{\mathcal{SU}}((u, a), (v, b)) = \mathcal{J}(ub'' - va'')$.

We denote by $(u, a, \chi, \alpha, \omega, \gamma, \delta)$ elements of $\mathcal{SU}(\mathcal{G})$ with $u \in C^\infty(S^1, \mathbb{R})$, $a \in C^\infty(S^1, \mathcal{G})$, and $(\chi, \alpha, \gamma) \in \mathbb{R}^3$. The Lie bracket of $\widetilde{\mathcal{SU}}(\mathbb{R})$ is given by

$$\left[(u, a, \phi, \alpha, \gamma), (v, b, \xi, \beta, \delta)\right] = (uv' - u'v, [a, b] - ub' + va', \mathcal{J}(u''v), \mathcal{J}((a', b)), \mathcal{J}(ub'' - va'')).$$

In this paper we discuss a few questions. Let us mention the main results. First, in Section 2 we consider Kirillov-Kostant Poisson brackets [7] of the regular dual of the semi-direct product of Virasoro algebra with the Affine Kac–Moody Lie algebra. Let us denote by $\mathcal{SU}(\mathcal{G})^\prime$ the subset of $\mathcal{SU}(\mathcal{G})$ of elements $(u, a, \chi, \alpha)$ with non-vanishing $\beta$. We denote by $\mathcal{SU}(\mathcal{G})^\prime \subset \text{Vect}(S^1) \otimes \mathcal{L}G$ composed of elements $(u, a, \chi, \alpha, \omega, \gamma, \delta)$ with $\beta \neq 0$. Then introduce two new maps $\mathcal{I}(u, a, \chi, \alpha, \omega, \gamma, \delta) : \mathcal{SU}(\mathcal{G})^\prime \rightarrow (\text{Vect}(S^1) \otimes \mathcal{L}G)^\prime$, and $\mathcal{I}(u, a, \chi, \alpha, \omega, \gamma, \delta) : \mathcal{SU}(\mathcal{G})^\prime \rightarrow \mathcal{SU}(\mathcal{G})^\prime \otimes \mathcal{L}R$. We prove that $\mathcal{I}(u, a, \chi, \alpha, \omega, \gamma, \delta)$ and $\mathcal{I}(u, a, \chi, \alpha, \omega, \gamma, \delta)$ are Poisson maps. In Section 3 we discuss coadjoint orbits and Casimir functions for $\mathcal{SU}(\mathcal{G})$. Let $\mathcal{H}$ be a central extension of a Lie algebra $\mathcal{H}$ and $H$ be a Lie group with Lie algebra is $\mathcal{H}$. We find explicit form for the the coadjoint actions of the groups $\text{Diff}(S^1) \ltimes \mathcal{L}G$ and $\text{Diff}(S^1) \ltimes \mathcal{L}R$. As a result we obtain the following new theorem. We prove that a coadjoint orbit of $\mathcal{SU}(\mathcal{G})$ is mapped by $\mathcal{I}$ to a coadjoint orbit of $V(\mathcal{SU}(\mathcal{G})) \otimes \mathcal{L}G$ to a coadjoint orbits of $\text{Vect}(S^1) \otimes \mathcal{L}G$. We prove that the map $\mathcal{I}$ sends the coadjoint orbits of $\mathcal{SU}(\mathcal{G})$ to coadjoint orbits of $\mathcal{SU}(\mathcal{G})^\prime \otimes \mathcal{L}G$. Previously, we determined Casimir functions on $\mathcal{SU}(\mathcal{G})^\prime$ and $\mathcal{SU}(\mathbb{R})$. We then prove new propositions concerning the explicit form of Casimir functions on $\mathcal{SU}(\mathcal{G})^\prime$, and in particular on on $\mathcal{SU}(\mathbb{R})$. This paper was partially inspired by the construction of bi-Hamiltonian systems as natural generalization of the classical Korteweg-de Vries equation. [1, 8–11]. It has been showed in [1], that the dispersive water waves system equation [9, 10, 12] is a bi–Hamiltonian system related to the semi-direct product of a Kac–Moody and Virasoro Lie algebras, and the hierarchy for this system was found. In Section 4 some results of [1] are obtained from another point of view. We prove new proposition for pairwise commuting functions under certain brackets. In section 5 we discuss properties of the universal enveloping algebra of $\mathcal{SU}(\mathcal{G})$. In subsection 5.1 we consider a decomposition of the enveloping algebra of a semi-direct product. We introduce the notion of realizability of the action of $\mathcal{K}$ on $H$ in $\mathcal{U}_{\omega_{\mathcal{K}}}(\mathcal{H})$. Then we show (Theorem 5.1) that the realizability of the action of $\mathcal{K}$ in $\mathcal{U}_{\omega_{\mathcal{K}}}(\mathcal{H})$ leads to the isomorphism

