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Research Article

Juan de Dios Pérez*

On the structure vector field of a real hypersurface in complex quadric

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Abstract: From the notion of Jacobi type vector fields for a real hypersurface in complex quadric Q^m we prove that if the structure vector field is of Jacobi type it is Killing when the real hypersurface is either Hopf or compact. In such cases we classify real hypersurfaces whose structure vector field is of Jacobi type.

Keywords: Real hypersurfaces, Complex quadric, Jacobi type vector field, Structure vector field, Killing vector field

MSC: 53C40, 53C15

1 Introduction

The complex quadric $Q^m = SO_{m+2}/SO_mSO_2$ is a compact Hermitian symmetric space of rank 2. It is also a complex hypersurface in complex projective space $\mathbb{C}P^{m+1}$, [1]. Q^m is equipped with two geometric structures: a complex conjugation A and a Kähler structure J .

Real hypersurfaces M in Q^m are immersed submanifolds of real codimension 1. The Kähler structure J of Q^m induces on M an almost contact metric structure (ϕ, ξ, η, g) , where ϕ is the structure tensor field, ξ is the structure (or Reeb) vector field, η is a 1-form and g is the induced Riemannian metric on M .

Real hypersurfaces M in Q^m whose Reeb flow is isometric are classified in [2]. They obtain tubes around the totally geodesic $\mathbb{C}P^k$ in Q^m when $m = 2k$. The condition of isometric Reeb flow is equivalent to the commuting condition of the shape operator S with the structure tensor field ϕ of M .

It is known that a Killing vector field X on a Riemannian manifold (\bar{M}, \bar{g}) satisfies $\mathcal{L}_X \bar{g} = 0$, where \mathcal{L} denotes the Lie derivative. Killing vector fields are a powerful tool in studying the geometry of a Riemannian manifold. A Killing vector field is a Jacobi vector field along any geodesic. However the converse is not true: the position vector on the euclidean space \mathbb{R}^n is a Jacobi field along any geodesic of \mathbb{R}^n but it is not Killing. Studying when the structure vector field of a complex projective space is Killing, Deshmukh, [3], introduced the notion of Jacobi type vector fields on a Riemannian manifold. A vector field Y on \bar{M} is of Jacobi type if it satisfies

$$\bar{\nabla}_X \bar{\nabla}_X Y + \bar{R}(Y, X)X = 0 \quad (1)$$

for any vector field X tangent to \bar{M} , where $\bar{\nabla}$ denotes the Levi-Civita connection on \bar{M} and \bar{R} its Riemannian curvature tensor. Naturally any Jacobi type vector field on \bar{M} is a Jacobi vector field along any geodesic of \bar{M} .

As on a real hypersurface M in Q^m we have a special vector field, the structure one ξ , it is interesting to see if it is Killing when it is of Jacobi type. In this sense we will prove the following

*Corresponding Author: Juan de Dios Pérez: Departamento de Geometría y Topología, Universidad de Granada, 18071-Granada, Spain, E-mail: jdperez@ugr.es

Theorem 1.1. *Let M be a real hypersurface in Q^m , $m \geq 3$. If M is either compact or Hopf and the structure vector field is of Jacobi type, it is a Killing vector field.*

By this Theorem and the classification of real hypersurfaces with geodesic Reeb flow we obtain

Corollary 1.2. *Let M be a compact or Hopf real hypersurface in Q^m , $m \geq 3$. Then the structure vector field is of Jacobi type if and only if m is even, say $m = 2k$, and M is locally congruent to a tube around a totally geodesic $\mathbb{C}P^k$ in Q^m .*

Similar results for real hypersurfaces of complex two-plane Grassmannians were obtained in [4].

2 The space Q^m

For the study of Riemannian geometry of Q^m see [1]. All the notations we will use since now are the ones in [2].

