Restricted triangulation on circulant graphs

Abstract: The restricted triangulation existence problem on a given graph decides whether there exists a triangulation on the graph’s vertex set that is restricted with respect to its edge set. Let $G = C(n, S)$ be a circulant graph on $n$ vertices with jump value set $S$. We consider the restricted triangulation existence problem for $G$. We determine necessary and sufficient conditions on $S$ for which $G$ admitting a restricted triangulation. We characterize a set of jump values $S(n)$ that has the smallest cardinality with $C(n, S(n))$ admits a restricted triangulation. We present the measure of non-triangulability of $K_n - G$ for a given $G$.

Keywords: Triangulation, Circulant Graph

MSC: 32C25

1 Introduction

A graph is an ordered pair $G = (V, E)$, where $V$ is a set of vertices, and $E$ is a set of edges. An order of $G$ is the number of its vertices while a size of $G$ is the number of its edges. A graph is called geometric, if its edges are straight-line segments.

A triangulation $T_n$ of a finite set of points $V$ in the plane is a maximally connected, straight-line planar graph with vertex set $V$. Each bounded face is a triangle, and the triangulation includes the boundary of the convex hull.

Let $n \geq 3$. A circulant graph $G = C(n, S)$ is a graph on the vertex set $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ such that each vertex $v_i$ is adjacent to vertices $v_{i+a}$ where $i = 0, 1, \ldots, n - 1$ and the subscript index $i \pm a$ is reduced modulo $n$ for all $a \in S$. That is, $v_i v_j$ is an edge of $C(n, S)$ if and only if $|j - i| \in S$ or $n - |j - i| \in S$. A set $S \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$ is called a set of jump values of $G = C(n, S)$. When discussing circulant graphs, we will often assume that the vertices are the corners of a regular $n$-gon, labeled in clockwise order and the edges are straight line segments. Hence, circulant graphs can be considered as geometric graphs.

Circulant graphs include the family of cycles $C(n, \{1\})$ and the family of complete graphs $K_n = C(n, \{1, 2, \ldots, \lfloor n/2 \rfloor\})$. Clearly, when $G_1 = C(n, S_1)$ and $G_2 = C(n, S_2)$ are two circulant graphs such that $|S_1| < |S_2|$, then the size of $G_1$ is smaller than the size of $G_2$ (i.e., $|E(G_1)| < |E(G_2)|$). Hence, for simplicity we shall say that $G_1 = C(n, S_1)$ is a smaller size circulant graph than $G_2 = C(n, S_2)$ when $|S_1| < |S_2|$. Let $E$ be some set of edges spanned by $V$. We say that a triangulation $T_n$ of $V$ is restricted with respect to $E$ if $E(T_n) \subseteq E$. The restricted triangulation existence problem, on a given graph $G(V, E)$, is to decide whether there exists a triangulation of $V$ that is restricted with respect to $E$. This problem was proven to be NP-complete.
(see [1, 2]). In section 2, we solve this problem for a certain geometric graph – a circulant graph. We give a characterization of a circulant graph \(G\) to admit a restricted triangulation.

Another related problem is beginning with a problem presented by Micha Perles on DIMACS Workshop on Geometric Graph Theory in 2002, which asks to determine the largest possible number \(h(n)\) such that every geometric graph on \(n\) vertices with at least \(\binom{n}{2} - h(n)\) edges has a non-crossing Hamiltonian path which has been studied by Černý et al. [3]. The authors in [4] motivated by Perles problem proved that if \(F_n\) is a subgraph of a convex complete graph \(K_n\), where \(F_n\) contains no boundary edges of \(K_n\) and \(|E(F_n)| \leq n - 3\), then \(K_n - F_n\) admits a restricted triangulation.

In section 3, we characterize a circulant graph \(G\) as a subgraph of a convex complete graph \(K_n\) and \(|E(G)| \leq \ell(n)\) such that \(K_n - G\) allows a restricted triangulation.

We obtain a set of jump values \(S(n) = \{a_1, a_2, \ldots, a_t\}\) that has the smallest cardinality \(|S(n)|\) such that \(C(n, S(n))\) admits a restricted triangulation. That is, \(C(n, S(n))\) is the smallest size circulant graph admitting a restricted triangulation.

Let \(v_i v_j\) be an edge in a convex graph \(G\), the distance between \(v_i\) and \(v_j\) in \(G\) is the length of a shortest \((v_i, v_j)\)-path in \(G\). A span of \(v_i v_j\) is defined as the distance between its two end points \(v_i\) and \(v_j\). In other words, a natural number \(d = \min\{j - i, n - j - i\}\) such that \(v_i v_j\) can be written as \(v_i v_{i+d}\) or \(v_i v_{j-d}\), a span of \(v_i v_j\).

Hence, we can see that \(H = C(n, \{d\})\) is a circulant graph in which \(v_i v_j \in E(H)\) and each edge in \(H\) has a span \(d\). Let \(E(T_n)\) be a set of edges of \(T_n\) and \(D(T_n)\) be a set of spans of all edges in \(E(T_n)\).

### 1.1 Importance of the triangulations and circulant graphs

Circulant graphs are an important class of interconnection networks in parallel and distributed computing. It can be used in the design of local area networks (see [5, 6]). On the other hand, computing a triangulation on a given graph has several important applications in different areas such as nondense matrix computations [7], database management [8] and artificial intelligence [9]. Moreover, triangulations are used in many areas of engineering and scientific applications such as finite element methods, approximation theory, numerical computation, computer-aided geometric design, computational geometry, etc. (see [10, 11]).

### 2 Restricted triangulation on \(G\)

In this section, we characterize a set of jump values \(S^*\) to a circulant graph \(C(n, S^*)\) admitting a restricted triangulation (Theorem 2.8). Also, we define the ‘smallest’ cardinality set of jump values \(S(n)\) such that \(C(n, S(n))\) still admits a restricted triangulation. We give a characterization of a circulant graph \(C(n, S)\) that can be redrawn to admit a restricted triangulation (Corollary 2.13). These results are then applied to determine the convex skewness of the circulant graphs \(G\) in section (2.2).

**Lemma 2.1.** If \(T_n\) is a restricted triangulation of a circulant graph \(G = C(n, S)\), then \(S\) contains \(D(T_n)\).

**Proof.** Suppose \(T_n\) is a restricted triangulation of \(G = C(n, S)\). Let \(d \in D(T_n)\). By definition of span of edges, there is an edge \(v_i v_j\) in \(T_n\) such that \(v_i v_j = v_i v_{i+d}\) where \(d = \min\{j - i, n - j - i\}\). Since \(T_n\) is a restricted triangulation of \(G\), then \(E(T_n) \subset E(G)\). Hence, \(v_i v_{i+d} \in E(G)\). Thus, \(d \in S\) (by definition of \(S\))

The following proposition proves that any circulant graph \(C(n, S)\) does not admit a restricted triangulation unless \(1, 2 \in S\).

**Proposition 2.2.** Let \(n \geq 4\) be a natural number. Suppose the circulant graph \(C(n, S)\) admits a restricted triangulation. Then \(1, 2 \in S\).

**Proof.** Suppose that \(C(n, S)\) is a circulant graph. Let \(T_n\) be a restricted triangulation of \(C(n, S)\).
It is well known that any triangulation on a set of point in the plane includes the boundary of the convex hull. For our case, \( v_iv_{i+1} \in E(T_n) \) for each \( i = 0, 1, \ldots, n-1 \) which yields \( 1 \in D(T_n) \) and then \( 1 \in S \) by Lemma 2.1.