$$\mathcal{U}_{\omega_{\mathcal{K}}}(\mathcal{K} \ltimes \mathcal{H}) \cong \mathcal{U}_{\omega_{\mathcal{K}}}(\mathcal{K}) \otimes \mathcal{U}_{\omega_{\mathcal{K}}}(\mathcal{H}).$$

In subsection 5.2 the case of $\mathcal{SU}(\mathcal{G})$ is considered. In subsection 5.3 we discuss representations of $\mathcal{SU}(\mathcal{G})$. We prove that positive energy representation $V$ of $\mathcal{SU}(\mathcal{G})$ with non-vanishing $\beta \partial_\cdot$ action of the cocycle $c_{\mathcal{K-M}}$ delivers a pair of commuting representations of Virasoro and affine Kac–Moody Lie algebras. This proposition determines whether a $\mathcal{SU}(\mathcal{G})$ Verma module is a sub-module of another Verma module of $\mathcal{SU}(\mathcal{G})$. We also prove a proposition regarding a linear form over $\mathfrak{h}$ with non-vanishing $\lambda(c_{K-M})$. In this paper we present proofs for corresponding theorems and lemmas.
2 The Kirillov-Kostant structure of $SU(G)$

Now we consider Kirillov-Kostant Poisson brackets of the regular dual of the semi-direct product of Virasoro Lie algebra with the Affine Kac–Moody Lie algebra. Let $K$ be a Lie algebra with a non-degenerated bilinear form $(\cdot, \cdot)$. A function $f : K \to \mathbb{R}$ is called regular at $x \in K$ if there exists an element $\nabla f(x)$ such that

$$f(x + \epsilon a) = f(x) + \epsilon (\nabla f(x), a) + o(\epsilon),$$

for any $a \in K$. For two regular functions $f, g : K \to \mathbb{R}$, we define the Kirillov-Kostant structure as a Poisson structure on $K$ with

$$\{f, g\} (x) = \langle x, [\nabla f(x), \nabla g(x)] \rangle.$$

Then for any $e \in G$, the second Poisson structure $\{f, g\}_e (x)$ compatible with the Kirillov-Kostant Poisson structure is defined by

$$\{f, g\}_e (x) = \langle e, [\nabla f(x), \nabla g(x)] \rangle.$$

A non-degenerated bilinear form on $SU(G)$ and $\text{Vect}(\mathcal{S}^1) \oplus \mathcal{L}G$ is defined by

$$\langle (u_1, a_1, \beta_1, \xi_1), (u_2, a_2, \beta_2, \xi_2) \rangle = \int_S u_1 u_2 + \int_S \{a_1, a_2\} + \xi_1 \xi_2 + \beta_1 \beta_2.$$ 

We denote by $SU(G)'$ the subset of $SU(G)$ of elements $(u, a, \xi, \beta)$ with non-vanishing $\beta$. Let $u = u - \frac{1}{2\beta^2}$. We denote by $(\text{Vect}(\mathcal{S}^1) \oplus \mathcal{L}G)'$ the subset of $\text{Vect}(\mathcal{S}^1) \oplus \mathcal{L}G$ composed of elements $(u, a, \xi, \beta)$ with $\beta \neq 0$. Let us introduce a new map $\mathcal{I}(u, a, \xi, \beta) = (u', a', \xi, \beta)$ from $SU(G)'$ to $(\text{Vect}(\mathcal{S}^1) \oplus \mathcal{L}G)'$. Then for non-vanishing $\beta$, let us introduce another new map $\mathcal{I}(u, a, \xi, \beta, \gamma) = (u' - \frac{1}{\beta^2}, a, \xi - \frac{2}{\beta}, \beta)$ from $SU(G)$ to $\text{Vect}(\mathcal{S}^1) \oplus \mathcal{L}R$. Here we give a proof for the following new theorem:

