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Rank relations between a \{0, 1\}-matrix and its complement

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Abstract: Let \( A \) be a \{0, 1\}-matrix and \( r(A) \) denotes its rank. The complement matrix of \( A \) is defined and denoted by \( A^c = J - A \), where \( J \) is the matrix with each entry being 1. In particular, when \( A \) is a square \{0, 1\}-matrix with each diagonal entry being 0, another kind of complement matrix of \( A \) is defined and denoted by \( \overline{A} = J - I - A \), where \( I \) is the identity matrix. We determine the possible values of \( r(A) \pm r(A^c) \) and \( r(A) \pm r(\overline{A}) \) in the general case and in the symmetric case. Our proof is constructive.

Keywords: \{0, 1\}-matrix, Complement matrix, Rank

MSC: 15A03, 15B36, 05B20, 05C50

1 Introduction

A \{0, 1\}-matrix is an integer matrix with each entry being 0 or 1. \{0, 1\}-matrices are closely related to graph theory and combinatorial mathematics [1-4]. They also have a wide range of practical applications in statistics and probability [5-8].

Denote by \( M_{m,n}(0, 1) \) the set of \( m \times n \) \{0, 1\}-matrices, and we abbreviate \( M_{n,n}(0, 1) \) as \( M_n(0, 1) \). Let \( A \in M_{m,n}(0, 1) \). Then the matrix \( A^c = J_{m,n} - A \) is called the complement of \( A \), where \( J_{m,n} \) is the \( m \times n \) matrix with each entry being 1. It is clear that \( A \) and \( A^c \) are mutually complementary, i.e., \( (A^c)^c = A \).

Recall that the adjacency matrix of a digraph \( D \) is the square matrix \( A = (a_{ij}) \), where \( a_{ij} \) is the number of arcs \( (i, j) \) in \( D \). A digraph is called strict if it has no loops or parallel arcs. Thus the adjacency matrix of a strict digraph is a \{0, 1\}-matrix with each diagonal entry being 0. The complement of a strict digraph \( D \), denoted by \( \overline{D} \), is also a strict digraph on the same vertices such that \( (i, j) \) is an arc in \( \overline{D} \) if and only if \( (i, j) \) is not an arc in \( D \). Let \( A \) be the \( n \times n \) adjacency matrix of a strict digraph \( D \). Then the adjacency matrix of \( \overline{D} \) is \( J_n - I_n - A \), where \( J_n = J_{n,n} \) and \( I_n \) is the identity matrix of order \( n \). Denote by \( \Omega_n(0, 1) \) the set of \( n \times n \) \{0, 1\}-matrices with each diagonal entry being 0. Thus for \( A \in \Omega_n(0, 1) \), we define another kind of complement matrix of \( A \) to be \( \overline{A} = J_n - I_n - A \). It is also clear that \( A \) and \( \overline{A} \) are mutually complementary, i.e., \( \overline{(\overline{A})} = A \).

In this paper, we mainly investigate the rank relations between a \{0, 1\}-matrix and its complement. Denote by \( r(A) \) the rank of a matrix \( A \). In Section 2, we determine the possible values of \( r(A) \pm r(A^c) \) for \( A \in M_{m,n}(0, 1) \) in the general case and in the symmetric case. In Section 3, we determine the possible values of \( r(A) \pm r(\overline{A}) \) for \( A \in \Omega_n(0, 1) \) in the general case and in the symmetric case.

We use \( O_{m,n} \) to denote the \( m \times n \) zero matrix. \( O_{n,n} \) will be abbreviated as \( O_n \). Denote by \( E_{ij} \) the matrix with its entry in the \( i \)-th row and \( j \)-th column being 1 and with all other entries being 0.
2 Rank relations between $A$ and $A^c$

First we determine the possible values of $r(A) \pm r(A^c)$ in the general case.

**Theorem 2.1.** Let $m, n \geq 2$ be positive integers. Then there exists $A \in M_{m,n}\{0, 1\}$ with $r(A) - r(A^c) = k$ if and only if $-1 \leq k \leq 1$.

**Proof.** Since $r(A^c) = r(I_{m,n} - A) \leq r(I_{m,n}) + r(A) = 1 + r(A)$. Let $A^c = A + r(A)$ and $A^c = A + r(A)$. Likewise, $r(A) - r(A^c) \leq 1$. This proves the necessity.