Every maximal outer planar graph (as a special case, triangulation on a convex polygon) has at least two vertices of degree 2 (see [12]). Hence, if \( v_i+1 \) is a vertex of degree 2 in \( T \), then the diagonal edge \( v_iv_{i+2} \) must be in \( E(T_n) \) because \( v_i+1 \) is incident just on two edges of \( T \), which are the boundary edges \( v_iv_{i+1} \) and \( v_{i+1}v_{i+2} \). Thus, \( 2 \in D(T_n) \) since 2 is the distance between \( v_i \) and \( v_{i+1} \). Then \( 2 \in S \) by Lemma 2.1.

It is not difficult to verify that the converse of Proposition 2.2 is not true. For instance, in \( E \) hull. For our case, \( \frac{360}{2} \two \tf vertices of degree 2.1.

Thus, \( a \) which \( a \) and \( a \) admits a restricted triangulation. Hence, the question arises: what conditions on \( S \) do guarantee that \( C(n, S) \) admits a restricted triangulation and what is the smallest cardinality \( |S| \) of the set of jump values \( S \) for which \( C(n, S) \) admits a restricted triangulation?

The following proposition proves that the circulant graph \( C(n, S_n) \) admits a restricted triangulation where \( S_n \) is a set of ascending values \( \{a_1, a_2, \ldots, a_s\} \) in which \( a_1 = 1, a_2 = 2, a_s = \lfloor \frac{n}{2} \rfloor \) and \( a_{i+1} - a_i \in S_n \) for each \( i \in \{1, 2, \ldots, s - 1\} \).

**Proposition 2.3.** Let \( n \geq 4 \) be a natural number. Suppose \( S_n \) is a set of ascending values \( \{1, 2, a_3, \ldots, a_s\} \) in which \( a_s = \lfloor \frac{n}{2} \rfloor \) and \( S_n \) has the property \( a_{i+1} - a_i \in S_n \) for each \( i \in \{1, 2, \ldots, s - 1\} \). Then the circulant graph \( C(n, S_n) \) admits a restricted triangulation.

**Proof.** Suppose that \( C(n, S_n) \) is a circulant graph. Since \( a_s \in S_n \) is a jump value, then \( v_0v_a \) and \( v_{n-a}v_0 \) are two edges in \( C(n, S_n) \). Thus, we can define \( G_1 \) and \( G_2 \) to be two subgraphs of \( C(n, S_n) \), induced by the vertices \( v_0, v_1, v_2, \ldots, v_a \), and \( v_{n-a}, v_{n-a+1}, \ldots, v_{n-1}, v_0 \), respectively. See Figure 1.

![Diagram](attachment:figure1.png)

We shall construct the triangulation \( T_{G_1} \) of \( G_1 \), and then \( G_2 \) can be triangulated in a similar way.

(*) It is important to mention that in the following argument the property \( a_{i+1} - a_i \in S_n \) for each two consecutive values \( a_i \) and \( a_{i+1} \) in \( S_n \), and \( 1, 2 \in S_n \), are basic tools to construct \( T_{G_1} \).
Let $a_i, a_{i-1}$ be two consecutive vertices in $S_n$ and $a_{i+1} - a_i = a_{j_i} \in S_n$ for some $a_{j_i} \in \{1, 2, a_3, \ldots, a_i\}$ (since $S_n$ is a set of ascending values). Let $v_0v_1v_2\ldots v_{n-1}v_0$ be a convex $(a_1 + 2)$-gon in $G_1$ denoted by $Q(a_{j_1}v_0)$. Clearly $G_1 = \cup_{i=1}^{n-1} Q(a_{j_i}v_0)$. First, we prove $Q(a_{j_i}v_0)$ can be triangulated by edges of $C(n, S_n)$ for each $i = 1, \ldots, s - 1$.

Note that $a_1 = 1, a_2 = 2$ and $a_3 \in \{3, 4\}$ (otherwise, if $a_3 = 5$ then $a_3 - a_2 = 5 - 2 = 3 \notin S_n$, a contradiction). It is easy to see that $Q(v_1)(= v_0v_1v_2v_0)$ is triangulated by $v_0v_1v_2v_0$ and when $a_3 = 3$ we have $Q(v_1)(= v_0v_2v_3v_0)$ is triangulated by the boundary edges $v_0v_2v_3v_0$ together with the diagonal edge $v_2v_4$.

Now we obtain a triangulation for $Q(a_{j_i}v_0), i = 3, \ldots, s - 1$.

Let $Q(a_{j_{i+1}}v_0) = v_0v_1\ldots v_{a_{j_{i+1}}-1}v_0$ be a convex $(a_{j_{i+1}})$-gon in $G_1$. Clearly, $A(v_0) := v_0v_1v_2v_0$ is a triangle that is trivially triangulated by $T_1 := v_0v_1v_2v_0$. If $i = 3$, then either $a_3 = 3$ and $A_3(v_0) := v_0v_1v_2v_3v_0$ is a quadrilateral that is triangulated by $T_2 := v_0v_1v_2v_3v_0$, or $a_3 = 4$ and $A_3(v_0) := v_0v_1v_2v_3v_4v_0$ is a pentagon triangulated by $T_2 = v_0v_2v_4v_0v_1v_2v_3v_0$.

Let $B(a_{j_i}v_0) = Q(a_{j_i}v_0) - v_0$. Then $B(a_{j_i}v_0) = v_0v_1\ldots v_{a_{j_i}+1}v_0$ (since $a_{i+1} - a_i = a_{j_i}$) which is equivalent to the convex polygon $v_0v_1\ldots v_{a_{j_i}+1}v_0$ (by considering $a_i = 0$). Thus, $B(a_{j_i}+1(v_0))$ is equivalent $A(a_{j_i}+1(v_0))$.

If $i = 3$, we have $a_4 - a_3 = a_{j_i} \in S_n$ for some $a_{j_i} \in \{1, 2, a_3\}$. Then, $B(a_{j_i}+1(a_3)) := v_0v_1\ldots v_{a_{j_i}+1}v_0$ (since $a_{i+1} - a_i = a_{j_i}$) is equivalent to $A(a_{j_i}+1(a_3)) := v_0v_1\ldots v_{a_{j_i}+2}v_0$ which is triangulated by $T \in \{T_1, T_2\}$. Then, $B(a_{j_i}+1(a_3))$ can be triangulated by $T'$ equivalent to $T$. Hence, $T_{3-2} = T' \cup v_0v_1v_2v_3v_0$ is a triangulation of $Q(a_{j_i}+2(v_0))$.

Recursively, if we have $i = s - 1$ then $a_s - a_{s-1} = a_{j_{s-1}} \in S_n$ for some $a_{j_{s-1}} \in \{1, 2, a_3, \ldots, a_{s-1}\}$. Then, $B(a_{j_{s-1}}+1(v_{a_{j_{s-1}}}))$ is equivalent to $A(a_{j_{s-1}}+1(v_{a_{j_{s-1}}}))$ which is triangulated by $T \in \{T_1, T_2, T_3, \ldots, T_{s-2}\}$ (depending on the value of $a_{j_{s-1}} \in \{1, 2, a_3, \ldots, a_{s-1}\}$). Then, $B(a_{j_{s-1}}+1(v_{a_{j_{s-1}}})))$ can be triangulated by $T'$ equivalent to $T$. Hence, $T_{s-1} = T' \cup v_0v_1v_2v_3v_0v_4v_5v_0$ is a triangulation of $Q(a_{j_{s-1}+2(v_{a_{j_{s-1}}})})$.

