On algebraic characterization of SSC of the Jahangir’s graph \(J_{n,m}\)

Abstract: In this paper, some algebraic and combinatorial characterizations of the spanning simplicial complex \(\Delta_s(J_{n,m})\) of the Jahangir’s graph \(J_{n,m}\) are explored. We show that \(\Delta_s(J_{n,m})\) is pure, present the formula for \(f\)-vectors associated to it and hence deduce a recipe for computing the Hilbert series of the Face ring \(k[\Delta_s(J_{n,m})]\). Finally, we show that the face ring of \(\Delta_s(J_{n,m})\) is Cohen-Macaulay and give some open scopes of the current work.

Keywords: Simplicial complexes, Spanning trees, Face ring, Hilbert series, \(f\)-vectors, Cohen Macaulay

MSC: 13P10, 13H10, 13F20, 13C14

1 Introduction

The concept of spanning simplicial complex (SSC) associated with the edge set of a simple finite connected graph is introduced by Anwar, Raza and Kashif in [1]. They revealed some important algebraic properties of SSC of a unicyclic graph. Kashif, Raza and Anwar further established the theory and explored algebraic characterizations of some more general classes of \(n\)-cyclic graphs in [10, 11]. The problem of finding the SSC of SSC of a unicyclic graph. Kashif, Raza and Anwar further established the theory and explored algebraic properties of SSC of another class \(n\)-cyclic graphs with a common edge.

In this article, we discuss some algebraic and combinatorial properties of the spanning simplicial complex \(\Delta_s(J_{n,m})\) of a certain class of cyclic graphs, \(J_{n,m}\). For simplicity, we fixed \(n = 2\) in our results. Here, \(J_{n,m}\) is the class of Jahangir’s graph defined in [12] as follows:

The Jahangir’s graph \(J_{n,m}\), for \(m \geq 3\), is a graph on \(nm + 1\) vertices i.e., a graph consisting of a cycle \(C_{nm}\) with one additional vertex which is adjacent to \(m\) vertices of \(C_{nm}\) at distance \(n\) to each other on \(C_{nm}\).

More explicitly, it consists of a cycle \(C_{nm}\) which is further divided into \(m\) consecutive cycles \(C_i\) of equal length such that all these cycles have one vertex common and every pair of consecutive cycles has exactly one edge common. For example the graph \(J_{2,3}\) is given in Figure 1. We fix the edge set of \(J_{2,m}\) as follows:

\[
E = \{e_{11}, e_{12}, e_{13}, e_{21}, e_{22}, e_{23}, \ldots, e_{m1}, e_{m2}, e_{m3}\}.
\]

Here, \(\{e_{k1}, e_{k2}, e_{k3}, e_{(k+1)1}\}\) is the edge set of the cycle \(C_k\) for \(k \in \{1, 2, \ldots, m-1\}\) and \(\{e_{m1}, e_{m2}, e_{m3}, e_{11}\}\) is the edge set of cycle \(C_m\). Also \(e_{k1}\) always represents the common edge between \(C_{k-1}\) and \(C_k\) for \(k \in \{1, 2, \ldots, m-1\}\) and \(e_{11}\) is the common edge between the cycle \(C_m\) and \(C_1\).

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2 Preliminaries

In this section, we give some background and preliminaries of the topic and define some important notions to make this paper self-contained. However, for more details of the notions we refer the reader to [3–7, 13, 14].

**Definition 2.1.** A spanning tree of a simple connected finite graph \( G(V, E) \) is a subtree of \( G \) that contains every vertex of \( G \).

We represent the collection of all edge-sets of the spanning trees of \( G \) by \( s(G) \), in other words;

\[
s(G) := \{ E(T_i) \subseteq E \mid \text{where } T_i \text{ is a spanning tree of } G \}.
\]

**Lemma 2.2.** Let \( G = (V, E) \) be a simple finite connected graph containing \( m \) cycles. Then its spanning tree contains exactly \(|E| - m\) edges.

**Proof.** A spanning tree of a graph is its spanning subgraph containing no cycles and no disconnection. If \( G \) is a unicyclic graph then deletion of one edge from it results in a spanning tree. If more than one edge is removed from the cycle in \( G \) then a disconnection is obtained which is not a spanning tree. Therefore, spanning tree has exactly \(|E| - 1\) edges.

If \( G \) has \( m \) disjoint cycles in it i.e. cycles sharing no common edges, then its spanning tree is obtained by removing exactly \( m \) edges from it, one from each of its cycle. Therefore, its spanning tree has \(|E| - m\) edges in it.

If any two cycles of \( G \) share one or more common edges and remaining are disjoint cycles, then one edge is needed to be removed from each cycle of \( G \) to obtain a spanning tree. However, if a common edge between two cycles is removed then exactly one edge from non common edges must be removed from the resulting big cycle. Therefore, its spanning tree has \(|E| - m\) edges in it. This can be extended to any number of cycles in \( G \) sharing common edges. This completes the proof.