**Theorem 2.1.** $\mathcal{I}$ and $\mathcal{I}$ are Poisson maps.

**Proof.** For any regular function $f(u, a, \xi, \beta)$ from $\text{Vect}(\mathcal{S}^1) \oplus \mathcal{L}G$ to $\mathbb{R}$ let us define a regular function $\tilde{f}$ from $SU(G)'$ to $\mathbb{R}$ by $\tilde{f}(u, a, \xi, \beta) = f(u', a', \xi, \beta)$. For $f(u, a, \xi, \beta)$ a function on $SU(G)$ or $(\text{Vect}(\mathcal{S}^1) \oplus \mathcal{L}G)$, let us denote by $f_a$ the function of the variables $u$ and $\xi$ that we get when we fix $a$ and $\beta$. With the previous notations, one has for $\beta \neq 0$ for the bracket $(\cdot, \cdot)_{SU(G)}$

$$\{f, g\}_{SU(G)} (u, a, \xi, \beta) = \left[ (f, u_a)_{SU(G)} + (f, g_u)_{SU(G)} + (f, g_a)_{SU(G)} + (f, g_u)_{SU(G)} \right] (u, a, \xi, \beta),$$

and for the bracket $(\cdot, \cdot)_{\text{Vect}(\mathcal{S}^1) \oplus \mathcal{L}G}$ we have

$$\{f, g\}_{\text{Vect}(\mathcal{S}^1) \oplus \mathcal{L}G} (u, a, \xi, \beta) = (f, u_a)_{\text{Vect}(\mathcal{S}^1) \oplus \mathcal{L}G} + (f, g_u)_{\text{Vect}(\mathcal{S}^1) \oplus \mathcal{L}G}.$$

Then the map $\pi_1$ from $SU(G)$ onto $\text{Vect}(\mathcal{S}^1)$ which sends $(u, a, \xi, \beta)$ onto $(u', \xi)$ is a Poisson morphism. The map $\pi_2$ from $SU(G)$ onto $\mathcal{L}G$ which sends $(u, a, \xi, \beta)$ to $(a, \beta)$ is a Poisson morphism. For any regular function $f$ on $\text{Vect}(\mathcal{S}^1)$ and any regular function $g$ on $\mathcal{L}G$ we have

$$\{\pi_1 f, \pi_2 g\} = 0.$$

Indeed, for $i = 1, 2$, $(\frac{\delta a}{\beta^i} u_a)f_i(u, \xi) = 0$. We have:

$$\{f_1(u, \xi), f_2(u, \xi)\}_{\xi, \beta} (u, a, \xi, \beta) = \mathcal{J}([\xi, \delta f_1(u, \xi), \xi, \delta f_2(u, \xi)] + 2(\delta f_1(u, \xi)) a \parallel \delta f_2(u, \xi) u$$

$$+ \delta f_{1, u}(\xi, \xi) a \parallel \delta f_{2, u}(\xi, \xi) - \beta^{-1}(\delta f_{1, u}(\xi, \xi) a) \parallel \delta f_{2, u}(\xi, \xi)$$

$$- (\delta f_{1, u}(\xi, \xi)) a \parallel \delta f_{2, u}(\xi, \xi) + \beta^{-1}(\delta f_{1, u}(\xi, \xi)) a \parallel \delta f_{2, u}(\xi, \xi) a).$$

This gives

$$\{f_1(u, \xi), f_2(u, \xi)\}_{\xi, \beta} (u, a) = \mathcal{J}([\xi, \delta f_{1, u}(\xi, \xi)]) a \parallel \delta f_{2, u}(\xi, \xi)$$
Proposition 3.1. The coadjoint actions of the groups \( H \)
Let \( g_1(\alpha, \beta) \), \( i = 1, 2 \) be two regular functions on the affine Kac-Moody algebra. One notes that \( \delta g_{1, u} = \delta g_{2, u} = 0 \). Therefore,

\[
\{g_1, g_2\} \xi, \beta(u, a) = \beta J((dx(\delta^1 g_1(a, \beta)), \delta^2 g_2(a, \beta)) + ([a, \delta^1 g_1(a, \beta)], \delta^2 g_2(a, \beta))).
\]

