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Research Article

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The log-concavity of the $q$-derangement numbers of type $B$

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Abstract: Recently, Chen and Xia proved that for $n \geq 6$, the $q$-derangement numbers $D_n(q)$ are log-concave except for the last term when $n$ is even. In this paper, employing a recurrence relation for $D_B^n(q)$ discovered by Chow, we show that for $n \geq 4$, the $q$-derangement numbers of type $B D_B^n(q)$ are also log-concave.

Keywords: The $q$-derangement numbers of type B, Unimodality, Log-concavity

MSC: 05A15, 05A19, 05A20

1 Introduction

Let $D_n$ denote the set of derangements on $\{1, 2, \ldots, n\}$ and let

$$D(\pi) := \{i \mid 1 \leq i \leq n - 1, \pi(i) > \pi(i + 1)\}$$

denote the descent set of a permutation $\pi$. Define the major index of $\pi$ by

$$\text{maj}(\pi) := \sum_{i \in D(\pi)} i.$$ (1)

The $q$-derangement number $D_n(q)$ is defined by

$$D_n(q) := \sum_{\pi \in D_n} q^{\text{maj}(\pi)}.$$ (2)

Gessel [1] (see also [2]) discovered the following formula

$$D_n(q) := \left[\frac{n!}{n}!\right] \sum_{k=0}^{n} (-1)^k q^k \frac{1}{[k]!},$$ (3)

where $[n] = 1 + q + q^2 + \cdots + q^{n-1}$ and $[n]! = [1][2][3][n]$. Combinatorial proofs of (3) have been found by Wachs [3] and Chen and Xu [4]. Chen and Rota [5] showed that the $q$-derangement numbers are unimodal, and conjectured that the maximum coefficient appears in the middle. Zhang [6] confirmed this conjecture by showing that the $q$-derangement numbers satisfy the spiral property. Recently, Chen and Xia [7] introduced the notion of ratio monotonicity for polynomials with nonnegative coefficients, and they proved that, for $n \geq 6$, the $q$-derangement numbers $D_n(q)$ are strictly ratio monotone except for the last term when $n$ is even. The ratio monotonicity implies the spiral property and log-concavity.

Let $B_n$ denote the hyperoctahedral group of rank $n$, consisting of the signed permutations of $\{1, 2, \ldots, n\}$. Let $D_B^n$ denote the set of derangements on $B_n$, which is defined as

$$D_B^n := \{\pi | \pi \in B_n, \pi(i) \neq i \text{ for all } i \in \{1, 2, \ldots, n\}\}.$$ (4)

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Let \( N(\pi) := \# \{ i | 1 \leq i \leq n, \pi(i) < 0 \} \) be the number of negative letters of \( \pi \) and let \( \text{maj}(\pi) \) be defined as before. In [8], Chow considered the \( q \)-derangement number of type \( B \) \( D_n^B(q) \), which is defined as

\[
D_n^B(q) := \sum_{\pi \in D_n^B} q^{\text{maj}(\pi)},
\]

where \( \text{maj}(\pi) := 2\text{maj}(\pi) + N(\pi) \). Chow [8] (see also [9]) established the following formula

\[
D_n^B(q) := [2][4] \cdots [2n] \sum_{k=0}^{n} \frac{(-1)^k q^{4k}}{[2][4] \cdots [2k]},
\]

where \([n]\) is defined as before. Furthermore, Chow [8] discovered that for all integers \( n \geq 1 \),

\[
D_{n+1}^B(q) = (1 + q + \cdots + q^{2n+1}) D_n^B(q) + (-1)^{n+1} q^{n^2+n}.
\]

Chen and Wang [10] proved the normality of the limiting distribution of the coefficients of the usual \( q \)-derangement numbers of type \( B \).