The complex projective space $\mathbb{C}P^{m+1}$ is considered as the Hermitian symmetric space of the special unitary group SU_{m+2} , namely $\mathbb{C}P^{m+1} = SU_{m+2}/S(U_{m+1}U_1)$. The symbol $o=[0,\dots,0,1]$ in $\mathbb{C}P^{m+1}$ is the fixed point of the action of the stabilizer $S(U_{m+1}U_1)$. The action of the special orthogonal group $SO_{m+2} \subset SU_{m+2}$ on $\mathbb{C}P^{m+1}$ is of cohomogeneity one. A totally geodesic projective space $\mathbb{R}P^{m+1} \subset \mathbb{C}P^{m+1}$ is an orbit containing o . The second singular orbit of this action is the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. It is a homogeneous model which interprets geometrically the complex quadric Q^m as the Grassmann manifold $G_2^+(\mathbb{R}^{m+2})$ of oriented 2-planes in \mathbb{R}^{m+2} . For $m = 1$ the complex quadric is isometric to a sphere S^2 of constant curvature. For $m = 2$ the complex quadric Q^2 is isometric to the Riemannian product of two 2-spheres with constant curvature. Therefore we assume the dimension of the complex quadric Q^m to be greater than or equal to 3.

Moreover, the complex quadric Q^m is the complex hypersurface in $\mathbb{C}P^{m+1}$ defined by the equation $z_1^2 + \dots + z_{m+2}^2 = 0$, where z_i , $i = 1, \dots, m + 2$, are homogeneous coordinates on $\mathbb{C}P^{m+1}$. The Kähler structure of complex projective space induces canonically a Kähler structure (J, g) on Q^m , where g is the Riemannian metric induced by the Fubini-Study metric of $\mathbb{C}P^{m+1}$.

A point $[z]$ in $\mathbb{C}P^{m+1}$ is the complex span of z , that is $[z] = \{\lambda z/\lambda \in \mathbb{C}\}$, where z is a nonzero vector of \mathbb{C}^{m+2} . For each $[z]$ in $\mathbb{C}P^{m+1}$ the tangent space $T_{[z]}\mathbb{C}P^{m+1}$ can be identified canonically with the orthogonal complement of $[z] \oplus [\bar{z}]$ in \mathbb{C}^{m+2} .

The shape operator $A_{\bar{z}}$ of Q^m with respect to the unit normal vector \bar{z} is given by

$$A_{\bar{z}}w = \bar{w}$$

for all $w \in T_{[z]}Q^m$. Then $A_{\bar{z}}$ is a complex conjugation restricted to $T_{[z]}Q^m$. Thus $T_{[z]}Q^m$ is decomposed into

$$T_{[z]}Q^m = V(A_{\bar{z}}) \oplus JV(A_{\bar{z}})$$

where $V(A_{\bar{z}})$ is the (+1)-eigenspace of $A_{\bar{z}}$ and $JV(A_{\bar{z}})$ is the (-1)-eigenspace of $A_{\bar{z}}$. Geometrically, it means that $A_{\bar{z}}$ defines a real structure on the complex vector space $T_{[z]}Q^m$. The set of all shape operators $A_{\lambda\bar{z}}$ defines a parallel rank 2 subbundle \mathfrak{A} of the endomorphism bundle $End(TQ^m)$ which consists of all the real structures of the tangent space of Q^m . For any $A \in \mathfrak{A}$, $A^2 = I$ and $AJ = -JA$.

The Gauss equation of Q^m in $\mathbb{C}P^{m+1}$ yields that the Riemannian curvature tensor \bar{R} of Q^m is given by

$$\begin{aligned} \bar{R}(X, Y)Z &= g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ \\ &\quad + g(AY, Z)AX - g(AX, Z)AY + g(JAY, Z)JAX - g(JAX, Z)JAY \end{aligned} \tag{2}$$

where J is the complex structure and A is a real structure in \mathfrak{A} .