Since, $G_1 = \cup_{i=1}^{n-1} Q(a_{j_i}v_0)$, then let $T_{G_1} = \cup_{i=1}^{n-1} T_i$ be the triangulation of $G_1$. In a similar way we obtain $T_{G_3}$. Thus, $T_n = T_{G_1}\cup T_{G_2}\cup \{v_0v_n\}$, where $a_n = n - a_s$; otherwise $\{v_0v_{n-a_s}\} = \emptyset$ is a restricted triangulation of $G$.

**Corollary 2.4.** Let $n \geq 4$ be a natural number. Suppose $G = C(n, S)$ is a circulant graph. Then $G$ admits a restricted triangulation if one of the following conditions hold.

1. $S = \{1, 2, a_1, a_2, \ldots, [\frac{n}{2}]\}$ where $a_i's$ are consecutive odd values.
2. $S = \{1, 2, a_1, a_2, \ldots, [\frac{n}{2}]\}$ where $a_i's$ are consecutive even values.

**Proof.** In both cases $a_1 = 1, a_2 = 2$ and $[\frac{n}{2}]$ are in $S$. Further, for each $a_{i+1}, a_i \in S_n, a_{i+1} - a_i \in \{1, 2\} \subseteq S_n$. Then the circulant graph $C(n, S)$ admits a restricted triangulation by Proposition 2.3.

Now, we shall define a smallest size circulant graph that admits a restricted triangulation. Before proceeding, we present some ingredients that will be used to prove the main results in this section.

- When $n$ is even number, then $n$ can be written as $n = t.2^r$ for some positive natural number $r'$ and some positive odd natural number $t$.
- Let $S_\alpha$ be a set of jump values obtaining by next algorithm where $\alpha = [\frac{r'}{2}]$ and $\beta = [\frac{t}{2}]$ with $t = \{n, n$ is odd ; $t, \ n = t.2^r$.
- Let $S_1 = \{1, 2, 4, 8, \ldots, c\}$ with $c = 2^r$ where $r = \begin{cases} 0, & n$ is odd ; \\
\lfloor r - 1, t = 1; \\
\lfloor r', t \geq 3.
\end{cases}$
- If we have a set $B = \{b_1, b_2, \ldots, b_k\}$ of positive integers, then define $a.B$ to be $\{a.b_1, a.b_2, \ldots, a.b_k\}$ where $a \geq 1$ is an integer number.
Algorithm (A)
1. If $\alpha = 0$, then let $S_\alpha = \{\beta\}$ and Stop. If $\alpha = 1$, then let $S_\alpha = \{1, \beta\}$ and Stop. If $2 \leq \alpha \leq 4$, then let $S_\alpha = \{1, 2, \alpha, \beta\}$ and Stop. Otherwise, let $L = 0$ be the number of starting level. Let $S(L) = \{1, 2, \beta\}$, $a_0 = \alpha$, and let $L = L + 1$.
2. If $a_0$ is odd and divisible by 3, then let $a_{1,L} = \frac{a_0}{3}$ and $a_{2,L} = 2a_{1,L}$. Otherwise, let $a_{1,L} = \lfloor \frac{a_0}{2} \rfloor$ and $a_{2,L} = \lceil \frac{a_0}{2} \rceil$.
3. Let $S(L) = \{a_{1,L}, a_{2,L}, a_0\}$. Let $S_\alpha = \bigcup_{l=0}^{L} S(l)$.
4. If $3 \in \{a_{1,L}, a_{2,L}\}$ and each of $\alpha$ and $\lfloor \frac{a_0}{2} \rfloor$ is odd and not divisible by 3, then do the following:
   (i) Let $L = 0$ be the number of starting level. Let $S(L) = \{1, 2, \alpha, \beta, a_0\}$. $a_0 = \alpha - 3$, and let $L = L + 1$.
   (ii) If $a_0$ is odd and not divisible by 3, then let $a_{1,L} = a_{2,L} = a_0 - 3$. If $a_0$ is odd and divisible by 3, then let $a_{1,L} = \frac{a_0}{3}$ and $a_{2,L} = 2a_{1,L}$. Otherwise, $a_{1,L} = \frac{a_0}{3}$.
   (iii) Let $S(L) = \{a_{1,L}, a_{2,L}, a_0\}$. Let $S_\alpha = \bigcup_{l=0}^{L} S(l)$.
   (iv) If $a_{1,L} \leq 3$, then arrange $S_\alpha$ to be a set of ascending values and stop. Otherwise, let $a_0 = a_{1,L}$ and $L = L + 1$ and then repeat Step (ii).
5. If $a_{1,L} \leq 4$, then arrange $S_\alpha$ to be a set of ascending values and stop. Otherwise, let $a_0 = a_{1,L}$ and $L = L + 1$ and then repeat Step (2).

Note that, $S_\alpha$ that obtained by Algorithm (A) is a set of ascending values $1, 2, a_{i-1}, a_i, a_{i+1}, \ldots, \alpha, \beta$.

Lemma 2.5. Suppose $S_\alpha = \{a_1, a_2, \ldots, \alpha, \beta\}$ is a set obtained by Algorithm (A). Then $1, 2 \in S_\alpha$ and $a_{i+1} - a_i \in S_\alpha$ for each $a_{i+1}, a_i$ belong to $S_\alpha$.

Proof. By Algorithm (A) step (1), we get $1, 2 \in S_\alpha$. Suppose that $a_i, a_{i+1}$ are two consecutive values belonging to $S_\alpha$.

If $a_{i+1} = a_0$ is an odd and divisible by 3 number, then by Algorithm (A) Step (2), $a_i = a_{2,L} = \frac{2a_{i+1}}{3}$. Thus, $a_{i+1} - a_i = a_{i+1} - \frac{2a_{i+1}}{3} = \frac{a_{i+1}}{3} = a_0 \in S_\alpha$ (where $a_{i+1} = a_{1,L} = \frac{a_{i+1}}{3} \in S_\alpha$).

If $3 \in S_\alpha$ and $a_{i+1} = a_0$ is an odd and not divisible by 3 number, then by Algorithm (A) Step (4), $a_i = a_{i+1} - 3$. Thus, $a_{i+1} - a_i = 3 \in S_\alpha$.

Otherwise, by Algorithm (A) Step (2), $a_i = \lfloor \frac{a_{i+1}}{2} \rfloor$ (where $\lfloor \frac{a_{i+1}}{2} \rfloor = a_{2,L} \in S_\alpha$). Hence, $a_{i+1} - a_i = a_{i+1} - \lfloor \frac{a_{i+1}}{2} \rfloor = \lfloor \frac{a_{i+1}}{2} \rfloor = a_{i+1} - a_i \in S_\alpha$ (where $a_{i+1} = \lfloor \frac{a_{i+1}}{2} \rfloor = a_{1,L} \in S_\alpha$). In case when, $\lfloor \frac{a_{i+1}}{2} \rfloor < \frac{a_{i+1}}{2}$, then we have $a_i = \lfloor \frac{a_{i+1}}{2} \rfloor$. Thus, $a_{i+1} - a_i = a_{i+1} - \lfloor \frac{a_{i+1}}{2} \rfloor = a_i \in S_\alpha$.