Applying Lemma 2.2, we can obtain the spanning tree of the Jahangir’s graph \( J_{2,m} \) by removing exactly \( m \) edges from it keeping in view the following:

- Not more than one edge can be removed from the non common edges of any cycle.
- If a common edge between two or more consecutive cycles is removed then exactly one edge must be removed from the resulting big cycle.
- Not all common edges can be removed simultaneously.

This method is referred as the cutting-down method. For example, by using the cutting-down method for the graph \( J_{2,3} \) given in Fig. 1 we obtain:

\[
s(J_{2,3}) = \{ \{e_{11}, e_{21}, e_{31}, e_{12}, e_{22}, e_{32}\}, \{e_{11}, e_{21}, e_{31}, e_{12}, e_{22}, e_{32}\}, \{e_{11}, e_{21}, e_{31}, e_{12}, e_{22}, e_{32}\}, \{e_{11}, e_{21}, e_{31}, e_{12}, e_{22}, e_{32}\}, \{e_{21}, e_{31}, e_{12}, e_{22}, e_{32}\}, \{e_{21}, e_{31}, e_{12}, e_{22}, e_{32}\}, \{e_{21}, e_{31}, e_{12}, e_{22}, e_{32}\}, \{e_{21}, e_{31}, e_{12}, e_{22}, e_{32}\}, \{e_{21}, e_{31}, e_{12}, e_{22}, e_{32}\}, \{e_{21}, e_{31}, e_{12}, e_{22}, e_{32}\} \}
\]
Definition 2.3. A simplicial complex $\Delta$ over a finite set $[n] = \{1, 2, \ldots, n\}$ is a collection of subsets of $[n]$, with the property that $\{i\} \in \Delta$ for all $i \in [n]$, and if $F \in \Delta$ then $\Delta$ will contain all the subsets of $F$ (including the empty set). An element of $\Delta$ is called a face of $\Delta$, and the dimension of a face $F$ of $\Delta$ is defined as $|F| - 1$, where $|F|$ is the number of vertices of $F$. The maximal faces of $\Delta$ under inclusion are called facets of $\Delta$. The dimension of the simplicial complex $\Delta$ is:

$$\dim \Delta = \max \{\dim F | F \in \Delta\}.$$ 

We denote the simplicial complex $\Delta$ with facets $\{F_1, \ldots, F_q\}$ by

$$\Delta = \{F_1, \ldots, F_q\}$$

Definition 2.4. For a simplicial complex $\Delta$ having dimension $d$, its $f$ – vector is a $d + 1$-tuple, defined as:

$$f(\Delta) = (f_0, f_1, \ldots, f_d)$$

where $f_i$ denotes the number of $i$ – dimensional faces of $\Delta$.

Definition 2.5 (Spanning Simplicial Complex). Let $G(V, E)$ be a simple finite connected graph and $s(G) = \{E_1, E_2, \ldots, E_k\}$ be the edge-sets of all possible spanning trees of $G(V, E)$, then we defined (in [1]) a simplicial complex $\Delta_s(G)$ on $E$ such that the facets of $\Delta_s(G)$ are precisely the elements of $s(G)$, we call $\Delta_s(G)$ as the spanning simplicial complex of $G(V, E)$. In other words;

$$\Delta_s(G) = \{E_1, E_2, \ldots, E_k\}.$$
3 Spanning trees of $\mathcal{J}_{2,m}$ and Face ring $\Delta_s(\mathcal{J}_{2,m})$

In this section, we give two lemmas which give an important characterization of the graph $\mathcal{J}_{2,m}$ and its spanning simplicial complex $s(\mathcal{J}_{2,m})$. We present a proposition which gives the $f$-vectors and the dimension of the $\mathcal{J}_{2,m}$. Finally, in Theorem 3.13 we give the formulation for the Hilbert series of the Face ring $k[\Delta_s(\mathcal{J}_{2,m})]$.

**Definition 3.1.** Let $C_{i_1}, C_{i_2}, \ldots, C_{i_k}$ be consecutive cycles in the Jahangir’s graph $\mathcal{J}_{2,m}$. Then the cycle obtained by deleting the common edges between the consecutive cycles $C_{i_1}, C_{i_2}, \ldots, C_{i_k}$ is a new cycle of the Jahangir’s graph $\mathcal{J}_{2,m}$ is denoted by $C_{i_1, i_2, \ldots, i_k}$. The cardinality count of the number of edges in the cycle $C_{i_1, i_2, \ldots, i_k}$ is denoted by $\beta_{i_1, i_2, \ldots, i_k} = |C_{i_1, i_2, \ldots, i_k}|$.