For every vector field W tangent to Q^m there is a complex conjugation $A \in \mathfrak{A}$ and orthonormal vectors $X, Y \in V(A)$ such that

$$W = \cos(t)X + \sin(t)JY$$

for some $t \in [0, \frac{\pi}{4}]$.

3 Real hypersurfaces in Q^m

Let M be a real hypersurface in Q^m , that is, a submanifold of Q^m with real codimension one. The induced Riemannian metric on M will also be denoted by g , and ∇ denotes the Riemannian connection of (M, g) . Let N be a unit normal vector field of M and S the shape operator of M with respect to N . For any X tangent to M we write

$$JX = \phi X + \eta(X)N$$

where ϕX denotes the tangential component of JX and $\eta(X)N$ its normal component. The structure vector field (or Reeb vector field) ξ is defined by $\xi = -JN$. The 1-form η is given by $\eta(X) = g(X, \xi)$ for any vector field X tangent to M . Therefore, on M we have an almost contact metric structure (ϕ, ξ, η, g) . Thus,

$$\phi^2 X = -X + \eta(X)\xi, \eta(\xi) = 1, g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \phi\xi = 0 \quad (3)$$

for all tangent vector fields X, Y on M . Moreover, the parallelism of J yields

$$(\nabla_X \phi)Y = \eta(Y)SX - g(SX, Y)\xi \quad (4)$$

and

$$\nabla_X \xi = \phi SX \quad (5)$$

for any X, Y tangent to M .

At each point $[z] \in M$ we choose a real structure $A \in \mathfrak{A}_{[z]}$ such that

$$\begin{aligned} N_{[z]} &= \cos(t)Z_1 + \sin(t)JZ_2 \\ AN_{[z]} &= \cos(t)Z_1 - \sin(t)JZ_2 \\ \xi_{[z]} &= -\cos(t)JZ_1 + \sin(t)Z_2 \\ A\xi_{[z]} &= \cos(t)JZ_1 + \sin(t)Z_2 \end{aligned} \quad (6)$$

where Z_1, Z_2 are orthonormal vectors in $V(A)$ and $0 \leq t \leq \frac{\pi}{4}$. Therefore $g(AN, \xi) = 0$.

Let $X \in T_{[z]}M$. Then AX is decomposed into

$$AX = BX + \rho(X)N \quad (7)$$

where BX is the tangential component of AX and $\rho(X)N$ is its normal component, with $\rho(X) = g(AX, N)$. As seen above $\rho(\xi) = 0$.

From (2) the curvature tensor R of M is given by

$$\begin{aligned} R(X, Y)Z &= g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &\quad - 2g(\phi X, Y)\phi Z + g(AY, Z)(AX)^T - g(AX, Z)(AY)^T + g(JAY, Z)(JAX)^T - g(JAX, Z) \\ &\quad (JAY)^T + g(SY, Z)SX - g(SX, Z)SY \end{aligned} \quad (8)$$

for any X, Y, Z tangent to M , where $(\cdot)^T$ denotes the tangential component of the correspondent vector field. From (8) the Ricci tensor of M is given (see [5]) by

$$\text{Ric}(X) = (2m - 1)X - 3\eta(X)\xi + \eta(B\xi)BX - \rho(X)\phi B\xi + \eta(BX)B\xi + (\text{trace}S)SX - S^2X \quad (9)$$

for any X tangent to M . Moreover, the Codazzi equation is given by

$$\begin{aligned} g((\nabla_X S)Y - (\nabla_Y S)X, Z) &= \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) \\ &\quad - 2g(\phi X, Y)\eta(Z) + g(X, AN)g(AY, Z) - g(Y, AN)g(AX, Z) \\ &\quad + g(X, A\xi)g(JAY, Z) - g(Y, A\xi)g(JAX, Z) \end{aligned} \quad (10)$$

for any X, Y, Z tangent to M .

The real hypersurface M is called Hopf if the Reeb vector field is an eigenvector of the shape operator S , that is

$$S\xi = \alpha\xi$$

where $\alpha = g(S\xi, \xi)$ is the Reeb function.