Theorem 2.6. Let $S(n) = S_1 \cup c.S_\alpha$, where $n \geq 4$. Then $C(n, S(n))$ admits a restricted triangulation.

Proof. Suppose that $C(n, S(n))$ is a circulant graph and let $\alpha, \beta, c$ and $t$ is defined as above. We shall construct a restricted triangulation $T_n(S(n))$ of $C(n, S(n))$.

When $n$ is even and $t = 1$, $S(n) = S_1$ (In this case, $\beta = 1$ and then $S_\alpha = \{1\}$ and then $c.S_\alpha = \{c\}$ and $c \in S_1$). Then let $T_n = \bigcup_{j=0}^{n/2} \{v_{j,2j}, v_{j+1,2j+2} \}$. The circulant graph when $n = 8$ is depicted in Figure 2.

When $n$ is even and $t \geq 3$, $S(n) = S_1 \cup c.S_\alpha$. In this case, and also when $n$ is odd (which means $t = n$), we have $t$-gon which is induced by the vertices $v_0, v_c, v_{2,c}, \ldots, v_{(t-2),c}, v_{(t-1),c}$ (the shaded part in Figure 3).

We shall triangulate this $t$-gon by $T'$ which is obtained by one of the following three cases depending on $S_\alpha$. According to $S_1$, let $T = \bigcup_{j=0}^{n/2} \{v_{j,2j}, v_{j+1,2j+2}, j = 0, 1, \ldots, \frac{n}{2} - 1\}$ which triangulates unshaded part, between $n$-gon and $t$-gon, in Figure 3. Then, let $T_n = T \cup T'$ or $T_n = T'$ be a triangulation to $C(n, S(n))$ when $n$ is even with $t \geq 3$ or when $n$ is odd, respectively.

Case (1) When $\alpha = \frac{t-1}{2}$, then $t = 2\beta + \alpha$ (since $\beta = \alpha + 1$). Let $\Delta = v_0v_c, v_{\alpha+\beta}v_0$ be a triangle. Then there are three $(\alpha + 1)$-gons $G_1, G_2$ and $G_3$ induced by the vertices $v_0, v_c, v_{2,c}, \ldots, v_{(\alpha+1),c}, v_{(\alpha+1)+c}, \ldots, v_{2\alpha,c}$ and $v_{\alpha+\beta}, v_{(2\alpha+1),c}, v_{(3\alpha+1),c}$, respectively. See Figure 4.

Case (2) When $\alpha = \frac{t-1}{2}$, then $t = 2\alpha + \beta$ (since $\beta = \alpha + 1$). Let $\Delta = v_0v_c, v_{2\alpha}v_0$ be a triangle. Then there are three $(\alpha + 1)$-gons $G_1, G_2$ and $G_3$ induced by the vertices $v_0, v_c, v_{2,c}, \ldots, v_{(\alpha+1),c}, v_{(\alpha+1)+c}, \ldots, v_{2\alpha,c}$ and $v_{(2\alpha),c}, v_{(2\alpha+1),c}, \ldots, v_{(3\alpha),c}$, respectively. See Figure 5.
Fig. 2. \( n = 8 = 2^3, t = 1, r = r' - 1 = 2, \)
\( T_8 = \{ v_j v_{j+1}, j = 0, \ldots, 7 \} \cup \{ v_2 v_2 v_4, j = 0, \ldots, 3 \} \cup \{ v_6 v_4, j = 0, 1 \} . \)

Fig. 4. \( n = 164, t = 41, r = 2, \alpha = 13, \) and \( \beta = 14. \)

Fig. 3. \( n = 56 = 7 \cdot 2^3, t = 7, r = r' = 3, \) \( T_{56} = \{ v_j v_{j+1}, j = 0, \ldots, 5, 55 \} \cup \{ v_2 v_2 v_2, j = 0, \ldots, 27 \} \cup \{ v_4 v_4, j = 0, \ldots, 13 \} \cup \{ v_8 v_8, j = 0, \ldots, 6 \} . \)

Fig. 5. \( n = 152, t = 19, r = 3, \alpha = 6, \) and \( \beta = 7. \)

**Case (3)** When \( \alpha = \frac{r}{3}, \) then \( t = 3 \alpha \). There are three \((\alpha + 1)\)-gons \( G_1, G_2 \) and \( G_3 \) induced by the vertices \( v_0, v_r, v_2, v_4, \ldots, v_r, v_r \) and \( v_2, v_4, v_6, v_8, v_{10}, \ldots, v_r \); \( v_2, v_4, v_6, v_8, v_{10}, \ldots, v_r \); \( v_2, v_4, v_6, v_8, v_{10}, \ldots, v_r \); \( v_2, v_4, v_6, v_8, v_{10}, \ldots, v_r \); \( v_2, v_4, v_6, v_8, v_{10}, \ldots, v_r \). See Figure 6.

In each case, \( G_1, G_2 \) and \( G_3 \) are of the same order. Suppose that \( R \) is an \( r \)-gon in \( G_i \), for some \( i \in \{ 1, 2, 3 \} \) such that \( V(R) = \{ v_h, v_{h+a}, v_{h+a+1}, \ldots, v_{h+a_1} \} \) for some \( v_h \in V(G_i) \) and two consecutive jump values \( a_i \) and \( a_{i+1} \) in \( S_\alpha \).

Without loss of generality assume that \( R \) is a subgraph of \( G_1 \) and \( h = 0. \)

By Lemma 2.5, we have \( 1, 2 \in S_\alpha \) and \( a_{i+1} = a_i \in S_\alpha \) for each \( a_{i+1}, a_i \) belonging to \( S_\alpha \). Hence, we can use the argument (***) of Proposition 2.3 to show that \( R \) can be triangulated by \( T_r \). Moreover, similar as in the proof of Proposition 2.3 we can consider \( G_1 \) as a finite union of such polygons. Then we can assume that \( T_{G_1} \) is a finite union of the restricted triangulation of these polygons.

Obtain \( T_{G_1} \) (on \( G_2 \)) and \( T_{G_1} \) (on \( G_3 \)) by “rotating” the edges of \( T_{G_1} \), see Figures 4, 5 and 6. Let \( T'' = T_{G_1} \cup T_{G_2} \cup T_{G_3} \cup \Delta \) for case (1) and case (2), and let \( T'' = T_{G_1} \cup T_{G_2} \cup T_{G_3} \) for case (3).

This completes the proof.
**Definition 2.7.** Let \( n \geq 4 \) and \( t \) be defined as above, and let \( a_k \) be a natural number such that \( \left\lfloor \frac{t}{2} \right\rfloor \cdot c \leq a_k \leq \left\lceil \frac{n}{2} \right\rceil \). Then define \( S^* \) to be a set of ascending natural numbers such that,

1. \( S^* = \{1, 2, a_3, a_4, \ldots, a_k\} \),
2. \( \{x, z\} \subset S^* \) such that \( n = x + y + z \) where \( y \in \{0, a_i\} \) for some \( i \in \{1, \ldots, k\} \),
3. \( a_i + 1 - a_i \in S^* \) for each \( i = 1, \ldots, k - 1 \).