The following lemma computes the total number of cycles in the Jahangir’s graph $\mathcal{J}_{2,m}$ and the cardinality count of the edges in these cycles.

**Lemma 3.2 (Characterization of $\mathcal{J}_{2,m}$).** Let $\mathcal{J}_{2,m}$ be the graph with the edges $E$ as is defined in eq. (1) and $C_1, C_2, \ldots, C_m$ be its $m$ consecutive cycles of equal lengths, then the total number of cycles in the graph are

$$\tau = m^2$$

such that $\beta_{i_1, i_2, \ldots, i_k} = 2(k + 1)$.

*Proof.* The Jahangir’s graph $\mathcal{J}_{2,m}$ contains more than just $m$ consecutive cycles. The remaining cycles can be obtained by deleting the common edges between any number (included) of consecutive cycles and getting a cycle by their remaining edges. The cycle obtained in this way by adjoining consecutive cycles $C_{i_1}, C_{i_2}, \ldots, C_{i_k}$ is denoted by $C_{i_1, i_2, \ldots, i_k}$. Therefore, we get the following cycles

$$C_{1, 2}, C_{2, 3}, \ldots, C_{m-1, m}, C_{m, 1}, C_{1, 2, 3, \ldots, m}, C_{m-2, m-1, m}, C_{m-1, m-1},$$

$$C_{m, 1, 2, \ldots, m}, C_{2, 3, 4, \ldots, m}, C_{3, 4, 5, \ldots, m}, C_{m-1, 2, \ldots, m-1}.$$

Combining these with $m$ cycles given we have total cycles in the graph $\mathcal{J}_{2,m}$,

$$C_{i_1, i_2, \ldots, i_k} \quad i_j \in \{1, 2, \ldots, m\} \text{ and } 1 \leq k \leq m,$$

such that $i_{j+1} = i_j + 1$ if $i_j \neq m$ and $i_{j+1} = 1$ if $i_j = m$.

Now for a fixed value of $k$, simple counting reveals that the total number of cycles $C_{i_1, i_2, \ldots, i_k}$ is $m$ for $i_k < m$. Hence the total number of cycles in $\mathcal{J}_{2,m}$ is $\tau$. Also it is clear from the construction above that $C_{i_1, i_2, \ldots, i_k}$ is obtained by deleting common edges between consecutive cycles $C_{i_1}, C_{i_2}, \ldots, C_{i_k}$ which are $k - 1$ in number. Therefore, the order of the cycle $C_{i_1, i_2, \ldots, i_k}$ is obtained by adding orders of all $C_{i_1}, C_{i_2}, \ldots, C_{i_k}$ that is, $4k$ and subtracting $2(k - 1)$ from it, since the common edges are being counted twice in sum. This implies

$$\beta_{i_1, i_2, \ldots, i_k} = |C_{i_1, i_2, \ldots, i_k}| = \sum_{i=1}^{k} |C_{i_j}| - 2(k - 1) = 2(k + 1).$$

In the following results, we fix $C_{u_1, u_2, \ldots, u_p}$, $C_{v_1, v_2, \ldots, v_q}$ to represent any two cycles from the cycles

$$C_{i_1, i_2, \ldots, i_k} \quad i_j \in \{1, 2, \ldots, m\} \text{ and } 1 \leq k \leq m,$$

such that $i_{j+1} = i_j + 1$ if $i_j \neq m$ and $i_{j+1} = 1$ if $i_j = m$, of the graph $\mathcal{J}_{2,m}$. Also we fix the notation “$a \rightarrow b$” if $b$ immediately proceeds $a$ i.e., the very next in order of preferences.

**Proposition 3.3.** Let $\mathcal{J}_{2,m}$ be the graph with the edges $E$ as defined in eq. (1) such that $\{u_1, u_2, \ldots, u_p\} \subseteq \{v_1, v_2, \ldots, v_q\}$ then we have

$$|C_{u_1, u_2, \ldots, u_p} \cap C_{v_1, v_2, \ldots, v_q}| = \begin{cases} \beta_{u_1, u_2, \ldots, u_p} - 2, & \{u_1, u_p\} \notin \{v_1, v_q\} \\ \beta_{u_1, u_2, \ldots, u_p} - 1, & u_1 \in \{v_1, v_q\} \text{ and } u_p \notin \{v_1, v_q\} \\ \beta_{u_1, u_2, \ldots, u_p} - 1, & u_p \in \{v_1, v_q\} \text{ and } u_1 \notin \{v_1, v_q\} \\ \beta_{u_1, u_2, \ldots, u_p}, & u_1 = v_1 \text{ and } u_p = v_q \text{ or } u_1 = v_q \text{ and } u_p = v_q \end{cases}$$
Proposition 3.6. Let $J_{2,m}$ be the graph with the edges $E$ as defined in eq. (1) such that $\{u_1, u_2, \ldots, u_p\} \cap \{v_1, v_2, \ldots, v_q\} \neq \emptyset$ and $u \in \{u_1, u_2, \ldots, u_p\} \& v \in \{v_1, v_2, \ldots, v_q\}$ then we have