4 Proof of Theorem 1.1

Let us suppose that ξ is of Jacobi type. Then $\nabla_X \nabla_X \xi + R(\xi, X)X = 0$ for any X tangent to M .

Take an orthonormal basis $\{e_1, \dots, e_{2m-1}\}$ of vector fields tangent to M . As ξ is of Jacobi type, $\sum_{i=1}^{2m-1} \nabla_{e_i} \nabla_{e_i} \xi + Ric(\xi) = 0$. That is,

$$\sum_{i=1}^{2m-1} \nabla_{e_i} \phi S e_i + Ric(\xi) = 0 \tag{11}$$

From (9) $Ric(\xi) = 2(m - 2)\xi + \eta(B\xi)B\xi - \rho(\xi)\phi B\xi + \eta(B\xi)B\xi + (traceS)S\xi - S^2\xi$. As $\rho(\xi) = 0$ and $\eta(B\xi) = g(A\xi, \xi)$ we obtain

$$Ric(\xi) = 2(m - 2)\xi + 2g(A\xi, \xi)B\xi + (traceS)S\xi - S^2\xi. \tag{12}$$

From (11) and (12) we get $\sum_{i=1}^{2m-1} \nabla_{e_i} \phi S e_i + 2(m - 2)\xi + 2g(A\xi, \xi)B\xi + (traceS)S\xi - S^2\xi = 0$. Taking its scalar product with ξ and bearing in mind that $g(B\xi, \xi) = g(A\xi, \xi)$ we obtain

Lemma 4.1. *Let M be a real hypersurface in Q^m , $m \geq 3$, such that ξ is of Jacobi type. Then*

$$-traceS^2 + 2(m - 2) + 2g(A\xi, \xi)^2 + (traceS)\eta(S\xi) = 0$$

Now we compute

$$\begin{aligned} \|\phi S - S\phi\|^2 &= \sum_{i=1}^{2m-1} g((\phi S - S\phi)e_i, (\phi S - S\phi)e_i) \\ &= \sum_{i,j=1}^{2m-1} g((\phi S - S\phi)e_i, e_j)g((\phi S - S\phi)e_i, e_j) \\ &= 2 \sum_{i,j=1}^{2m-1} g(\phi S e_j, e_i)g(\phi S e_j, e_i) + 2 \sum_{i,j=1}^{2m-1} g(\phi S e_j, e_i)g(\phi S e_i, e_j) \\ &= -2 \sum_{j=1}^{2m-1} g(\phi S e_j, S\phi e_j) + 2 \sum_{j=1}^{2m-1} g(\phi S e_j, \phi S e_j) \\ &= 2traceS^2 - 2g(S^2\xi, \xi) - 2 \sum_{j=1}^{2m-1} g(\nabla_{e_j} \xi, S\phi e_j) \end{aligned} \tag{13}$$

where we have used (4).

Take now $U = \nabla_\xi \xi = \phi S \xi$. Then we have

$$\begin{aligned} div(U) &= \sum_{i=1}^{2m-1} g(\nabla_{e_i} U, e_i) = \sum_{i=1}^{2m-1} g(\nabla_{e_i} \phi S \xi, e_i) \\ &= \sum_{i=1}^{2m-1} g((\nabla_{e_i} \phi)S\xi, e_i) + \sum_{i=1}^{2m-1} g(\phi \nabla_{e_i} S\xi, e_i) \\ &= \sum_{i=1}^{2m-1} g(\eta(S\xi)S e_i - g(S e_i, S\xi)\xi, e_i) - \sum_{i=1}^{2m-1} g(\nabla_{e_i} S\xi, \phi e_i), \end{aligned} \tag{14}$$

that is

Lemma 4.2. *Let M be a real hypersurface in Q^m , $m \geq 3$, and $U = \phi S \xi$. Then*

$$div(U) = (traceS)\eta(S\xi) - \eta(S^2\xi) - \sum_{i=1}^{2m-1} g(\nabla_{e_i} S\xi, \phi e_i).$$