It is clear that, \( S^* = S_n \) when \( a_k = \left\lceil \frac{n}{2} \right\rceil \). Our main result, Theorem 2.8, proves the sufficient and necessary condition for a circulant graph \( C(n, S) \) to admit a restricted triangulation.

**Theorem 2.8.** Suppose \( G = C(n, S) \) is a circulant graph. \( G \) admits a restricted triangulation if and only if \( S \) contains \( S^* \).

**Proof.** Let \( G \) have a restricted triangulation \( T_n \). We shall prove that \( S \) contains \( S^* \). By Lemma 2.1, it is enough to prove \( S^* = D(T_n) \).

Arrange the spans of \( D(T_n) \) to be ascending values.

Since, the circulant graph \( C(n, D(T_n)) \) admits \( T_n \), then by Proposition 2.2, we have \( \{1, 2\} \subseteq D(T_n) \).

Let \( |D(T_n)| = s \), then \( d_s \) is the maximum span in \( D(T_n) \).

(*) Let \( t \) be defined as before. Suppose that \( d_s < \left\lfloor \frac{t}{2} \right\rfloor \cdot c \). Without loss of generality assume that, \( c = 1 \) and \( d_s = \left\lfloor \frac{t}{2} \right\rfloor - 1 \). Thus, \( d_s = \frac{t-b}{2} - 1 \), \( b \in \{0, 1, 2\} \) and then \( 3d_s + b \geq t \). Let \( R \) be a convex \( r \)-gon induced by the vertices \( v_{d_s}, v_{2d_s}, \ldots, v_{3d_s+b+1}, v_{3d_s+b+2}, v_t \) (recall that, \( v_t = v_{3d_s+b+3} \)). Suppose that \( T_r = T_n(R) \) is a subgraph of \( T_n \) that triangulates \( R \) and let \( e \) be an edge in \( T_r \) such that the span of \( e \) is the maximum with respect to \( D(T_r) \). Then:

- either \( e = v_{2d_s}, v_t \) which yields that its span \( t - 2d_s \in D(T_r) \subset D(T_n) \); but \( t - 2d_s > d_s + 1 \) (by above assumption, \( t - 2d_s = d_s + b + 3 > d_s + 1 \)), which is a contradiction with maximality of \( d_s \in D(T_n) \);

- or, \( e = v_{d_s}, v_{3d_s+b+1} \) which yields that \( 3d_s - d_s = 2d_s \in D(T_r) \subset D(T_n) \); but \( 2d_s > d_s \), which also contradicts the maximality of \( d_s \in D(T_n) \).

Hence, \( R \) is not triangulated by \( T_n \) which is a contradiction with \( C(n, S) \) admitting a restricted triangulation \( T_n \). Thus, \( d_s \geq \left\lfloor \frac{t}{2} \right\rfloor \cdot c \).

Now, in order to check property (ii) of \( S^* \) we have to consider two cases:

**Case (1)** If \( d_s = \frac{t}{2} \), then \( n = 2. d_s \). Hence let \( x = z = d_s \) and \( y = 0 \).
Case (2) If \( \left[ \frac{1}{2} \right] \cdot c \leq d_x \leq \frac{n-1}{2} \), then \( T_n \) contains a triangle \( \triangle = v_h v_{h+d}, v_{h+d+2d}, v_h \) for some \( d_i \in D(T_n) \) and \( v_h \in V(G) \). The span of the edge \( v_{h+d}, v_h \in E(T_n) \) is \( n - (d_x + d_i) \in D(T_n) \) (by definition of the span of edge). Hence let \( x = d_s, y = d_i \) and \( z = n - (d_x + d_i) \), then \( n = x + y + z \).

To check the property (iii) of \( S^* \), assume that \( d_i \) and \( d_{i+1} \), \( i \in \{1, 2, \ldots, s - 1\} \) are two consecutive spans in \( D(T_n) \).

If \( d_x = \frac{n}{2} \), then let \( T_1 \) and \( T_2 \) be two subgraphs of \( T_n \) induced by the vertices \( v_0, v_1, v_2, \ldots, v_s \); \( v_x, v_{x+1}, \ldots, v_{x+y-1}, v_0 \) respectively (\( x = \frac{n}{2} \)). Let \( R \) be a polygon induced by the vertices \( v_j, v_{j+d}, \ldots, v_{j+d_i} \) where \( v_j \in V(T_1) \) and \( v_{j+d}, v_{j+d_i} \) are two diagonals in \( T_i \) for some \( i \in \{1, 2\} \).

If \( \left[ \frac{1}{2} \right] \cdot c \leq d_x \leq \frac{n-1}{2} \), then let \( T_1, T_2 \) and \( T_3 \) be three subgraphs of \( T_n \) induced by the vertices \( v_0, v_1, v_2, \ldots, v_s \); \( v_x, v_{x+1}, \ldots, v_{x+y}, v_{x+y+1}, \ldots, v_{x+y+z-1}, v_0 \) respectively. Let \( R \) be a polygon induced by the vertices \( v_j, v_{j+d}, \ldots, v_{j+d_i} \), where \( v_j \in V(T_1) \) and \( v_{j+d}, v_{j+d_i} \) are two diagonals in \( T_i \) for some \( i \in \{1, 2, 3\} \).

Assume without loss of generality that \( v_{j+d}, v_{j+d_i} \) and \( v_{j+d_i+1} \) are two diagonals in \( T_1 \). Then, \( j < j + d_i < j + d_{i+1} \) (by definitions of \( T_1 \) and \( R \)). It is clear that, \( v_{j+d}, \ldots, v_{j+d_i}, v_j \in T_1 \). Let \( T_r = T_1(R) \) be a subgraph of \( T_1 \) that triangulates \( R \) with the boundary edges \( v_j v_{j+d}, \ldots, v_{j+d_i} v_j \) of \( R \).

Suppose that \( v_{j+d}, v_{j+d_i+1} \notin E(T_r) \). Then there is \( d_i < d < d_{i+1} \) such that \( v_{j+d} v_{j+d_i} \in E(T_r) \). This implies that, \( d_i \notin D(T_r) \subset D(T_n) \), a contradiction with \( d_i \) and \( d_{i+1} \) are two consecutive spans in \( D(T_n) \).

Hence, \( v_{j+d}, v_{j+d_i+1} \in E(T_r) \in E(T_n) \). Then \( d_{i+1} - d_i \notin D(T_n) \).

Thus, \( D(T_n) = S^* \). This completes the proof of the necessity.

To show the sufficiency, suppose that \( S^* \subseteq S \). Then, \( C(n, S^*) \) is a subgraph of \( C(n, S) \).

Let \( \Delta = v_0 v_x v_{x+y} v_{x+y+z} \) be a triangle (since, \( x + y + z = n \)). Clearly, \( E(\Delta) \subset E(G) \). Then there are three polygons \( G_1, G_2 \) and \( G_3 \), induced by the vertices \( v_0, v_1, v_2, \ldots, v_s \); \( v_x, v_{x+1}, \ldots, v_{x+y}, v_{x+y+1}, \ldots, v_{x+y+z} \) respectively, and \( G_2 = \emptyset \) where \( y \in \{0, 1\} \).

Suppose that \( R \) is an \( r \)-gon in \( G_i \), for some \( i \in \{1, 2, 3\} \) such that \( V(R) = \{v_h, v_{h+a_i}, v_{h+a_j}, \ldots, v_{h+a_k}\} \) for some \( v_h \in V(G) \) and two consecutive jump values \( a_i \) and \( a_{i+1} \) in \( S^* \).