For the sufficiency, note that $r(A) - r(A^c) = \pm 1$ if $A = O_{m,n}$ or $I_{m,n}$, and $r(A) = r(A^c)$ if $A = \begin{bmatrix} I_{m,n} & O_{m,n-1} \end{bmatrix}$. This completes the proof.

**Theorem 2.2.** Let $m, n \geq 2$ be positive integers. Then there exists $A \in M_{m,n}\{0, 1\}$ with $r(A) + r(A^c) = k$ if and only if $1 \leq k \leq 2 \min\{m, n\}$.

**Proof.** The necessity is clear. Now we prove the sufficiency.

Suppose $m \leq n$. Let $A_1 = \begin{bmatrix} I_p & O_{p,n-p} \\ O_{m-p,p} & O_{m-p,n-p} \end{bmatrix} \in M_{m,n}\{0, 1\}$ with $0 \leq p \leq m - 1$. Clearly $r(A_1) = p$. It is easy to verify that $A_1^c = \begin{bmatrix} I_p - I_p & I_p \\ O_{m-p,p} & O_{m-p,n-p} \end{bmatrix}$ has rank $p + 1$. Then $r(A_1) + r(A_1^c) = 2p + 1$, $0 \leq p \leq m - 1$. Thus for every odd $k$ with $1 \leq k \leq 2m - 1$, there exists $A \in M_{m,n}\{0, 1\}$ such that $r(A) + r(A^c) = k$.

Let $A_2 = \begin{bmatrix} I_q & O_{q,m-q} \\ O_{m-q,p} & O_{m-q,n-m} \end{bmatrix} \in M_{m,n}\{0, 1\}$ with $1 \leq q \leq m - 1$. Clearly $r(A_2) = q + 1$. It is easy to verify that $A_2^c = \begin{bmatrix} I_q - I_q & I_q \\ O_{m-q,p} & O_{m-q,n-m} \end{bmatrix}$ has rank $q + 1$. Then $r(A_2) + r(A_2^c) = 2q + 1$, $1 \leq q \leq m - 1$. Thus for every even $k$ with $4 \leq k \leq 2m$, there exists $A \in M_{m,n}\{0, 1\}$ such that $r(A) + r(A^c) = k$.

Let $A_3 = \begin{bmatrix} J_{m,1} & O_{m,n-1} \end{bmatrix} \in M_{m,n}\{0, 1\}$. Then $r(A_3) + r(A_3^c) = 1 + 1 = 2$.

Thus for every integer $k$ with $1 \leq k \leq 2m$, there exists $A \in M_{m,n}\{0, 1\}$ such that $r(A) + r(A^c) = k$.

If $m > n$, the argument is similar. This completes the proof.

Next we consider the case when $A \in M_{n}\{0, 1\}$ is symmetric.

**Theorem 2.3.** Let $n \geq 2$ be a positive integer. Then there exists symmetric $A \in M_{n}\{0, 1\}$ with $r(A) - r(A^c) = k$ if and only if $-1 \leq k \leq 1$.

**Proof.** For the necessity, the argument is the same as that of Theorem 2.1.

For the sufficiency, note that $r(A) - r(A^c) = -1, 0, 1$ if $A = O_n, I_n, J_n$, respectively.

**Lemma 2.4.** Let $A \in M_{n}\{0, 1\}$ be symmetric. Then $r(A) + r(A^c) \neq 2$.

**Proof.** Assume that there exists symmetric $A \in M_{n}\{0, 1\}$ such that $r(A) + r(A^c) = 2$. If $r(A) = 0$, then $A = O_n$ and thus $A^c = J_n$, which contradicts the assumption that $r(A) + r(A^c) = 2$. If $r(A) = 2$, then $A^c = 0$. This implies $A = I_n$ and thus $r(A) = 1$, a contradiction. If $r(A) = 1$, by the proof of Theorem 4(i) in [9], it follows that $A$ is permutation similar to $\begin{bmatrix} I_p & O_{p,n-p} \\ O_{n-p,p} & O_{n-p} \end{bmatrix}$ with $1 \leq p \leq n$. Thus $A^c$ is permutation similar to $\begin{bmatrix} O_{p} & I_{p,n-p} \\ J_{n-p} \end{bmatrix}$, which implies that $r(A^c) = 0$ when $p = n$ and $r(A^c) = 2$ when $1 \leq p \leq n - 1$. Then $r(A) + r(A^c) = 1$ or 3, a contradiction. Thus $r(A) + r(A^c) \neq 2$ for any symmetric $A \in M_n\{0, 1\}$.