From (13) and Lemma 4.2 we obtain

$$div(U) - \frac{1}{2}\|\phi S - S\phi\|^2 = -traceS^2 + \eta(S\xi)(traceS) - \sum_{i=1}^{2m-1} g((\nabla_{e_i} S)\xi, \phi e_i). \tag{15}$$

Then

$$\begin{aligned} \sum_{i=1}^{2m-1} (g((\nabla_\xi S)e_i, \phi e_i) - \sum_{i=1}^{2m-1} g(\phi(\nabla_\xi S)e_i, e_i)) \\ = -trace(\phi(\nabla_\xi S)) = -trace((\nabla_\xi S)\phi) = -\sum_{i=1}^{2m-1} g((\nabla_\xi S)\phi e_i, e_i) \\ = -\sum_{i=1}^{2m-1} g((\nabla_\xi S)e_i, \phi e_i). \end{aligned} \tag{16}$$

Thus we conclude

$$\sum_{i=1}^{2m-1} g((\nabla_\xi S)e_i, \phi e_i) = 0. \tag{17}$$

Bearing in mind (17) Codazzi equation yields

$$\begin{aligned} \sum_{i=1}^{2m-1} g((\nabla_{e_i} S)\xi, \phi e_i) &= -\sum_{i=1}^{2m-1} g(\phi e_i, \phi e_i) \\ + \sum_{i=1}^{2m-1} g(e_i, AN)g(A\xi, \phi e_i) + \sum_{i=1}^{2m-1} g(e_i, A\xi)g(JA\xi, \phi e_i) \\ &\quad - \sum_{i=1}^{2m-1} g(\xi, A\xi)g(JAe_i, \phi e_i) \\ &= 2(m - 2) - g(AN, N)^2 + g(\xi, A\xi)g(AN, N) - g(\xi, A\xi)(traceA). \end{aligned} \tag{18}$$

From (6) $g(AN, N) = \cos(2t) = -g(A\xi, \xi)$. Moreover, as $\{e_1, \dots, e_{2m-1}, N\}$ is an orthonormal basis of vectors tangent to Q^m at any point of M , $\{Je_1, \dots, Je_{2m-1}, JN\}$ is also an orthonormal basis. Then $\text{trace}A = \sum_{i=1}^{2m-1} g(AJe_i, Je_i) + g(AJN, JN) = -\sum_{i=1}^{2m-1} g(JAe_i, Je_i) - g(JAN, AN) = -\sum_{i=1}^{2m-1} g(Ae_i, e_i) - g(AN, N) = -\text{trace}A$. Thus $\text{trace}A = 0$ and (18) becomes

$$\sum_{i=1}^{2m-1} g((\nabla_{e_i} S)\xi, \phi e_i) = -2(m-2) - 2g(A\xi, \xi)^2. \quad (19)$$

From this, Lemma 4.1, Lemma 4.2 and (17) we get

$$\text{div}(U) = \frac{1}{2} \|\phi S - S\phi\|^2. \quad (20)$$

Now if M is Hopf, $U = 0$ and then $\phi S - S\phi = 0$.

If M is compact, $\frac{1}{2} \int_M \|\phi S - S\phi\|^2 dV = 0$. Thus again $\phi S - S\phi = 0$.

In both cases as $(\mathcal{L}_\xi g)(X, Y) = g((\phi S - S\phi)X, Y)$, for any $X, Y \in TM$, we conclude $\mathcal{L}_\xi g = 0$ and ξ is Killing, obtaining our Theorem.

As $\phi S = S\phi$ we have, [2], that $m = 2k$ and M must be locally congruent to a tube around a totally geodesic $\mathbb{C}P^k$ in Q^m .

Bearing in mind the expression of the shape operator S of such a real hypersurface, [2], it is immediate to see that its structure vector field is of Jacobi type and we conclude the proof of our Corollary.

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