Without loss of generality assume that \( R \) is a subgraph of \( G_1 \) and \( h = 0 \).

Since \( S^* = \{1, 2, a_3, a_4, \ldots, a_k\} \) and satisfies that \( a_{i+1} - a_i \in S^* \) for each \( i = 1, \ldots, k - 1 \), then we can use argument (**) of Proposition 2.3 to show that \( R \) can be triangulated by \( T_r \). Consider \( G_1 \) as a finite union of such polygons. Then we can assume that \( T_{G_1} \) is a finite union of the restricted triangulation of those polygons.

Obtain \( T_{G_1} \) (on \( G_2 \)) and \( T_{G_2} \) (on \( G_3 \)) in a similar way. Then \( T_{G_1} \cup T_{G_2} \cup T_{G_3} \) is a triangulation of \( C(n, S^*) \).

Since \( C(n, S^*) \) is a spanning subgraph of \( G \), then \( T_{G_1} \cup T_{G_2} \cup T_{G_3} \) is a triangulation of \( G \).

This completes the proof.

Corollary 2.9. \( S(n) \) is \( S^* \).

Proof. By definition of \( S(n) \), either \( S(n) = S_1 \) (when \( n \) is even and \( t = 1 \)) or \( S(n) = S_\alpha \) (when \( n \) is odd) or else \( S(n) = S_1 \cup c.S_\alpha \). By definition of \( S_1 \), we have that \( S_1 \) is a set of ascending values and contains \( 1, 2 \); also \( S_\alpha \) is a set of ascending values by Algorithm (A) and contains \( 1, 2 \) by Lemma 2.5. Thus \( S(n) \) is a set of ascending values containing \( 1, 2 \).

Now, let \( \ell \) denote the cardinality of \( S(n) \).

When \( S(n) = S_1, a_\ell = 2^\ell = \frac{n}{2} \). Let \( x = z = a_\ell \) and \( y = 0 \) then we have \( n = x + y + z \).

When \( S(n) = S_\alpha \) or \( S(n) = S_1 \cup c.S_\alpha \), we have by definition of \( \beta, a_\ell = \left[ \frac{\ell}{2} \right] \cdot c \). To get \( n = x + y + z \) we have three cases. When \( \alpha = \frac{t}{2}, \beta = \alpha + 1 \) and \( t = 3, \alpha + 2 \). Let \( x = c, \alpha, y = z = c, \beta \) (since, \( n = c.t \)). When \( \alpha = \frac{t}{2}, \beta = \alpha + 1 \) and \( t = 3, \alpha + 1 \). Let \( x = y = c, \alpha, z = c, \beta \). When \( \alpha = \frac{t}{2}, \beta = \alpha + 1 \) and \( t = 3, \alpha + 1 \). Let \( x = y = c, \alpha, z = c, \beta \).

Let \( a_1 \) and \( a_i \) be any two consecutive values in \( S(n) \). If \( a_1, a_{i+1} \in S_1 \), then \( a_1 = 2^1 \) and \( d_{i+1} = 2^{i+1} \) and then \( a_{i+1} - a_1 = 2^{i+1} - 2^1 = 2^i \). If \( a_1 = 2^1 \), then \( a_1 \) is the last value in \( S_1 \) and the first in \( c.S_\alpha \) (recall that, \( c = 2^1 \)) and if \( a_1, a_{i+1} \in S_\alpha \) then by Lemma 2.2, \( a_{i+1} - a_1 \in S_\alpha \). Thus, \( a_{i+1} - a_1 \in S(n) \) for each \( a_{i+1}, a_1 \in S(n) \).

This completes the proof.

Corollary 2.10. Let \( n \geq 4 \) be a natural number. Suppose \( G = C(n, S) \) is a circulant graph. Then \( G \) admits a restricted triangulation if one of the following conditions hold.
(1) When \( n \) is odd, \( S \) contains \( \{2\} \) and all odd values \( a_i \leq a_j \) where \( \left\lfloor \frac{2}{3} \right\rfloor \leq a_i \leq \left\lfloor \frac{2}{3} \right\rfloor \).

(2) When \( n \) is even, \( S \) contains \( \{1\} \) and all even values \( a_i \leq a_j \) where \( \left\lfloor \frac{1}{2} \right\rfloor \cdot c \leq a_i \leq \left\lfloor \frac{1}{2} \right\rfloor \).

**Proof.** In both cases \( a_1 = 1 \), \( a_2 = 2 \) belong to a set of ascending values \( S \), and \( \left\lfloor \frac{1}{2} \right\rfloor \cdot c \leq a_i \leq \left\lfloor \frac{1}{2} \right\rfloor \) (when \( n \) is an odd natural number, \( \left\lfloor \frac{2}{3} \right\rfloor \cdot c \leq a_i \leq \left\lfloor \frac{2}{3} \right\rfloor \)). Further, for each \( a_{i+1}, a_i \in S \), \( a_{i+1} - a_i \in \{1, 2\} \subseteq S \).

To show the property (2) of \( S^* \) we have to consider two cases.

**Case (1)** When \( n \) is an odd natural number. Let \( n \geq 7 \) (When \( n = 5 \), then \( S = \{1, 2\} \) and clearly \( C(5, S) \) admits a restricted triangulation \( T_5 = \{v_0v_1v_2v_3v_4v_5\} \cup \{v_0v_2, v_0v_3\} \)). If \( \frac{n}{2} \) is an odd number, then \( \left\lfloor \frac{n}{2} \right\rfloor = a_i \) for some \( i \in \{3, \ldots, s\} \). If \( \frac{n}{2} \) is an even number, then \( \left\lfloor \frac{n}{2} \right\rfloor + 1 = a_i \) for some \( i \in \{4, \ldots, s\} \).

Whether \( \left\lfloor \frac{n}{2} \right\rfloor \) is odd or even, we have \( n - 2a_i \) is odd and \( 1 \leq n - 2a_i \leq a_i \). Then \( n - 2a_i = a_j \) for some \( j \in \{1, \ldots, i\} \).

Let \( x = y = a_i, z = a_i \). Thus, \( n = x + y + z \).

**Case (2)** When \( n \) is an even natural number, we have \( r \geq 1 \) and then \( c(\geq 2) \) is even. Hence, \( \left\lfloor \frac{1}{2} \right\rfloor \cdot c \) is even and then \( \left\lfloor \frac{1}{2} \right\rfloor \cdot c = a_i \) for some \( i \in \{2, \ldots, s\} \). Now, we have \( n - 2a_i \) is even and either \( n - 2a_i = 0 \) or \( 2 \leq n - 2a_i \leq a_i \). If \( n - 2a_i = 0 \), let \( x = z = a_i, y = 0 \). If \( 2 \leq n - 2a_i \leq a_i \), then \( n - 2a_i = a_j \) for some \( j \in \{2, \ldots, i\} \) and let \( x = z = a_i, y = a_j \). Thus, \( n = x + y + z \).

By Theorem 2.8, the circulant graph \( C(n, S) \) admitting a restricted triangulation.

**Proposition 2.11** ([13]). Suppose \( G = C(n, \{a_1, a_2, \ldots, a_k\}) \) and \( H = C(n, \{b_1, b_2, \ldots, b_k\}) \) with \( \{a_1, a_2, \ldots, a_k\} = q \{b_1, b_2, \ldots, b_k\} \), where the multiplication is reduced modulo \( n \) and \( \gcd(q, n) = 1 \). Then \( G \) is isomorphic to \( H \).