Note that the matrices $A_1, A_2 \in M_{m,n}\{0, 1\}$ in the proof of Theorem 2.2 are symmetric when $m = n$. By Theorem 2.2 and Lemma 2.4, we have the following result.
Theorem 2.5. Let \( n \geq 2 \) be a positive integer. Then there exists symmetric \( A \in M_n(0, 1) \) with \( r(A) + r(A^c) = k \) if and only if \( 1 \leq k \leq 2n \) with \( k \neq 2 \).

3 Rank relations between \( A \) and \( \overline{A} \)

In this section, we only consider \( A \in \Omega_n(0, 1) \) which corresponds to the adjacency matrix of a strict digraph. Recall that for an \( n \times n \) matrix \( A = (a_{ij}) \), the main diagonal of \( A \) is the list of entries \( a_{11}, a_{22}, \ldots, a_{nn} \), and the secondary diagonal of \( A \) is the list of entries \( a_{1n}, a_{2,n-1}, \ldots, a_{n1} \). Let \( C_1 \) be the square matrix whose entries above the main diagonal are all 1’s, while other entries are all 0’s. The size of \( C_1 \) will be clear from the context.

First we determine the possible values of \( r(A) \pm r(\overline{A}) \) in the general case.

Theorem 3.1. Let \( n \geq 2 \) be a positive integer. Then there exists \( A \in \Omega_n(0, 1) \) with \( r(A) - r(\overline{A}) = k \) if and only if

(i) \( k = 0, \pm 2 \) when \( n = 2 \);

(ii) \( -n \leq k \leq n \) when \( n \geq 3 \).

Proof. (i) The case \( n = 2 \) is trivial.

(ii) The necessity is clear. Now we prove the sufficiency.

Let \( B_1 = \begin{bmatrix} C_1 & I_{n-n-p} \\ O_{n-p,n} & O_{n-p} \end{bmatrix} \in \Omega_n(0, 1) \), where \( 1 \leq p \leq n - 1 \). It is easy to verify that \( r(B_1) = p, \ r(\overline{B_1}) = n - 1 \). Thus for every integer \( k \) with \( -n + 2 \leq k \leq n - 2 \), there exists \( A \in \Omega_n(0, 1) \) such that \( r(A) - r(\overline{A}) = k \).

Let \( B_2 = \begin{bmatrix} C_2 \\ O_{n-1,n} \end{bmatrix} \in \Omega_n(0, 1) \), where \( C_2 = [0, 1, \ldots, 1, 0] \). Then \( r(B_2) = 1, \ r(\overline{B_2}) = n \). Note that \( r(O_n) = 0, \ r(\overline{O_n}) = n \). Thus for \( k = z(n - 1), zn \), there exists \( A \in \Omega_n(0, 1) \) such that \( r(A) - r(\overline{A}) = k \). This proves the sufficiency. \( \square \)

Theorem 3.2. Let \( n \geq 2 \) be a positive integer. Then there exists \( A \in \Omega_n(0, 1) \) with \( r(A) + r(\overline{A}) = k \) if and only if

(i) \( k = 2 \) when \( n = 2 \);

(ii) \( n \leq k \leq 2n \) when \( n \geq 3 \).

Proof. (i) Trivial.

(ii) \( r(A) + r(\overline{A}) \leq 2n \) is clear. Since \( A + \overline{A} = I_n - I_n, \ n = r(A + \overline{A}) \leq r(A) + r(\overline{A}) \). This proves the necessity.

For the sufficiency, first note that the matrix \( B_1 \in \Omega_n(0, 1) \) in the proof of Theorem 3.1(ii) implies that for every integer \( k \) with \( n \leq k \leq 2n - 2 \), there exists \( A \in \Omega_n(0, 1) \) such that \( r(A) + r(\overline{A}) = k \).