Then,

\[
\{g_1, g_2\} \xi, \beta(u, a) = \{f, g\}_{\xi, \beta}(a, \beta).
\]

We have:

\[
\{f(\bar{\alpha}, \xi), g(a, \beta)\} = \mathcal{J}((\delta f_0(a, \beta), \delta g, a(a, \beta))- \beta dx(\delta f_0(\bar{\alpha}, \xi)a), \delta g, a(a, \beta)) + ([a, \delta f_0 a], \delta g)).
\]

The sum of the first two terms is equal to 0. The last term is \( \mathcal{J}([a, a], \delta g) \), and is equal to zero. One can proceed similarly for \( \bar{T} \).

\[\square\]

3 Coadjoint orbits Casimir functions and for \( SU(\mathcal{G}) \)

Let \( \widetilde{\mathcal{H}} \) be a central extension of a Lie algebra \( \mathcal{H} \), and \( H \) be a Lie group with Lie algebra is \( \mathcal{H} \). Then \( H \) acts on \( \widetilde{\mathcal{H}}^* \) by the coadjoint action along coadjoint orbits.

Proposition 3.1. The coadjoint actions of the groups \( Diff(S^1) \times LG \) and \( Diff(S^1) \times L\mathbb{R}^* \) are given by

\[
Ad^*(\phi, g)^{-1}(u, a, \xi, \beta) = (\phi \circ \phi)^{-2} + \xi S(\phi) + (g^{-1} g', a) \phi^{-2} + \frac{1}{2} \beta \| g^{-1} \| ^2, \phi' Ad(g^{-1} a \circ \phi + \beta g^{-1} g', \xi, \beta),
\]

\[
((u \circ \phi)^{-2} + \xi S(\phi) + (g' g^{-1}, a) \phi^{-2} + \frac{1}{2} \beta (g' g^{-1})^2 + \gamma g'' g^{-1}, \phi' Ad(g^{-1} a \circ \phi + \beta g^{-1} g' - \gamma g'' g^{-1}, \xi, \beta, \gamma).
\]

The classification of coadjoint orbits of \( Vect(S^1) \times LG \) can be known from the classification of coadjoint orbits of the Virasoro and affine Kac-moody algebra. Here we obtain the following new

Theorem 3.2. A coadjoint orbit of \( SU(\mathcal{G}) \) is mapped by \( T \) to a coadjoint orbit of \( Vect(S^1) \otimes \widetilde{\mathcal{G}} \) to a coadjoint orbits of \( Vect(S^1) \).

In other words, this means that if \( \beta_1 \neq 0 \), the elements \( (u_1, a_1, \xi_1, \beta_1) \) and \( (u_1, a_1, \xi_2, \beta_2) \) are in the same coadjoint orbit if and only if: \( \xi_1 = \xi_2, \beta_1 = \beta_2, (a_1, \beta_1) \) and \( (a_2, \beta_2) \) are on the same coadjoint orbit of \( \widetilde{\mathcal{G}}, (u_1 - \frac{\| a_1 \|}{2\beta_1^2}, \xi_1) \) and \( (u_2 - \frac{\| a_2 \|}{2\beta_2^2}, \xi_2) \) are elements of the same coadjoint orbit of \( Vect(S^1) \).

Proof. For any \( \phi \in Diff(S^1) \), there exists \( h \in LG \) such that

\[
hah^{-1} + \beta \frac{\partial h(x)}{\partial x} . h^{-1} = a \circ \phi \circ \phi'.
\]

By direct computation we check that

\[
T(Ad^*(\phi, g)(u, a, \xi, \beta) = (Ad^*(\phi, g.h)T(u, a, \xi, \beta).
\]

This implies Theorem 3.2.
Proposition 3.3. The map $\mathcal{T}$ sends the coadjoint orbits of $\mathcal{SU}(G)$ to coadjoint orbits of $\text{Vect}(S^1) \otimes \mathcal{L}G$.