**Example 2.12.** Let \( n = 9 \) and \( S(n) = \{1, 2, 3\} \).

\( S_1 = \{2, 3, 4\} \), then we have \( S_1 = \{2, 6, 4\} = 2(S(n)) \).

\( S_2 = \{1, 3, 4\} \), then we have \( S_2 = \{8, 12, 4\} = 4(S(n)) \).

The next corollary considers circulant graph \( G = C(n, S) \) admits a restricted triangulation when there is an integer \( q \geq 1 \) with \( \gcd(q, n) = 1 \) such that \( S^* \subseteq q \cdot S \).

**Corollary 2.13.** let \( G = C(n, S) \) be a circulant graph. Suppose there is an integer \( q \geq 1 \) with \( \gcd(q, n) = 1 \) such that \( S^* \subseteq q \cdot S \) or \( q \cdot S^* \subseteq S \) where the multiplication is reduced modulo \( n \). Then \( G \) has a configuration that admits a restricted triangulation.

**Proof.** Suppose that, \( q \geq 1 \) is an integer such that \( \gcd(q, n) = 1 \) and \( S^* \subseteq q \cdot S \) or \( q \cdot S^* \subseteq S \). Then, there is a set \( S' \subseteq S \) such that \( S^* = q \cdot S' \) or \( q \cdot S^* = S' \). Thus, by Proposition 2.11, the subgraph \( H = C(n, S') \) of \( C(n, S) \) is isomorphic to \( C(n, S^*) \). By Theorem 2.8, \( H = C(n, S^*) \) admits a restricted triangulation. Thus, \( G \) has a configuration that admits a restricted triangulation. This completes the proof.

**2.1 An application**

The skewness of a graph \( G \), denoted \( sk(G) \), is the minimum number of edges to be deleted from \( G \) such that the resulting graph is planar. The convex skewness of a convex graph \( G \), denoted \( sk_c(G) \) is the minimum number of edges to be removed from \( G \) so that the resulting graph is a convex plane graph (see [14]).

**Proposition 2.14.** Let \( G = C(n, S) \) be a circulant graph and let \( q \geq 1 \) be an integer such that \( \gcd(q, n) = 1 \). Then \( sk_c(G) = E(G) - (2n - 3) \) if \( S^* \subseteq S \), or \( qS^* \subseteq S \), or \( S^* \subseteq qS \).

**Proof.** If \( S^* \subseteq S \), then \( G = C(n, S) \) admits a restricted triangulation, by Theorem 2.8. If \( qS^* \subseteq S \) or \( S^* \subseteq qS \), then \( G = C(n, S) \) admits a restricted triangulation, by Corollary 2.13.

It is known that, any triangulation \( T \) of a convex \( n \)-gon has \( 2n - 3 \) edges \((n - 3 \text{ of them are non-boundary edges})\). If any new straight line segment is added to the triangulation, it will intersect with some non-boundary edge of \( T \). Hence, we have \( sk_c(G) = E(G) - (2n - 3) \).
3 $K_n - G$ admits a restricted triangulation

In this section we turn to another question: which circulant graph $G$ on $n$ vertices does satisfy that $K_n - G$ admits a restricted triangulation and what is the largest size of $G$ such that $K_n - G$ still admits a restricted triangulation? We answered the first question by Corollary 3.1 and Corollary 3.3. We show that $C(n, S(n))$ is a smallest size circulant graph that admits a restricted triangulation, in order to answer the second question by Theorem 3.5. In what follows, let $\mathcal{N} = \{1, 2, \ldots, \lfloor n/2 \rfloor\}$.

**Corollary 3.1.** Let $G = C(n, S)$ be a circulant graph. Then $K_n - G$ admits a restricted triangulation if one of the following conditions holds.

1. If there is $S^*$ such that $S \cap S^* = \emptyset$.
2. $S$ contains all odd $i$ of $\mathcal{N}$ except $\{1, \lfloor n/2 \rfloor\}$.
3. $S$ contains all even $i$ of $\mathcal{N}$ except $\{2, \lfloor n/2 \rfloor\}$.

**Proof.** First, it is well known that $K_n - C(n, S) = C(n, \mathcal{N} - S)$.

Suppose that $S \cap S^* = \emptyset$. Then, $\mathcal{N} - S$ contains $S^*$ which yields that the circulant graph $C(n, \mathcal{N} - S)$ admits the restricted triangulation by Theorem 2.8, and this shows the property (1).

In (2), $\mathcal{N} - S$ contains all even values of $\mathcal{N}$ together with $\{1, \lfloor n/2 \rfloor\}$. In (3), $\mathcal{N} - S$ contains all odd values of $\mathcal{N}$ together with $\{2, \lfloor n/2 \rfloor\}$. By Corollary 2.4, $C(n, \mathcal{N} - S)$ admits the restricted triangulation for both cases. This completes the proof. \hfill $\Box$

**Definition 3.2 ([14]).** Let $K_n$ be a convex complete graph with $n$ vertices. $F$ is said to be potentially triangulable in $K_n$ if there exists a configuration of $F$ in $K_n$ such that $K_n - F$ admits a triangulation.

**Corollary 3.3.** Let $G = C(n, S)$ be a circulant graph. Suppose $S^* \subseteq q.(\mathcal{N} - S)$ for some integer $q \geq 1$ with $\text{gcd}(q, n) = 1$. Then $G$ is potentially triangulable in $K_n$.

**Proof.** Suppose that $q \geq 1$ is an integer such that $\text{gcd}(q, n) = 1$ and $S^* \subseteq q.(\mathcal{N} - S)$ for some $S^*$. Then, there is a set $S' \subseteq \mathcal{N} - S$ such that $S^* = q.S'$. Then by Proposition 2.11, the subgraph $C(n, S')$ of $C(n, \mathcal{N} - S)$ is isomorphic to $C(n, S^*)$. By Theorem 2.8, $C(n, S')$ has a configuration that admits a restricted triangulation. Thus, $C(n, \mathcal{N} - S) (= K_n - G)$ admits a restricted triangulation. This completes the proof. \hfill $\Box$

To answer the second part of the question, we shall determine the largest size $L(n)$ of $G$ for which $K_n - G$ admits a restricted triangulation. Before proceeding, let first $|E(C(n, S(n)))| = E_\ell$ where $|S(n)| = \ell$. Then we deduce that,

$$E_\ell = \begin{cases} n\ell - \frac{n}{\ell}, & t = 1; \\ n\ell, & \text{otherwise}. \end{cases}$$

The next result shows that whenever $C(n, S)$ admits a restricted triangulation then $|S| \geq |S(n)|$. That is, $C(n, S)$, is not a smaller size than $C(n, S(n))$.

**Theorem 3.4.** $C(n, S(n))$ is the smallest size circulant graph admitting a restricted triangulation if $n \geq 4$.

**Proof.** Recall that, $S(n) = S_1 \cup c \cdot S_2$. By Theorem 2.6, $C(n, S(n))$ admits a restricted triangulation. Hence, we just show that $S(n)$ is the smallest cardinality set for which the conclusion remains true.

Assume that $C(n, S)$ is a circulant graph that admits a restricted triangulation where $S = \{b_1, b_2, b_3, \ldots, b_3\}$ is a set of ascending jump values to $C(n, S)$.