Let \( B_3 = C_1 - E_{1n} \in \Omega_n(0, 1) \). Then \( r(B_3) = n - 1, \ r(\overline{B_3}) = n \).

Let \( B_4 = B_3 + E_{1n} \in \Omega_n(0, 1) \). Then \( r(B_4) = r(\overline{B_4}) = n \).

Thus for \( k = 2n - 1, 2n \), there exists \( A \in \Omega_n(0, 1) \) such that \( r(A) + r(\overline{A}) = k \). This proves the sufficiency. \( \square \)

Next we consider the case when \( A \in \Omega_n(0, 1) \) is symmetric.

Lemma 3.3. Let \( G = \begin{bmatrix} G_1 & I_{p,n-p} \\ J_{n-p,n} & J_{n-p} - I_{n-p} \end{bmatrix} \in \Omega_n(0, 1) \) with \( 1 \leq p \leq n - 1 \), where \( G_1 \) is the \( p \times p \) matrix whose entries on the main diagonal, on the secondary diagonal and above the secondary diagonal are all 0’s, while other entries are all 1’s. Then \( r(G) = n \) if \( p \) is odd, and \( r(G) = n - 1 \) if \( p \) is even.

Proof. First consider the case when \( p \) is odd. If \( p = n - 1 \), then \( n = p + 1 \) is even and it is clear that \( \det G \neq 0 \).

If \( p < n - 1 \), for \( i = p + 1, p + 2, \ldots, n - 1 \), subtract the last row of \( G \) from the \( i \)-th row, and then add the \( i \)-th column to the last column. Using Laplace expansion formula, we deduce that \( \det G \neq 0 \).

When \( p \) is even, the \( \frac{p}{2} \)-th row (column) of \( G \) is identical to the \( (\frac{p}{2} + 1) \)-th row (column). Denote by \( G_2 \) the submatrix of \( G \) obtained by deleting the \( \frac{p}{2} \)-th row and the \( \frac{p}{2} \)-th column. Note that \( G_2 = \)
\[
\begin{bmatrix}
G_3 \\
J_{p-1,n-p}
\end{bmatrix}
\end{bmatrix}
\in \Omega_{n-1}(0, 1),
\text{where } G_3 \text{ is the } (p - 1) \times (p - 1) \text{ matrix which has the same form as } G_1.
\text{Since } p - 1 \text{ is odd, } r(G) = r(G_3) = n - 1 \text{ by what we have just proved.}
\]

**Lemma 3.4.** Let \( H \in \Omega_n(0, 1) \) be the matrix whose entries on the main diagonal, on the diagonal above and the diagonal below are all 0's, while other entries are all 1's. Then

(i) \( r(H + E_{1, \frac{n}{2}} + E_{2, \frac{n}{2}}) = n \) for odd \( n \geq 5 \), \( r(H + E_{1, \frac{n}{2}} + E_{2, \frac{n}{2}} - E_{n, \frac{2n}{3}}) = n \) for odd \( n \geq 7 \), \( r(H - E_{1, \frac{n}{2}} - E_{2, \frac{n}{2}} + E_{\frac{3n}{4}, \frac{n}{2}} + E_{n, \frac{2n}{3}}) = n \) for odd \( n \geq 7 \);

(ii) \( r(H + E_{1, \frac{n}{2}} + E_{2, \frac{n}{2}}) = n \) for even \( n \geq 4 \), \( r(H + E_{1, \frac{n}{2}} + E_{2, \frac{n}{2}} - E_{\frac{3n}{4}, \frac{n}{2}} - E_{n, \frac{2n}{3}}) = n - 1 \) for even \( n \geq 6 \).

**Proof.** (i) When \( n \geq 5 \) is odd, \( \frac{n+1}{2} > 1 \). Then \( E_{1, \frac{n}{2}} \neq E_{n, \frac{2n}{3}} \). Using Laplace expansion formula, it is easy to verify that \( \det(H + E_{1, \frac{n}{2}} + E_{2, \frac{n}{2}}) \neq 0 \) for odd \( n \geq 5 \) and \( \det(H + E_{1, \frac{n}{2}} + E_{2, \frac{n}{2}} - E_{n, \frac{2n}{3}}) \neq 0 \) for odd \( n \geq 7 \).