In other words, this means that if $\beta_1 \neq 0$ the elements $(u_1, a_1, \xi_1, \beta_1, \gamma_1)$ and $(u_1, a_1, \xi_2, \beta_2, \gamma_2)$ are in the same coadjoint orbit if and only if $\gamma_1 = \gamma_2$, $\xi_1 = \xi_2$, $\beta_1 = \beta_2$, $(a_1, \beta_1)$ and $(a_2, \beta_2)$ are on the same coadjoint orbit of $\mathcal{L}G$, $(u_1 - \frac{a_1^2}{2\beta_1}, \xi_1 - \frac{\gamma_1}{\beta_1})$ and $(u_2 - \frac{a_1^2}{2\beta_2}, \xi_2 - \frac{\gamma_2}{\beta_2})$ are elements of the same coadjoint orbit of $\text{Vect}(S^1)$. In a particular case, if $\beta_1 = \beta_2 = 0$, then:

Proposition 3.4. If the elements $(u_1, a_1, \xi_1, \beta_1, \gamma_1)$ and $(u_1, a_1, \xi_2, \beta_2, \gamma_2)$ are in the same coadjoint orbit then $\gamma_1 = \gamma_2$, $(a_1^2 + \gamma_1 a_1')$ and $(a_2^2 + \gamma_2 a_2')$ are in the same coadjoint orbit of the Virasoro Lie algebra.

Proof. We have: $\text{Ad}(\phi, g)(a_1^2 + \gamma_1 a_1') = (a_1^2 + \gamma_1 a_1') \circ \phi + \gamma_1 S(\phi)$.

Previously, we determined Casimir functions on $\mathcal{SU}(G)'$ and $\mathcal{SU}(\mathbb{R})$. We gave the following proposition:

Proposition 3.5. Let $C_{\text{Vir}}, C_{\text{K–M}}, C_A$ be Casimir functions for Virasoro, affine Kac–Moody, and the Heisenberg Lie algebras $A$ correspondingly. Let $\mathcal{SU}(G)'$, $\mathcal{SU}(\mathbb{R})$ be Poisson submanifolds of $\mathcal{SU}(G)$ and $\mathcal{SU}(\mathbb{R})$ defined by $\xi = 0$. Then the functions $C_{\text{Vir}}(u', \xi), C(u, a, \beta, \xi) = C_{\text{K–M}}(a, \beta)$, and $\int_{S^1} |u'|^{1/2}$, are Casimir functions on $\mathcal{SU}(G)'$. In particular, the functions $C_A(u, a, \beta, \xi) = C_A(a, \beta)$, $C_{\text{Vir}}(u' - \frac{\gamma a'}{\beta}, \xi)$, and $\int_{S^1} |u' - \frac{\gamma a'}{\beta}|^{1/2}$, are Casimir functions on $\mathcal{SU}(\mathbb{R})$.

4 Bi–hamiltonian dispersive water waves systems associated to $\mathcal{SU}(G)$

It has been showed in [1], that the dispersive water waves system equation [9, 10, 12] is a bi–Hamiltonian system related to the semi-direct product of a Kac–Moody and Virasoro Lie algebras, and the hierarchy for this system was found. In this section some results of [1] are obtained from another point of view. We obtain new

Proposition 4.1. The functions $\{ \phi_1(A(u + B \frac{da}{dt} + C)) | \lambda \in \mathbb{R} \}$ commute pairwise for the Sugawara $\{., .\}_{\text{Sug}}$ and e-braket $\{., .\}_e$ with $e = (1, 0, 0, 2, 0)$, and $A = \left( \xi - \frac{\gamma}{\beta - 2\lambda} \right)^2, B = -\frac{\gamma}{\beta - 2\lambda}, C = -\frac{|a|^2}{\beta - 2\lambda} - \lambda$.

The function $\lambda \mapsto \phi_1(A(u + B \frac{da}{dt} + C))$ has an asymptotic development. The coefficients of this development form a hierarchy. The first term of this development is $\int_{S^1} u$, and the second one is $\int_{S^1} (u^2 + \gamma u + \|a\|^2)$. A linear combination of these two terms gives the Hamiltonian of equations $H(u, a) = \int_{S^1} (u^2 + \|a\|^2)$.

Let $\{\phi_i, i \in I\}$ be a set of Casimir functions and $e \in G$. Define $x_\chi = x - \chi e$, for some $\chi \in \mathbb{R}$.

Lemma 4.2. For any $(i, j) \in I^2$ and any $(\lambda, \mu) \in \mathbb{R}^2$ we have $\{\phi_i(x_\lambda), \phi_j(x_\mu)\}_e = \{\phi_i(x_\lambda), \phi_j(x_\mu)\}_e = 0$.