Recall that a positive sequence \( a_0, a_1, \ldots, a_n \) or the polynomial \( a_0 + a_1 x + \cdots + a_n x^n \) is called log-concave if the ratios

\[
\frac{a_0}{a_1}, \frac{a_1}{a_2}, \ldots, \frac{a_{n-1}}{a_n}
\]

form an increasing sequence. Clearly, if a positive sequence is log-concavity, then it is unimodality. In this paper, we prove that for \( n \geq 4 \), the \( q \)-derangement numbers of type \( B \) \( D_n^B(q) \) are log-concave.

Suppose that \( n \) is given. It is easy to prove that the degree of \( D_n^B(q) \) is \( n^2 \) and the coefficient of \( q^{n^2} \) is 1. Set

\[
D_n^B(q) = B_n(1) q + B_n(2) q^2 + \cdots + B_n(n^2) q^{n^2}.
\]

The log-concavity of \( D_n^B(q) \) can be stated as the following theorem.

**Theorem 1.1.** For all integers \( n \geq 4 \), the \( q \)-derangement numbers of type \( B \) \( D_n^B(q) \) are log-concave, namely,

\[
\frac{B_n(1)}{B_n(2)} < \frac{B_n(2)}{B_n(3)} < \cdots < \frac{B_n(n^2-2)}{B_n(n^2-1)} < \frac{B_n(n^2)}{B_n(n^2+1)}.
\]

For example, by (6), we have

\[
D_4^B(q) = q + 4q^2 + 8q^3 + 13q^4 + 18q^5 + 22q^6 + 26q^7 + 28q^8 + 28q^9 + 25q^{10} + 21q^{11} + 17q^{12} + 11q^{13} + 7q^{14} + 3q^{15} + q^{16}.
\]

It is easy to check that

\[
\frac{1}{4} < \frac{4}{8} < \frac{8}{13} < \frac{18}{18} < \frac{22}{26} < \frac{26}{28} < \frac{28}{28} < \frac{28}{25} < \frac{25}{21} < \frac{21}{17} < \frac{11}{17} < \frac{7}{7} < \frac{3}{3} = 1.
\]

**2 Some lemmas**

To prove Theorem 1.1, we first present some lemmas. By (7), it is easy to check that

**Lemma 2.1.** For \( n \geq 4 \),

\[
B_{n+1}(k) = \begin{cases} \sum_{i=1}^{k} B_n(i), & 1 \leq k \leq 2n + 2, \\ \sum_{i=k-2n-1}^{k-n^2} B_n(i), & 2n + 2 < k \leq n^2, \\ \sum_{i=n^2-n-1}^{n^2} B_n(i) + (-1)^{n+1}, & k = n^2 + n, \\ \sum_{i=k-2n-1}^{n^2} B_n(i), & n^2 \leq k \leq (n + 1)^2 \text{ and } k \neq n^2 + n. \end{cases}
\]
Based on recurrence relation (11), it is easy to verify the following lemma.

**Lemma 2.2.** Let \( n \geq 4 \) be an integer. Then \( B_n(i) \) are positive integers for \( 1 \leq i \leq n^2 \) and

\[
B_n(n^2) = 1, \quad B_n(n^2 - 1) = n - 1, \tag{12}
\]

\[
B_n(n^2 - 2) = \frac{n^2 - n + 2}{2}, \quad B_n(n^2 - 3) = \frac{n^3 + 5n - 18}{6}. \tag{13}
\]

To prove Theorem 1.1, we require the following lemma.

**Lemma 2.3.** For positive integers \( a_1, a_2, \ldots, a_{k+1}, a_{k+2} \) \( (k \geq 1) \) satisfying

\[
a_i < \frac{a_{i+1}}{a_{i+2}}, \quad 1 \leq i \leq k, \tag{14}
\]

we have

\[
\sum_{i=1}^{k} \frac{a_i}{a_{i+1} a_{i+2}} < \frac{\sum_{i=1}^{k+1} a_i}{\sum_{i=1}^{k+2} a_i}, \tag{15}
\]

\[
\sum_{i=1}^{k} \frac{a_i}{a_{i+2} a_{i+3}} < \frac{\sum_{i=1}^{k+1} a_i}{\sum_{i=1}^{k+2} a_i}, \tag{16}
\]