When \( n \geq 7 \) is odd, subtract the second row of \( H - E_{1, \frac{n}{2}} - E_{2, \frac{n}{2}} + E_{\frac{3n}{4}, \frac{n}{2}} + E_{n, \frac{2n}{3}} \) from the first row, and then subtract the second column from the first column. Using Laplace expansion formula, we deduce that \( \det(H + E_{1, \frac{n}{2}} + E_{2, \frac{n}{2}}) \neq 0 \) for even \( n \geq 4 \).

(ii) When \( n \geq 4 \) is even, \( \frac{n}{2} > 1 \). Then \( E_{1, \frac{n}{2}} \neq E_{\frac{n}{2}, \frac{n}{2}} \). Using Laplace expansion formula, it is easy to verify that \( \det(H + E_{1, \frac{n}{2}} + E_{2, \frac{n}{2}} - E_{n, \frac{2n}{3}} - E_{n, \frac{2n}{3}}) \neq 0 \).

When \( n \geq 6 \) is even, \( \frac{n}{2} - 1 > 1 \). Note that the \( (\frac{n}{2} + 2) \)-th row (column) of \( H + E_{1, \frac{n}{2}} + E_{2, \frac{n}{2}} - E_{n, \frac{2n}{3}} + E_{\frac{3n}{4}, \frac{n}{2}} - E_{n, \frac{2n}{3}} \) is the sum of the first row (column) and the \( (\frac{n}{2} - 1) \)-th row (column). Then \( \det(H + E_{1, \frac{n}{2}} + E_{2, \frac{n}{2}} - E_{n, \frac{2n}{3}} + E_{\frac{3n}{4}, \frac{n}{2}}) = n - 1 \). Using Laplace expansion formula and the fact that \( J_0 - I_1 \) is nonsingular, we can deduce that the submatrix of \( H + E_{1, \frac{n}{2}} + E_{2, \frac{n}{2}} - E_{n, \frac{2n}{3}} + E_{\frac{3n}{4}, \frac{n}{2}} \) obtained by deleting the \( (\frac{n}{2} + 2) \)-th row and the \( (\frac{n}{2} + 1) \)-th column is nonsingular. Thus \( \det(H + E_{1, \frac{n}{2}} + E_{2, \frac{n}{2}} - E_{n, \frac{2n}{3}} + E_{\frac{3n}{4}, \frac{n}{2}}) = n - 1 \).

Finally we determine the possible values of \( r(A) \pm r(\overline{A}) \) in the symmetric case.

**Theorem 3.5.** Let \( n \geq 2 \) be a positive integer. Then there exists symmetric \( A \in \Omega_n(0, 1) \) with \( r(A) - r(\overline{A}) = k \) if and only if

(i) \( k = \pm 2 \) when \( n = 2 \);
(ii) \( k = 0, \pm 3 \) when \( n = 3 \);
(iii) \( -n \leq k \leq n \) with \( k \neq \pm (n - 1) \) when \( n \geq 4 \).

**Proof.** (i) and (ii) are easy to verify.

(iii) First we prove the necessity. It is clear that \( -n \leq r(A) - r(\overline{A}) \leq n \). If \( r(A) - r(\overline{A}) = -(n - 1) \), then either \( r(A) = 0 \) and \( r(\overline{A}) = n - 1 \), or \( r(A) = 1 \) and \( r(\overline{A}) = n \). For the former case, \( A = O_n \) and \( \overline{A} = J_n - I_n \) is nonsingular, a contradiction. For the latter case, note that any nonzero symmetric \( A \in \Omega_n(0, 1) \) always has a \( 2 \times 2 \) submatrix

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

Then \( r(A) \geq 2 \), a contradiction. Thus \( r(A) - r(\overline{A}) \neq -(n - 1) \). Likewise, \( r(A) - r(\overline{A}) \neq n - 1 \).

Next we prove the sufficiency. We will use the symmetric matrices \( \Omega_1 = \Omega_{p}(0, 1) \) and \( \Omega_2, H \in \Omega_3(0, 1) \) in Lemmas 3.3 and 3.4.