$$\left| C_{u_1, u_2, \ldots, u_p} \cap C_{v_1, v_2, \ldots, v_q} \right| = \begin{cases} \beta_{u_1, u_2, \ldots, u_p} - 1, & u_1 = v_1 & u_q \neq v_1 \& v_1 \neq u_1 \\ \beta_{u_1, u_2, \ldots, u_p} - 2, & u_1 = v_1 & u_q \neq v_1 & v_1 \neq u_1 \\ \beta_{u_1, u_2, \ldots, u_p} - 1, & u_1 = v_1 & u_2 \neq v_1 & u_1 \neq v_1 \\ \beta_{u_1, u_2, \ldots, u_p} - 2, & u_1 = v_1 & u_2 \neq v_1 & u_1 \neq v_1 \end{cases}$$

Proof. Here, the cycles $C_{u_1, u_2, \ldots, u_p}$ are amongst $\sigma$ consecutive adjoining cycles of the cycle $C_{u_1, u_2, \ldots, u_p}$ which are also overlapping with the $\sigma$ consecutive adjoining cycles of the cycle $C_{v_1, v_2, \ldots, v_q}$. If the adjoining cycle $C_{u_1, u_2, \ldots, u_p}$ overlaps with the first adjoining cycle $C_{v_1, v_2, \ldots, v_q}$ and the adjoining cycles $C_{u_1, u_2, \ldots, u_p}$ and $C_{u_1, u_2, \ldots, u_p}$ are consecutive then by previous proposition the order of the intersection $C_{u_1, u_2, \ldots, u_p} \cap C_{v_1, v_2, \ldots, v_q}$ is indeed $\beta_{u_1, u_2, \ldots, u_p} - 1$. Similarly, if the adjoining cycles $C_{u_1, u_2, \ldots, u_p}$ and $C_{u_1, u_2, \ldots, u_p}$ are not consecutive then they will have no common edge and the use of proposition 3.3 gives the order of the intersection $C_{u_1, u_2, \ldots, u_p} \cap C_{v_1, v_2, \ldots, v_q}$ as $\beta_{u_1, u_2, \ldots, u_p} - 2$. Similar can be done for the remaining cases. □

Remark 3.5. The case when there exists a $\tau_0 < \sigma < p$ such that $T_{\tau_0 - 1} \neq T_{\tau_0}$ in above proposition i.e., when cycles $C_{w_1, w_2, \ldots, w_p}, C_{w_1, w_2, \ldots, w_p}, C_{w_1, w_2, \ldots, w_p}, \ldots, C_{w_1, w_2, \ldots, w_p}$ are not amongst $\sigma$ consecutive adjoining cycles of the cycle $C_{w_1, w_2, \ldots, w_p}$, the order of the intersection $C_{w_1, w_2, \ldots, w_p} \cap C_{v_1, v_2, \ldots, v_q}$ can be calculated by applying proposition 3.4 on the overlapping portions.

Proposition 3.6. Let $J_{2,m}$ be the graph with the edges $E$ as defined in (1) such that $\{u_1, u_2, \ldots, u_p\} \cap \{v_1, v_2, \ldots, v_q\} = \emptyset$ and $p \leq q$. Then we have

$$\left| C_{u_1, u_2, \ldots, u_p} \cap C_{v_1, v_2, \ldots, v_q} \right| = \begin{cases} 1, & u_p \rightarrow v_1 & v_q \neq u_1 \\ 1, & u_p \neq v_1 & v_q \rightarrow u_1 \\ 2, & u_p \rightarrow v_1 & v_q \neq u_1 \\ 0, & \text{otherwise} \end{cases}$$

Proof. In this case the adjoining cycles of $C_{u_1, u_2, \ldots, u_p}$ and $C_{v_1, v_2, \ldots, v_q}$ have no common cycle. However, if the adjoining cycle on one of the extreme ends of the cycle $C_{u_1, u_2, \ldots, u_p}$ is consecutive with the adjoining cycles on one of the extreme ends of the other cycle $C_{v_1, v_2, \ldots, v_q}$ then the intersection $C_{u_1, u_2, \ldots, u_p} \cap C_{v_1, v_2, \ldots, v_q}$ will have only one edge. The remaining cases are easy to see. □
Proposition 3.8. A subset \( E(T_{(j_1,i_1,j_2,i_2,\ldots,j_{m},i_{m})}) \) of \( E \) with \( j_\alpha = j_\alpha 1 \) for any \( \alpha \) will belong to \( s(J_{2,m}) \) if and only if

\[
E(T_{(j_1,i_1,j_2,i_2,\ldots,j_{m},i_{m})}) = E \setminus \{ e_{j_1,i_1}, e_{j_2,i_2}, \ldots, e_{j_{m},i_{m}} \}
\]

where, \( \{ e_{j_1,i_1}, e_{j_2,i_2}, \ldots, e_{j_{m},i_{m}} \} \) will contain exactly one edge from \( C_{(j_{\alpha - 1})}(j_{\alpha}) \setminus \{ e_{(j_{\alpha - 1})1}, e_{(j_{\alpha})1} \} \) other than \( e_{(j_{\alpha})1} \).