\[
\sum_{i=1}^{k} \frac{a_i}{a_{i+2} a_{i+3}} < \frac{\sum_{i=1}^{k+1} a_i}{\sum_{i=1}^{k+3} a_i}, \tag{17}
\]

\[
\sum_{i=1}^{k} \frac{a_i}{a_{i+2} a_{i+3}} < \frac{\sum_{i=1}^{k+1} a_i}{\sum_{i=1}^{k+3} a_i}, \tag{18}
\]

\[
\sum_{i=1}^{k} \frac{a_i}{a_{i+1} a_{i+2}} < \frac{\sum_{i=1}^{k+1} a_i}{\sum_{i=1}^{k+3} a_i}. \tag{19}
\]

**Proof.** We only prove (15). The rest can be proved similarly and the details are omitted. Based on (14),

\[
a_i a_{k+2} < a_{i+1} a_{k+1} \quad (1 \leq i \leq k),
\]

and

\[
a_{k+2}(a_1 + a_2 + \cdots + a_k) < a_{k+1}(a_2 + a_3 + \cdots + a_{k+1}).
\]

Therefore,

\[
(a_1 + a_2 + \cdots + a_k)(a_1 + a_2 + \cdots + a_k + a_{k+1} + a_{k+1})
\]

\[
= (a_1 + a_2 + \cdots + a_k)^2 + a_{k+1}(a_1 + a_2 + \cdots + a_k) + a_{k+2}(a_1 + a_2 + \cdots + a_k)
\]

\[
< (a_1 + a_2 + \cdots + a_k)^2 + a_{k+1}(a_1 + a_2 + \cdots + a_k) + a_{k+1}(a_2 + a_3 + \cdots + a_{k+1})
\]

\[
< (a_1 + a_2 + \cdots + a_k)^2 + a_{k+1}(a_1 + a_2 + \cdots + a_k) + a_{k+1}(a_2 + a_3 + \cdots + a_{k+1}) + a_1 a_{k+1}
\]

\[
= (a_1 + a_2 + \cdots + a_k + a_{k+1})^2,
\]

which yields (15). This completes the proof of this lemma.

### 3 Proof of Theorem 1.1

We prove Theorem 1.1 by induction on \( n \). It is easy to check that Theorem 1.1 holds for \( 4 \leq n \leq 12 \). Thus, we always assume that \( n \geq 13 \) in the following proof. Suppose that Theorem 1.1 holds for \( n = m \), namely,

\[
\frac{B_m(i)}{B_m(i + 1)} < \frac{B_m(i + 1)}{B_m(i + 2)}, \quad 1 \leq i \leq m^2 - 2. \tag{20}
\]

...
We proceed to show that Theorem 1.1 holds for \( n = m + 1 \), that is,

\[
\frac{B_{m+1}(k)}{B_{m+1}(k+1)} < \frac{B_{m+1}(k+1)}{B_{m+1}(k+2)}, \quad 1 \leq k \leq (m+1)^2 - 2. \tag{21}
\]

Employing (11), (15) and (20), we see that (21) holds for \( 1 \leq k \leq 2m \). It follows from (11), (16) and (20) that (21) is true for the case \( k = 2m + 1 \). In view of (11), (17) and (20), we find that (21) holds for \( 2m + 2 \leq k \leq m^2 - 2 \).

From (11), (18) and (20), we deduce that (21) is true for the case \( k = m^2 - 1 \). By (11), (19) and (20), we can prove that (21) holds for \( m^2 \leq k \leq m^2 + m - 3 \) and \( m^2 + m + 1 \leq k \leq (m+1)^2 - 2 \).

Now, special attentions should be paid to three cases \( k = m^2 + m - 2 \), \( k = m^2 + m - 1 \) and \( k = m^2 + m \).