Note that \( r(\overline{G}) = r(G) \), \( p - 1 \) if \( p \) is odd, and \( r(G) = r(G_1) = p \) if \( p \geq 2 \) is even. Then by Lemma 3.3, \( r(G) - r(\overline{G}) = n - 1 \) for odd \( p \), and \( r(G) - r(\overline{G}) = n - p \) for even \( p \). When \( n \geq 5 \) is odd, for odd \( p \) with \( 1 \leq p \leq n - 2 \), \( r(G) - r(\overline{G}) = 3, 5, 7, \ldots, n - 2, n \); for even \( p \) with \( 2 \leq p \leq n - 1 \), \( r(G) - r(\overline{G}) = 0, 2, 4, \ldots, n - 5, n - 3 \). Thus \( k \leq 0, \pm 2, \pm 3, \ldots, \pm (n - 2), \pm n \) for odd \( n \geq 5 \). When \( n \geq 4 \) is even, for odd \( p \) with \( 1 \leq p \leq n - 1 \), \( r(G) - r(\overline{G}) = 2, 4, 6, \ldots, n - 2, n \); for even \( p \) with \( 2 \leq p \leq n - 2 \), \( r(G) - r(\overline{G}) = 1, 3, 5, \ldots, n - 5, n - 3 \). Thus \( k \leq \pm 1, \pm 2, \ldots, \pm (n - 2), \pm n \) for even \( n \geq 4 \).

By Lemma 3.4(i), for odd \( n \geq 5 \), \( r(H - E_{1, \frac{n}{2}} + E_{2, \frac{n}{2}}) = n \). Since \( r(H + E_{1, \frac{n}{2}} - E_{\frac{3n}{4}, \frac{n}{2}}) = r(H - E_{1, \frac{n}{2}} - E_{\frac{3n}{4}, \frac{n}{2}}) = n - 1 \), \( n \) is odd, and \( r(H + E_{1, \frac{n}{2}} - E_{\frac{3n}{4}, \frac{n}{2}} - E_{n, \frac{2n}{3}}) = 1 \). Thus \( k \) can be \( \pm 1 \) for odd \( n \geq 5 \). By Lemma 3.4(ii), for even \( n \geq 4 \), \( r(H + E_{1, \frac{n}{2}} + E_{2, \frac{n}{2}}) = n \). Since \( r(H - E_{1, \frac{n}{2}} - E_{2, \frac{n}{2}}) = r(H - E_{1, \frac{n}{2}} - E_{\frac{3n}{4}, \frac{n}{2}}) = n \) when \( n \) is even, \( k \) can be \( 0 \) for even \( n \geq 4 \).
Thus for \( k = 0, \pm 1, \pm 2, \ldots, \pm (n - 2), \pm n \) with \( n \geq 4 \), there exists symmetric \( A \in \Omega_n \) such that \( r(A) - r(\overline{A}) = k \). This completes the proof. \( \square \)

**Theorem 3.6.** Let \( n \geq 2 \) be a positive integer. Then there exists symmetric \( A \in \Omega_n \) with \( r(A) + r(\overline{A}) = k \) if and only if

(i) \( k = \pm 2 \) when \( n = 2 \);
(ii) \( k = 3, 4 \) when \( n = 3 \);
(iii) \( k = 4, 5, 6, 8 \) when \( n = 4 \);
(iv) \( n \leq k \leq 2n \) when \( n \geq 5 \).

**Proof.** (i) and (ii) are easy to verify.

(iii) Denote by \( f(A) \) the number of 1’s in \( A \). Then for symmetric \( A \in \Omega_n \), \( f(A) \) and \( f(\overline{A}) \) are even with \( f(A) + f(\overline{A}) = 12 \). Since \( A \) and \( \overline{A} \) are mutually complementary, we may suppose \( f(A) \leq 6 \).

If \( f(A) = 0 \), then \( A = O_4 \) and thus \( r(A) + r(\overline{A}) = 0 + 4 = 4 \).

If \( f(A) = 2 \), under permutation similarity, it suffices to consider the case \( A = E_{12} + E_{21} \). A direct computation shows that \( r(A) + r(\overline{A}) = 2 + 3 = 5 \).

If \( f(A) = 4 \), under permutation similarity, it suffices to consider the cases \( A = E_{12} + E_{21} + E_{13} + E_{31} \) and \( A = E_{12} + E_{21} + E_{32} + E_{43} \). A direct computation shows that \( r(A) + r(\overline{A}) = 2 + 4 = 6 \) in the first case and \( r(A) + r(\overline{A}) = 4 + 2 = 6 \) in the second case.