Proof. For a spanning tree of \( J_{2,m} \) such that exactly one common edge \( e_{j_{\alpha}1} \) is removed, we need to remove precisely \( m - 1 \) edges from the remaining edges using the cutting down process. However, we cannot remove more than one edge from the non-common edges of the cycle \( C_{(j_{\alpha - 1})}(j_{\alpha}) \) (since this will result in a disconnected graph). This explains the proof of the above case. \( \square \)

Proposition 3.9. A subset \( E(T_{(j_1,i_1,j_2,i_2,\ldots,j_{m},i_{m})}) \) of \( E \), where \( j_\alpha = j_\alpha 1 \) for \( \alpha \in \{ r_1, r_2, \ldots, r_p \} \subset \{ 1, 2, \ldots, m \} \), will belong to \( s(J_{2,m}) \) if and only if it satisfies any of the following:

1. if \( e_{j_1,1}, e_{j_2,1}, \ldots, e_{j_p,1} \) are common edges from consecutive cycles then

\[
E(T_{(j_1,i_1,j_2,i_2,\ldots,j_{m},i_{m})}) = E \setminus \{ e_{j_1,i_1}, e_{j_2,i_2}, \ldots, e_{j_{m},i_{m}} \}
\]

such that \( \{ e_{j_1,i_1}, e_{j_2,i_2}, \ldots, e_{j_{m},i_{m}} \} \) will contain exactly one edge from \( C_{(j_{\alpha - 1})}(j_{\alpha}) \) other than \( e_{j_\alpha,1} \), \( e_{j_{\alpha - 1},1} \), \( \ldots, e_{j_1,1} \), \( j_{\alpha} \rightarrow j_{\alpha - 1} \).

2. if none of \( e_{j_1,1}, e_{j_2,1}, \ldots, e_{j_p,1} \) are common edges from consecutive cycles then

\[
E(T_{(j_1,i_1,j_2,i_2,\ldots,j_{m},i_{m})}) = E \setminus \{ e_{j_1,i_1}, e_{j_2,i_2}, \ldots, e_{j_{m},i_{m}} \}
\]

such that for each edge \( e_{j_\alpha,1} \) proposition 3.6 holds.

3. if some of \( e_{j_1,1}, e_{j_2,1}, \ldots, e_{j_p,1} \) are common edges from consecutive cycles then

\[
E(T_{(j_1,i_1,j_2,i_2,\ldots,j_{m},i_{m})}) = E \setminus \{ e_{j_1,i_1}, e_{j_2,i_2}, \ldots, e_{j_{m},i_{m}} \}
\]

such that proposition 3.9.1 is satisfied for the common edges of consecutive cycles and proposition 3.9.2 is satisfied for remaining common edges.

Proof. For the case 1, we need to obtain a spanning tree of \( J_{2,m} \) such that \( |r_\rho - r_1|_m \) common edges must be removed from \( \rho \) consecutive cycles \( C_{j_\alpha}, C_{j_{\alpha - 1}}, \ldots, C_{j_1} \). The remaining \( m - |r_\rho - r_1|_m \) edges must be removed in such a way that exactly one edge is removed from the non-common edges of the adjoining cycles \( C_{j_\alpha}, C_{j_{\alpha - 1}}, \ldots, C_{j_1} \) and the remaining \( m - |r_\rho - r_1|_m \) cycles of the graph \( J_{2,m} \). This concludes the case.

The remaining cases of the proposition can be visualised in a similar manner using the propositions 3.7 and 3.8. This completes the proof. \( \square \)

Remark 3.10. If we denote the disjoint classes of subsets of \( E \) discussed in propositions 3.7, 3.8 and 3.9 by \( C_{J_1}, C_{J_2}, C_{J_3a}, C_{J_3b}, C_{J_3c} \) respectively, then, we can write \( s(J_{2,m}) \) as follows:

\[
s(J_{2,m}) = C_{J_1} \cup C_{J_2} \cup C_{J_3a} \cup C_{J_3b} \cup C_{J_3c}.
\]

In our next result, we give an important characterization of the \( f \)-vectors of \( \Delta s(J_{2,m}) \).