Lemma 4.3. Suppose $\phi_i(x_\lambda)$ can be expanded in terms of inverse powers of $\lambda$ with some extra function $f(\lambda)$, and modes $F_{i,k}(x)$, i.e.,

$$\phi_i(x_\lambda) = f(\lambda) \sum_{k \in \mathbb{R}} \lambda^{-k} F_{i,k}(x),$$

then $\{F_{i,k+1}, f\}_e = \{F_{i,k}, f\}_0$. We can choose $e$ so that the Hamiltonian $H(x) = \frac{1}{2} \{x, x\}$ commute with these functions.

Lemma 4.4. If an element $e \in G$ satisfies two conditions: (i) $ad^+(e)e = 0$; (ii) for any $u \in G$, $ad^+(u)e$ belongs to the tangent space to the coadjoint orbit of $u$ (i.e., for any $u \in G$ there exists $v \in G$ such that $ad^+(u)e = ad^+(v)u$), then the functions $\phi(a - \lambda e)$ commute with the Hamiltonian of the geodesics $H(a) = \frac{1}{2} \|a\|^2$ with respect to the brackets $\{., .\}_0$ and $\{., .\}_e$. 

5 The universal enveloping algebra of $\mathcal{SU}(G)$

When $\mathcal{H} = \sum_{k \in \mathbb{Z}} \mathcal{H}_k$ has a structure of graded algebra, its universal enveloping algebra $\mathcal{U}\mathcal{H}$ is also naturally endowed with a structure of a graded Lie algebra. Indeed, the weight of a product $h_1, \ldots, h_n \in \mathcal{U}\mathcal{H}$ of homogeneous elements is defined to be the sum of the weights of the elements $h_i, 1, \ldots, n$. The universal enveloping algebra $\mathcal{U}\mathcal{H}$ admits a filtration $\mathcal{U}\mathcal{H} = \bigcup_{n=0}^{\infty} F_k$ where $F_k$ is the vector space generated by the products of at most $k$ elements of $\mathcal{H}$. The generalized enveloping algebra is the algebra of the elements of the form $\sum_{k \leq n} u_k$ where $u_k$ is an element of weight $k$ of $\mathcal{U}\mathcal{H}$. The product of two such elements is defined by:

$$\sum_{k \leq n} u_k \cdot \sum_{k \leq m} v_k = \sum_{k \leq m} w_k,$$

where $w_k = \sum_{i \leq n} u_i v_k$, which is a finite sum. Let $\omega_1, \ldots, \omega_n$ be two-cocycles on the Lie algebra $\mathcal{H}$, let $\widetilde{\mathcal{H}}$ be the central extension associated with and let $e_1, \ldots, e_n$ be the central elements associated with these cocycles.

The modified generalized enveloping algebra $\mathcal{U}_{\omega_1,\ldots,\omega_n}^{\mathcal{H}}$ is defined to be the quotient of the generalized enveloping algebra of $\mathcal{H}$ by the ideal generated by the elements $\{ e_1 - 1, \ldots, e_n - 1 \}$. We denote again by $1$ the neutral element of $\mathcal{U}_{\omega_1,\ldots,\omega_n}^{\mathcal{H}}$. The algebra $\mathcal{U}_{\omega_1,\ldots,\omega_n}^{\mathcal{H}}$ is by construction a graded algebra and a filtered algebra. We denote by $F_n, n \in \mathbb{N}$ its filtration. Let us recall shortly the main properties of the modified generalized enveloping algebra. Let $V$ be a module over $\widetilde{\mathcal{H}}$ such that for any $\eta \in V$, there exists $\alpha, \beta \in \mathbb{C}$ such that for any $n > n_0$ and any $h \in \widetilde{\mathcal{H}}$ we have $h \eta = 0$. Such modules are called representations of positive energy, and $\epsilon_i$ acts on $V$ by $\lambda_i \cdot 1$. Then $V$ is a module over $\mathcal{U}_{\omega_1,\ldots,\omega_n}^{\mathcal{H}}$. Such modules are named modules of positive energy.

The anticommutator provides a structure of Lie algebra on $\mathcal{U}_{\omega_1,\ldots,\omega_n}^{\mathcal{H}}$. For this bracket $F_i$ is a Lie sub-algebra isomorphic to the central extension of $\mathcal{H}$ by the cocycle $\omega = \sum_{i=1}^n \omega_i$. We denote by $i$ be the natural inclusion of $\widetilde{\mathcal{H}}$ into $\mathcal{U}_{\omega_1,\ldots,\omega_n}^{\mathcal{H}}$ given by this identification.