By Theorem 2.8, $S$ contains $S^*$. Thus, $1, 2 \in S$ and $\left\lfloor \frac{n}{2} \right\rfloor \cdot c \leq b_2 \leq \left\lfloor \frac{n}{2} \right\rfloor$ and for any $i \in \{1, \ldots, s-1\}$, $b_{i+1} - b_i = b_j \in S$ where $j \in \{1, \ldots, i\}$.

In case when $b_{i+1}$ is even we have, according to $S_n$ (by Algorithm (A) step (2) where $b_{i+1} = a_0$ is even) and $S_1$ (by definition of $S_1$), the difference between the two consecutive values $b_i$ and $b_{i+1}$ is always $b_i$.

If $j \in \{1, \ldots, i-1\}$, then the number of values in $S(n)$ is less than the number of values in $S$. If $b_{i+1} - b_i = b_i \in S$, then the number of values in $S(n)$ is equal to the number of values in $S$. Thus, $|S| \geq |S(n)|$.\hfill $\Box$
In case when \( b_{i+1} \) is odd, then \( b_j \neq b_1 \) and there are three cases which are depending only on \( S_{a} \).

**Case (1):** \( b_1 \in \{1, 2\}. \)

In this case, the difference between the two consecutive values \( b_1 \) and \( b_{i+1} \) is at most 2. According to Algorithm (A), the difference between the two consecutive values is at most 2 only when \( b_1 = \lceil \frac{b_{i+1}}{2} \rceil \) and \( b_{i+1} = \lceil \frac{b_1}{2} \rceil \) or when \( b_1 = 3 \) and \( b_1 = 5 \). Then the number of values in \( S(n) \) is equal to or less than the number of values in \( S \). Whereas Step (2) and Step (4) of Algorithm (A) give the difference between any two consecutive values being at least 3, which implies \(|S| \geq |S(n)|\).

For instance, take \( b_{i+1} = 13 \) then \( b_i \in \{11, 12\} \). Take \( S \in \{S_1, S_2, S_3, S_4, S_5, S_6\} \) where \( S_1 = \{1, 2, 3, 5, 10, 12, 13\} \), \( S_2 = \{1, 2, 3, 6, 9, 12, 13\} \), \( S_3 = \{1, 2, 3, 5, 6, 11, 13\} \), \( S_4 = \{1, 2, 3, 6, 9, 11, 13\} \), \( S_5 = \{1, 2, 3, 5, 10, 11, 13\} \), or \( S_6 = \{1, 2, 3, 4, 7, 9, 11, 13\} \). Then the cardinality of \( S \) is 7. By Algorithm (A) step (2):
- either, \( b_1 = 7 \) and \( b_{i-1} = 6 \) and then \( S_{o} = \{1, 2, 3, 6, 7, 13\} \). Thus, the cardinality of \( S_{o} \) is 6;
- or, \( b_{i+2} = 25 \) and then \( b_{i+1} = 13 \) and \( b_i = 12 \). Then, \( S_{o} = \{1, 2, 3, 6, 12, 13, 25\} \). In this case, Take \( S \in \{S_1, S_2\} \) where \( S_1 = \{1, 2, 3, 5, 10, 12, 13, 25\} \), and \( S_2 = \{1, 2, 3, 6, 9, 12, 13, 25\} \). Note that \( S \) is of cardinality 8 while \( S_{o} \) is of cardinality 7.

**Case (2):** \( j \in \{3, \ldots, i-2\} \).

In \( S_{o} \), this case is satisfied in step (4) of Algorithm (A) (when \( b_1 = a_0 \) is odd and not divisible by 3). But in this case, Algorithm (A) states \( b_i \) to be \( b_{i+1} - 3 \) which is always even. Then by step (2) when \( b_i = a_0 \) is even we have that \( b_{i-1} = \frac{b_1}{2} \). That is, the difference between the two consecutive values \( b_i \) and \( b_{i+1} \) is always \( b_{i-1} \) (since \( b_1 = 2b_{i-1} \) ). While, in \( S \), we have either \( b_1 - b_{i-1} = b_{i-1} \) and then \( |S| = |S(n)| \) or \( b_1 - b_{i-1} = b_k \neq b_{i-1} \) and then \( |S| > |S(n)| \).

For instance, take \( b_{i+1} = 23 \) and \( S = \{1, 2, 3, 6, 8, 9, 17, 23\} \). By Algorithm (A) we have \( S_{o} = \{1, 2, 3, 5, 10, 20, 23\} \). Note that \( S \) is of cardinality 8 while \( S_{o} \) is of cardinality 7.

**Case (3):** \( j = i-1 \).

If \( b_{i+1} \) is not divisible by 3, then by Algorithm (A) step (2) we have \( b_i = b_{i-1} = \lceil \frac{b_{i+1}}{2} \rceil \) and \( b_1 = \lceil \frac{b_{i+1}}{2} \rceil \) (where \( b_{i-1} = b_{i-1} \) and \( b_1 = b_{2i} \)), and then the difference between the two consecutive values \( b_i \) and \( b_{i+1} \) is 1. While in \( S \), we have either \( b_1 - b_{i-1} = 1 \) and then \( |S| = |S(n)| \) or \( b_1 - b_{i-1} = b_k \neq 1 \) and then \( |S| > |S(n)| \).

If \( b_{i+1} \) is divisible by 3, then in \( S_{o} \) by Algorithm (A) step (2) (where \( b_1 = a_0 \) is odd and divisible by 3). Then \( b_{i-1} = \frac{b_{i+1}}{3} \) and \( b_1 = 2b_{i-1} \) (where \( b_{i-1} = a_{1i} \) and \( b_1 = a_{2i} \)). That is, the difference between the two consecutive values \( b_i \) and \( b_{i+1} \) is always \( b_{i-1} \) itself (since \( b_1 = 2b_{i-1} \) ). While in \( S \), either \( b_1 - b_{i-1} = b_{i-1} \) and then \( |S| = |S(n)| \) or \( b_1 - b_{i-1} = b_k \) for some \( k \in \{1, 2, \ldots, i-2\} \) and then \( |S| > |S(n)| \).

This completes the proof. \( \Box \)

Let \( L(n) = \binom{n}{2} - E_\ell \) (where \( \binom{n}{2} \) is the size of \( K_n \) and \( E_\ell \) is the size of \( C(n, S(n)) \)). Then, we conclude that

\[
L(n) = \begin{cases} 
\frac{n(n-2k)}{2}, & t = 1; \\
\frac{n(n-2k-1)}{2}, & \text{otherwise.} 
\end{cases}
\]

By Theorem 3.4, we have that \( E_\ell \) is the smallest number of edges of a circulant graph that admits a restricted triangulation. The following result is to measure the non-triangulability of \( K_n - G \).

**Theorem 3.5.** Let \( G = C(n, S) \) be a circulant graph. Then \( K_n - G \) admits no restricted triangulation if \(|E(G)| > L(n)|. \)

**Proof.** Let \(|E(G)| > L(n)|. \) Then \(|E(K_n - G)| = \binom{n}{2} - |E(G)| < \binom{n}{2} - L(n) = E_\ell. \) Then \(|N - S| < E_\ell. \) By Theorem 3.4, \( C(n, N - S) (= K_n - G) \) admits no restricted triangulation. \( \Box \)

**References**

Restricted triangulation on circulant graphs