Proposition 3.11. Let \( \Delta s(J_{2,m}) \) be a spanning simplicial complex of the graph \( J_{2,m} \), then the \( \dim(\Delta s(J_{2,m})) = 2m - 1 \) with \( f \)-vector \( f(\Delta s(J_{2,m})) = (f_0, f_1, \ldots, f_{2m-1}) \) and

\[
f_i = \binom{3m}{i + 1} + \sum_{i=1}^{\tau} (-1)^{t} \left[ \sum_{(i_1,i_2,\ldots,i_t) \in \mathcal{C}_t} \left( \binom{3m - \sum_{s=1}^{t} \beta_{i_s} + \sum_{(i_1,i_2,\ldots,i_t) \in \mathcal{C}_t} \mid C_{i_s} \cap C_{i_t} \mid}{i + 1 - \sum_{s=1}^{t} \beta_{i_s} + \sum_{(i_1,i_2,\ldots,i_t) \in \mathcal{C}_t} \mid C_{i_s} \cap C_{i_t} \mid} \right) \right]
\]

where \( 0 \leq i \leq 2m - 1 \) \( I = \{ i_1, i_2, \ldots, i_k \} \subseteq \{ 1, 2, \ldots, m \} \) and \( 1 \leq k \leq m \) such that \( i_{j+1} = i_j + 1 \) if \( i_j \neq m \) and \( i_{j+1} = 1 \) if \( i_j = m \) and \( \mathcal{C}_t = \{ \text{Subsets of } I \text{ of cardinality } t \} \).
Proof. Let $E$ be the edge set of $J_{2,m}$ and $C_{J_1}, C_{J_2}, C_{J_3}, C_{J_3b}, C_{J_3c}$ are disjoint classes of spanning trees of $J_{2,m}$ then from propositions 3.7, 3.8, 3.9 and the remark 3.10 we have

$$s(J_{2,m}) = C_{J_1} \cup C_{J_2} \cup C_{J_3} \cup C_{J_3b} \cup C_{J_3c}.$$ 

Therefore, by definition 2.5 we can write $\Delta_0(J_{2,m}) = \{C_{J_1} \cup C_{J_2} \cup C_{J_3} \cup C_{J_3b} \cup C_{J_3c}\}$. Since each facet $E(J_{2,m}) = E(T(J_{2,m}))$ is obtained by deleting exactly $m$ edges from the edge set of $J_{2,m}$, keeping in view the propositions 3.7, 3.8 and 3.9, therefore dimension of each facet is the same i.e., $2m - 1$ (since $|E(J_{2,m})| = 2m$) and hence dimension of $\Delta_0(J_{2,m})$ will be $2m - 1$.

Also it is clear from the definition of $\Delta_0(J_{2,m})$ that it contains all those subsets of $E$ which do not contain the given sets of cycles \{ $e_{k1}, e_{k2}, e_{k3}, e_{k(11)}$ \} for $k \in \{1, 2, \ldots, m - 1 \}$ and \{ $e_{m1}, e_{m2}, e_{m3}, e_{11}$ \} in graph as well as any other cycle in the graph $J_{2,m}$.

Now by lemma 3.2 the total cycles in the graph $J_{2,m}$ are

$$C_{i_1,i_2,\ldots,i_k} \quad i_j \in \{1, 2, \ldots, m\} \text{ and } 1 \leq k \leq m,$$

such that $i_{j+1} = i_j + 1$ if $i_j \neq m$ and $i_{j+1} = 1$ if $i_j = m$, and their total number is $\tau$. Let $F$ be any subset of $E$ of order $i + 1$ such that it does not contain any $C_{i_1,i_2,\ldots,i_k}$ \quad $i_j \in \{1, 2, \ldots, m\} \text{ and } 1 \leq k \leq m$, in it. The total number of such $F$ is indeed $f_i$ for $0 \leq i \leq 2m - 1$. We use inclusion exclusion principle to find this number. Therefore, $f_i = \text{Total number of subsets of $E$ of order $i + 1$} - \sum \left( \text{subset of $E$ of order $i + 1$ containing $C_{i_s}$ for } s = 1 \right) + \sum \left( \text{subset of $E$ of order $i + 1$ containing both $C_{i_s}$ for all } 1 \leq s \leq 2 \right) - \cdots + (-1)^{\tau} \sum \left( \text{subset of $E$ of order $i + 1$ simultaneously containing each $C_{i_s}$ for all } 1 \leq s \leq \tau \right)$.