By (12) and (13), it is easy to check that for \( m \geq 4 \),

\[
\frac{B_m(m^2 - 2) + B_m(m^2 - 1) + B_m(m^2) - m - 3}{B_m(m^2 - 1) + B_m(m^2) + 1} - \frac{B_m(m^2 - 3)}{B_m(m^2 - 2)} = \frac{m^6 - 13m^2 + 72m - 132}{6(m - 3)(m^2 - m + 2)} > 0. \tag{22}
\]

In view of (20) and (22),

\[
\frac{B_m(m^2 - m - 3)}{B_m(m^2 - m - 2)} \frac{B_m(m^2 - 3)}{B_m(m^2 - 2)} < \frac{B_m(m^2 - 2) + B_m(m^2 - 1) + B_m(m^2) - m - 3}{B_m(m^2 - 1) + B_m(m^2) + 1}. \tag{23}
\]

From (11), it is easy to prove that for \( m \geq 4 \),

\[
B_m(m^2 - m - 2) > B_m(m^2 - m - 1) > \cdots > B_m(m^2 - 1) > B_m(m^2). \tag{24}
\]

Thanks to (23) and (24),

\[
B_m(m^2 - m - 3)(B_m(m^2 - 1) + B_m(m^2) + (-1)^{m+1} + (-1)^{m+1} \sum_{i=0}^{m+2} B_m(m^2 - i)
\]
\[
< B_m(m^2 - m - 3)(B_m(m^2 - 1) + B_m(m^2) + 1) + (m + 3)B_m(m^2 - m - 2)
\]
\[
< B_m(m^2 - m - 2)(B_m(m^2 - 2) + B_m(m^2 - 1) + B_m(m^2)). \tag{25}
\]

By (20),

\[
B_m(m^2 - m - 3) \sum_{i=2}^{m+1} B_m(m^2 - i) < B_m(m^2 - m - 2) \sum_{i=3}^{m+2} B_m(m^2 - i). \tag{26}
\]

Combining (25) and (26) yields

\[
\frac{\sum_{i=m^2 - m -3}^{m^2} B_m(i)}{\sum_{i=m^2 - m -2}^{m^2} B_m(i)} < \frac{\sum_{i=m^2 - m -1}^{m^2} B_m(i)}{\sum_{i=m^2 - m -1}^{m^2} B_m(i) + (-1)^{m+1}}, \tag{27}
\]

which can be rewritten as

\[
\frac{B_{m+1}(m^2 + m - 2)}{B_{m+1}(m^2 + m - 1)} < \frac{B_{m+1}(m^2 + m - 1)}{B_{m+1}(m^2 + m)}. \tag{28}
\]

Therefore, (21) holds for the case \( k = m^2 + m - 2 \).

Based on (12) and (13), we deduce that for \( m \geq 13 \),

\[
\frac{B_m(m^2 - 2) + B_m(m^2 - 1) + B_m(m^2) - 2(m + 2)}{B_m(m^2 - 1) + B_m(m^2)} - \frac{B_m(m^2 - 3)}{B_m(m^2 - 2)} = \frac{m^6 - 12m^4 - 13m^3 + 36m^2 - 36}{6m(m^2 - m + 2)} > 0. \tag{29}
\]
By (20) and (29),
\[
\frac{B_m(m^2 - m - 2)}{B_m(m^2 - m - 1)} < \frac{B_m(m^2 - 3)}{B_m(m^2 - 2)} < \frac{B_m(m^2 - 2) + B_m(m^2 - 1) + B_m(m^2) - 2(m + 2)}{B_m(m^2 - 1) + B_m(m^2)}.
\] (30)

It follows from (24) and (30) that
\[
\frac{B_m(m^2 - m - 2)(B_m(m^2 - 1) + B_m(m^2))}{B_m(m^2 - m - 1)(B_m(m^2 - 2) + B_m(m^2 - 1) + B_m(m^2) - 2(m + 2))}
\]
\[
< \frac{B_m(m^2 - m - 1)(B_m(m^2 - 2) + B_m(m^2 - 1) + B_m(m^2)) - 2 \sum_{i=0}^{m+1} B_m(m^2 - i)}{B_m(m^2 - m - 1)(B_m(m^2 - 2) + B_m(m^2 - 1) + B_m(m^2))}
\]
\[
+ 2 \times (-1)^{m+1} \sum_{i=0}^{m+1} B_m(m^2 - i).
\] (31)