If \( f(A) = 6 \), under permutation similarity, it suffices to consider the cases \( A = E_{12} + E_{21} + E_{13} + E_{13} + E_{23} + E_{32} \), \( A = E_{12} + E_{21} + E_{31} + E_{41} \) and \( A = E_{12} + E_{21} + E_{13} + E_{24} + E_{42} \). A direct computation shows that \( r(A) + r(\overline{A}) = 3 + 2 = 5 \) in the first case, \( r(A) + r(\overline{A}) = 2 + 3 = 5 \) in the second case and \( r(A) + r(\overline{A}) = 4 + 4 = 8 \) in the third case.

Therefore, \( r(A) + r(\overline{A}) \) can only be 4, 5, 6, 8 for symmetric \( A \in \Omega_4 \).

(iv) The necessity has been proved in Theorem 3.2(ii). Now we prove the sufficiency.

By Lemma 3.3 and the proof of Theorem 3.5 (iii), \( k \) can be \( n - 1 + p \) for \( 1 \leq p \leq n - 1 \).

By Lemma 3.4 (i), for odd \( n \geq 5 \), \( r(H + E_{1,2} + E_{2,3} + E_{3,4} + E_{4,1}) + r(H - E_{1,2} - E_{2,3} - E_{3,4} - E_{4,1}) = n + (n - 1) = 2n - 1 \). By Lemma 3.4 (ii), for even \( n \geq 6 \), \( r(H + E_{1,2} + E_{2,3} + E_{3,4} + E_{4,1} + E_{5,1} + E_{5,2} + E_{5,3} + E_{5,4} + E_{5,5} + E_{6,1} + E_{6,2} + E_{6,3} + E_{6,4} + E_{6,5} + E_{6,6}) = (n - 1) + n = 2n - 1 \). Thus \( k \) can be \( 2n - 1 \) when \( n \geq 5 \).

By Lemma 3.4 (i), for odd \( n \geq 7 \), \( r(H + E_{1,2} + E_{2,3} + E_{3,4} + E_{4,1} + E_{5,1} + E_{5,2} + E_{5,3} + E_{5,4} + E_{5,5} + E_{6,1} + E_{6,2} + E_{6,3} + E_{6,4} + E_{6,5} + E_{6,6}) = n + n = 2n \). By Lemma 3.4 (ii), for even \( n \geq 4 \), \( r(H + E_{1,2} + E_{2,3} + E_{3,4} + E_{4,1} + E_{5,1} + E_{5,2} + E_{5,3} + E_{5,4} + E_{5,5} + E_{6,1} + E_{6,2} + E_{6,3} + E_{6,4} + E_{6,5} + E_{6,6}) = (n - 1) + (n - 1) = 2n - 2 \). Note that

\[
\begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 1 & 0
0 & 0 & 1 & 0 & 1 & 0 & 1
1 & 0 & 0 & 0 & 0 & 1 & 1
0 & 1 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}
+ \begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 0 & 1
0 & 0 & 1 & 0 & 1 & 0 & 1
1 & 1 & 0 & 0 & 0 & 1 & 1
1 & 1 & 0 & 0 & 0 & 1 & 1
\end{pmatrix} = 5 + 5 = 10.
\]

Thus \( k \) can be \( 2n \) when \( n \geq 4 \).

Then for symmetric \( A \in \Omega_n \) with \( n \geq 5 \), \( r(A) + r(\overline{A}) \) can be \( n, n + 1, \ldots, 2n \). \( \square \)

**4 Conclusion**

This paper considers two kinds of complement matrices \( A^c \) and \( \overline{A} \) of a \( \{0, 1\} \)-matrix \( A \). If \( A \) is a square \( \{0, 1\} \)-matrix with each diagonal entry being 0, then \( A \) and its complement \( \overline{A} \) correspond to a strict digraph \( D \) and its complement \( \overline{D} \). We mainly discuss their rank relations. As is shown in the proof, we construct a \( \{0, 1\} \)-matrix \( A \) for each possible value of \( r(A) \) in both general and symmetric cases.

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