This implies

$$f_i = \sum \left( \frac{3m}{i+1} - \left[ \sum |C_{i_1} \cap C_{i_2}| \right] \right) + \sum \left( \frac{3m - \frac{\tau}{s=1} \beta s}{i+1} + \sum |C_{i_1} \cap C_{i_2}| \right) + \cdots + (-1)^{\tau} \sum \left( \frac{3m - \frac{\tau}{s=1} \beta s}{i+1} + \sum |C_{i_1} \cap C_{i_2}| \right).$$

This implies

$$f_i = \sum \left( \frac{3m}{i+1} - \left[ \sum |C_{i_1} \cap C_{i_2}| \right] \right) + \sum \left( \frac{3m - \frac{\tau}{s=1} \beta s}{i+1} + \sum |C_{i_1} \cap C_{i_2}| \right).$$

Example 3.12. Let $\Delta_0(J_{2,3})$ be a spanning simplicial complex of the Jahangir’s graph $J_{2,m}$ given in Figure 1, then the $\dim(\Delta_0(J_{2,3})) = 5$ and $\tau = 3^2 = 9$. Therefore, $f$-vectors $f(\Delta_0(J_{2,3})) = (f_0, f_1, \ldots, f_5)$ and

$$f_i = \left( \frac{9}{i+1} - \left[ \sum \left( \frac{9 - \beta s}{i+1 - \beta s} \right) \right] \right).$$
Here, we present the main result of this section. Let \( \Delta \) be the face ring of a simplicial complex \( \Delta \) of dimension \( d \) and \( f(\Delta) = (f_0, f_1, \ldots, f_d) \) its \( f \)-vector, then the Hilbert series of the face ring \( k[\Delta] \) is given by

\[
H(k[\Delta], t) = 1 + \frac{d}{(1-t)^{d+1}} + \sum_{i=0}^{d} \sum_{k=1}^{r} (-1)^k \sum_{i \geq k, \beta_i \in \Delta} \left( \sum_{\{i_1, \ldots, i_k\} \subseteq \Delta} \beta_{i_1} \cdots \beta_{i_k} \right) t^{i+1}.
\]

Proof. From [14], we know that if \( \Delta \) is a simplicial complex of dimension \( d \) and \( f(\Delta) = (f_0, f_1, \ldots, f_d) \) its \( f \)-vector, then the Hilbert series of the face ring \( k[\Delta] \) is given by

\[
H(k[\Delta], t) = 1 + \frac{d}{(1-t)^{d+1}} + \sum_{i=0}^{d} \sum_{k=1}^{r} (-1)^k \sum_{i \geq k, \beta_i \in \Delta} \left( \sum_{\{i_1, \ldots, i_k\} \subseteq \Delta} \beta_{i_1} \cdots \beta_{i_k} \right) t^{i+1}.
\]

By substituting the values of \( f_i \)'s from Proposition 3.11 in this above expression, we get the desired result. \( \square \)

4 Cohen-Macaulayness of the face ring of \( \Delta_S(\mathcal{J}_{2,m}) \)

In this section, we present the Cohen-Macaulayness of the face ring of SSC \( \Delta_S(\mathcal{J}_{2,m}) \), using the notions and results from [2].

**Definition 4.1** ([2]). Let \( I \subseteq S = k[x_1, x_2, \ldots, x_n] \) be a monomial ideal. We say that \( I \) has linear residuals, if there exists an ordered minimal monomial system of generators \( \{ m_1, m_2, \ldots, m_r \} \) of \( I \) such that \( \text{Res}(I_i) \) is minimally generated by linear monomials for all \( 1 < i \leq r \), where \( \text{Res}(I_i) = \{ u_1, u_2, \ldots, u_{i-1} \} \) such that \( u_k = \frac{m_i}{\gcd(m_i, m_j)} \) for all \( 1 \leq k \leq i-1 \).

**Theorem 4.2** ([2]). Let \( \Delta \) be a simplicial complex of dimension \( d \) over \( [n] \). Then \( \Delta \) will be a shellable if and only if \( I_x(\Delta) \) has linear residuals.

**Corollary 4.3** ([2]). If the facet ideal \( I_x(\Delta) \) of a pure simplicial complex \( \Delta \) over \( [n] \) has linear residuals, then the face ring \( k[\Delta] \) is Cohen Macaulay.
Theorem 4.4. The face ring of $\Delta_s(J_2,m)$ is Cohen-Macaulay.

Proof. By corollary 4.3, it is sufficient to show that $I_{\mathcal{F}}(\Delta_s(J_2,m))$ has linear residuals in $S = k[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$. By propositions 3.7, 3.8, 3.9 and the remark 3.10, we have

$$s(J_2,m) = C_{J_1} \cup C_{J_2} \cup C_{J_3} \cup C_{J_4} \cup C_{J_5}.$$  

Therefore,

$$\Delta_s(J_2,m) = \{ E_{(i_1,i_2,i_3,i_4,i_5,i_6,i_7,i_8)} | e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}, e_{i_5}, e_{i_6}, e_{i_7}, e_{i_8} \in s(J_2,m) \}$$

and hence we can write,

$$I_{\mathcal{F}}(\Delta_s(J_2,m)) = \left( x_{E_{(i_1,i_2,i_3,i_4,i_5,i_6,i_7,i_8)}} \right| E_{(i_1,i_2,i_3,i_4,i_5,i_6,i_7,i_8)} \in s(J_2,m)).$$