In view of (20),
\[
B_m(m^2 - m - 2) \sum_{i=2}^{m} B_m(m^2 - i) < B_m(m^2 - m - 1) \sum_{i=3}^{m+1} B_m(m^2 - i).
\] (32)

It follows from (31) and (32) that
\[
\frac{\sum_{i=m+1-m-2}^{m^2} B_m(i)}{\sum_{i=m+1-m-1}^{m^2} B_m(i)} < \frac{\sum_{i=m^2-m}^{m} B_m(i) + (-1)^{m+1} \sum_{i=m^2-i}^{m^2} B_m(i)}{\sum_{i=m^2-i}^{m^2} B_m(i)}.
\] (33)

By (11), we can rewrite (33) as follows
\[
\frac{B_{m+1}(m^2 + m - 1)}{B_{m+1}(m^2 + m)} < \frac{B_{m+1}(m^2 + m)}{B_{m+1}(m^2 + m + 1)},
\] (34)

which implies that (21) holds for the case \( k = m^2 + m - 1 \).

In view of (12) and (13), we see that for \( m \geq 4 \),
\[
\frac{B_m(m^2 - 2) + B_m(m^2 - 1) + B_m(m^2) - m}{B_m(m^2 - 1) + B_m(m^2)} = \frac{B_m(m^2 - 3)}{B_m(m^2 - 2)} = \frac{(m + 1)(m^3 - 7m^2 + 12m + 12)}{6m(m^2 - m + 2)} > 0.
\] (35)

By (20) and (35), we find that for \( m \geq 4 \),
\[
\frac{B_m(m^2 - m - 1)}{B_m(m^2 - m)} < \frac{B_m(m^2 - 3)}{B_m(m^2 - 2)} < \frac{B_m(m^2 - 2) + B_m(m^2 - 1) + B_m(m^2) - m}{B_m(m^2 - 1) + B_m(m^2)}.
\] (36)

It follows from (24) and (36) that
\[
\frac{B_m(m^2 - m - 1)(B_m(m^2 - 1) + B_m(m^2)) + (-1)^{m+1} \sum_{i=0}^{m-1} B_m(m^2 - i)}{B_m(m^2 - m - 1)(B_m(m^2 - 2) + B_m(m^2 - 1) + B_m(m^2))}
\]
\[
< \frac{B_m(m^2 - m - 1)(B_m(m^2 - 1) + B_m(m^2)) + mB_m(m^2 - m)}{B_m(m^2 - m)(B_m(m^2 - 2) + B_m(m^2 - 1) + B_m(m^2))}.
\] (37)

By (20),
\[
B_m(m^2 - m - 1) \sum_{i=2}^{m} B_m(m^2 - i) < B_m(m^2 - m) \sum_{i=3}^{m} B_m(m^2 - i).
\] (38)
In view of (37) and (38), we can prove that
\[
\sum_{i=m^2-m}^{m^2} B_m(i) + (-1)^{m+1} \frac{\sum_{i=m^2-m}^{m^2} B_m(i)}{\sum_{i=m^2-m}^{m^2} B_m(i)} < \frac{\sum_{i=m^2-m}^{m^2} B_m(i)}{\sum_{i=m^2-m}^{m^2} B_m(i)}.
\] (39)

By (11), we can rewrite (39) as follows
\[
\frac{B_{m+1}(m^2 + m)}{B_{m+1}(m^2 + m + 1)} < \frac{B_{m+1}(m^2 + m + 1)}{B_{m+1}(m^2 + m + 2)},
\] (40)

which implies that (21) holds for the case \( k = m^2 + m \). This completes the proof.

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**References**