5.1 Decomposition of the enveloping algebra of a semi-direct product

In some very particular cases, the modified generalized enveloping algebra of a semi-direct product $\mathcal{K} \ltimes \mathcal{H}$ of two Lie algebras is isomorphic to the tensor product of some modified generalized enveloping algebras of $\mathcal{K}$ and of $\mathcal{H}$. Let $\widetilde{\mathcal{H}}$ be the central extension of $\mathcal{H}$ with the two-cocycle $\omega_\mathcal{H}$. Denote by $\cdot$ the action of the Lie algebra $\mathcal{K}$ on the Lie algebra $\widetilde{\mathcal{H}}$. Let us introduce the semi-direct product $\mathcal{K} \ltimes \widetilde{\mathcal{H}}$ which is a central extension of $\mathcal{K} \ltimes \mathcal{H}$ by a two-cocycle $\omega_{\mathcal{K} \ltimes \mathcal{H}}$ with

$$\omega_{\mathcal{K} \ltimes \mathcal{H}}((0, h_1), (0, h_2)) = \omega_{\mathcal{H}}(h_1, h_2).$$

A two-cocycle $\omega_\mathcal{K}$ on $\mathcal{K}$ defines also a two-cocycle $\omega'_{\mathcal{K}}$ by

$$\omega'_{\mathcal{K}}((g_1, h_1), (g_2, h_2)) = \omega_{\mathcal{K}}(g_1, g_2),$$

of $\mathcal{K} \ltimes \mathcal{H}$. Let $I$ be the natural inclusion of $\widetilde{\mathcal{H}}$ into $\mathcal{U}_{\omega_\mathcal{H}}(\mathcal{H})$ and $J$ be the natural inclusion of $\widetilde{\mathcal{H}}$ into $\mathcal{U}_{\omega'_{\mathcal{K}},\omega_{\mathcal{H}}}(\mathcal{K} \ltimes \mathcal{H})$.

We call the action of $\mathcal{K}$ on $\mathcal{H}$ realizable in $\mathcal{U}_{\omega_\mathcal{H}}(\mathcal{H})$ when there exists a map $F : \mathcal{K} \to \mathcal{U}_{\omega_\mathcal{H}}(\mathcal{H})$ and a two-cocycle $\alpha$ on $\mathcal{K}$ such that for any pair $(g_1, g_2)$ in $\mathcal{K}^2$

$$F([g_1, g_2]) = [F(g_1), F(g_2)] + \alpha(g_1, g_2) 1,$$

and the map $F$ satisfies the compatibility condition, i.e., for any $g \in \mathcal{K}$ and $h \in \widetilde{\mathcal{H}}$ with the anti-commutator $[F(g), I(h)] = I(g \cdot h)$, of the algebra $\mathcal{U}_{\omega_\mathcal{H}}(\mathcal{H})$.

**Theorem 5.1.** If the action of $\mathcal{K}$ is realizable in $\mathcal{U}_{\omega_\mathcal{H}}(\mathcal{H})$ then

$$\mathcal{U}_{\omega'_{\mathcal{K}},\omega_{\mathcal{H}}}(\mathcal{K} \ltimes \mathcal{H}) = \mathcal{U}_{\omega_\mathcal{K}-\alpha}(\mathcal{K}) \otimes \mathcal{U}_{\omega_\mathcal{H}}(\mathcal{H}).$$
Proof. Let $U_g = \{ g \mid g \in K \}$ with be the unitary subalgebra of $U_{\omega_K \omega_M}(K \ltimes H)$ generated by the elements $g = g - F(g)$, and $U_f = \{ j(h), h \in H \}$ be the unitary subalgebra of $U_{\omega_K \omega_M}(K \ltimes H)$. For any $(g, h)$ this implies that the generators of $U_g$ and $U_f$ commute, i.e., $[\bar{g}, j(h)] = 0$. The subalgebras $U_g$ and $U_f$ therefore commute. The subalgebra $U_g$ is isomorphic to $U_{\omega_K \omega_M}(K \ltimes H)$. Let us check that the generators $\{ g \mid g \in K \}$ of this algebra satisfy the relations of the generators of $U_{\omega_K \omega_M}(K)$:

$$\langle [\bar{g}, \bar{g}] \rangle = [g_1, g_2] + \omega_K(g_1, g_2)1 + [F(g_1), F(g_2)] - [F(g_1), g_2] - [g_1, F(g_2)].$$