Here, $I_{\mathcal{F}}(\Delta_s(J_2,m))$ is a pure monomial ideal of degree $2m - 1$ with $x_{E_{(i_1,i_2,i_3,i_4,i_5,i_6,i_7,i_8)}}$ as the product of all variables in S except $x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}, x_{i_7}, x_{i_8}$. Now we will show that $I_{\mathcal{F}}(\Delta_s(J_2,m))$ has linear residuals with respect to the following orders in its monomials:

$$\{ x_{E_{(i_1,i_2,i_3,i_4,i_5,i_6,i_7,i_8)}} | i_1 \neq 1; 1 \leq r_1 \leq m \& i_k = 1; k \neq r_1 \},$$

$$\{ x_{E_{(i_1,i_2,i_3,i_4,i_5,i_6,i_7,i_8)}} | i_1, i_2 \neq 1; 1 \leq r_1, r_2 \leq m \& i_k = 1; k \neq r_1, r_2 \},$$

$$\{ x_{E_{(i_1,i_2,i_3,i_4,i_5,i_6,i_7,i_8)}} | i_1, i_2, \ldots, i_m \neq 1; 1 \leq r_1, r_2 \ldots r_m \leq m \}.$$  

More explicitly, the monomials $\{ x_{E_{(i_1,i_2,i_3,i_4,i_5,i_6,i_7,i_8)}} | i_1 \neq 1; 1 \leq r_1 \leq m \& i_k = 1; k \neq r_1 \}$ in the order 2, consists of monomials of the form $x_{E_{(i_1,i_2,i_3,i_4,i_5,i_6,i_7,i_8)}}$, $x_{E_{(i_1,i_2,\ldots,(m-1),(m-1),(m-1)}}$, $x_{E_{(i_1,i_2,\ldots,(m-1),(m-1),(m-1)}}$, where $i_k \in \{2, 3\}$ and $1 \leq i_k \leq m$. Similarly, other monomials in order 2. Let us put

$$\text{Res}(x_{E_{(i_1,i_2,i_3,i_4,i_5,i_6,i_7,i_8)}}) = \left\{ \frac{\text{gcd}(m_k, x_{E_{(i_1,i_2,i_3,i_4,i_5,i_6,i_7,i_8)}})}{m_k} \left| x_{E_{(i_1,i_2,i_3,i_4,i_5,i_6,i_7,i_8)}} \right. \right\}$$

For instance, for $r_1 = m$ in $\text{Res}(x_{E_{(i_1,i_2,i_3,i_4,i_5,i_6,i_7,i_8)}})$ we have, $	ext{Res}(x_{E_{(i_1,i_2,\ldots,(m-1),(m-1),(m-1)}})$

where, $m_k$ in this case are all the monomials in $S$ of the form $x_{E_{(i_1,i_2,\ldots,(m-1),(m-1),(m-1)}}$ where $i_m \neq m$ and $j_m = 2, 3$. Since all the monomials $m_k$ differ from $x_{E_{(i_1,i_2,\ldots,(m-1),(m-1),(m-1)}}$ at only one position, therefore, $\text{Res}(x_{E_{(i_1,i_2,\ldots,(m-1),(m-1),(m-1)}})$ have all linear terms i.e., $\text{Res}(x_{E_{(i_1,i_2,\ldots,(m-1),(m-1),(m-1)}})$ is minimally generated by linear monomials.

Continuing the same process the order 2 of the monomials of $I_{\mathcal{F}}(\Delta_s(J_2,m))$ guarantees that $\text{Res}(x_{E_{(i_1,i_2,i_3,i_4,i_5,i_6,i_7,i_8)}})$ is minimally generated by linear monomials for all $x_{E_{(i_1,i_2,i_3,i_4,i_5,i_6,i_7,i_8)}} \in I_{\mathcal{F}}(\Delta_s(J_2,m))$. Hence, $I_{\mathcal{F}}(\Delta_s(J_2,m))$ has linear residuals, and by Corollary 4.3 $\Delta_s(J_2,m)$ is Cohen-Macaulay. \qed

5 Conclusions and Scopes

We conclude this paper with some perspectives for further study as well as some constraints related to our work.

- The results given in this paper can be naturally extended for any integer $n \geq 2$.
- The scope of SSC of a graph can be explored for some other classes of graphs like the wheel graph $W_n$ etc. However, since finding spanning trees of a general graph is a NP-hard problem, therefore the results given here are not easily extendable for a general class of graph.
- In view of the work done in [8, 9], we intend to find some perspectives for the SSC in studying sensor networks.
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