Since $F(g_1)$ is an element of $U_f$ and since the algebras $U_g$ and $U_f$ commute $[F(g_1), g_2] = [F(g_1), F(g_2)]$ and $[g_1, F(g_2)] = [F(g_1), F(g_2)]$. Therefore:

$$[\bar{g}, \bar{g}] = [g_1, g_2] + \omega_K(g_1, g_2)1 - [F(g_1), F(g_2)],$$

and finally

$$[\bar{g}, \bar{g}] = [g_1, g_2] - F([g_1, g_2]) + (\omega_K(g_1, g_2) - \alpha(g_1, g_2))1.$$
5.3 Representations of $SU(G)$

**Proposition 5.4.** A positive energy representation $V$ of $SU(G)$ with non-vanishing $\beta\text{Id}$-action of the cocycle $c_{K-M}$ brings about a pair of commuting representations of Virasoro and affine Kac–Moody Lie algebras.

This proposition determines whether a $SU(G)$ Verma module is a sub-module of another Verma module of $SU(G)$. Let $g$ be a Cartan algebra of $G$ with a basis $\{h_1, \ldots, h_n\}$. The Lie subalgebra $\mathfrak{t}$ of $SU(G)$ is generated by the elements $\{c_{\text{Vir}}, c_{K-M}, u_0, (h_1)_0, \ldots, (h_n)_0\}$. A Verma module $V_\lambda(SU(G))$ of $SU(G)$ is associated to any linear form $\lambda \in \mathfrak{t}^*$. Verma modules $V_{\nu}^{\text{Vir}}$, $V_{\mu}^{K-M}$, are associated to linear forms $\nu$, $\mu$ over the spaces generated by $c_{\text{Vir}}$ and $u_0$, $c_{K-M}$ and $\{(h_1)_0, \ldots, (h_n)_0\}$ correspondingly. For any $\lambda \in \mathfrak{t}^*$, the Verma module $V_\lambda(SU(G))$ is a positive energy representation. Thus, $V_\lambda(SU(G))$ is Virasoro and affine Kac–Moody algebra module. The generator $e$ of $V_\lambda(SU(G))$ brings about a Verma module $V_{\nu}^{\text{Vir}}$ for Virasoro algebra. It generates also a Verma module $V_{\mu}^{K-M}$ for the affine Kac–Moody algebra. The linear form $\nu$ satisfies $\nu(u_0)e = \lambda(u_0 - F(u_0))e$, i.e.,

$$(u_0 - (\beta + \eta)^{-1}K \cdot K^* e = \nu(u_0)e.)$$

Suppose the action of a Casimir element of $G$ is given by acts by $D(\lambda)$Id for $D(\lambda) \in \mathbb{C}$. We then have

$$(u_0 - (\beta + \eta)^{-1}K \cdot K^* e = (u_0 - (\beta + \eta)^{-1} \sum_{j=1}^{n} (K_j)_0 (K_j^*)_0 : e),$$

$$D(\lambda)(u_0) = \frac{D(\lambda)}{2\eta}. \nu(u_0) = \lambda(u_0) - \frac{D(\lambda)}{2\eta}.$$ The other values of $\mu$ and $\nu$ can be computed by the same method.

**Proposition 5.5.** Let $\lambda$ be a linear form over $\mathfrak{t}$ with non-vanishing $\lambda(c_{K-M})$. Then

$$(1)\quad V_\lambda(SU(G)) \cong V_{\nu}^{\text{Vir}} \otimes V_{\mu}^{K-M},$$

where $\mu(e_i) = \lambda(e_i)$, $i = 1, \ldots, n$, defines $\mu$, $\mu(c_{K-M}) = \lambda(c_{K-M})$, and $\nu(c_{\text{Vir}}) = \lambda(c_{\text{Vir}}) - \frac{\beta}{2\eta}$ defines $\nu$, $\nu(u_0) = \lambda(u_0) - \frac{D(\lambda)}{2\eta}$.

**References**

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