Open Mathematics

Research Article

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A parametric linearizing approach for quadratically inequality constrained quadratic programs

https://doi.org/10.1515/math-2018-0037

Received December 1, 2017; accepted February 20, 2018.

Abstract: In this paper we propose a new parametric linearizing approach for globally solving quadratically inequality constrained quadratic programs. By utilizing this approach, we can derive the parametric linear programs relaxation problem of the investigated problem. To accelerate the computational speed of the proposed algorithm, an interval deleting rule is used to reduce the investigated box. The proposed algorithm is convergent to the global optima of the initial problem by subsequently partitioning the initial box and solving a sequence of parametric linear programs relaxation problems. Finally, compared with some existing algorithms, numerical results show higher computational efficiency of the proposed algorithm.

Keywords: Quadratically inequality constrained quadratic programs, Global optimization, Parametric linearizing technique, Interval deleting rule

MSC: 90C20, 90C26, 65K05

1 Introduction

In this paper we consider the following quadratically inequality constrained quadratic programs:

\[(QICQP): \begin{aligned}
\min H_0(z) &= \sum_{k=1}^{n} d_k^0 z_k + \sum_{j=1}^{n} \sum_{k=1}^{n} p_{jk}^0 z_j z_k \\
\text{s.t. } H_i(z) &= \sum_{k=1}^{n} d_k^i z_k + \sum_{j=1}^{n} \sum_{k=1}^{n} p_{jk}^i z_j z_k \leq b_i, \quad i = 1, \ldots, m, \\
z &\in Z^0 = \{z \in \mathbb{R}^n : l^0 \leq z \leq u^0\},
\end{aligned}\]

where \(p_{jk}^i, d_k^i\) and \(b_i\) are all arbitrary real numbers; \(l^0 = (l_1^0, \ldots, l_m^0)^T\), \(u^0 = (u_1^0, \ldots, u_m^0)^T\). The investigated problem (QICQP) has a broad applications in investment portfolio, management decision, route optimization, engineering optimization, production planning and so on. In addition, the investigated problem (QICQP) usually owns multiple local optima which are not global optima, that is to say, in this kind of problems there are important theoretical and computational complexities. Therefore, it is very necessary to present an effective global optimization algorithm for solving the (QICQP).

In last decades, for the problem (QICQP) and its special cases many methods have been developed and described in the existent literature. For example, semi-definite relaxation method [1], reformulation-
In this paper, we will present a new branch-and-bound algorithm for globally solving the (QICQP). Firstly, we present a new parametric linearizing technique. By utilizing this method, we can convert the (QICQP) into a parametric linear programs relaxation problem, which can be used to compute the lower bounds of the optimal values of the initial problem (QICQP) and its subproblems. Secondly, based on the branch-and-bound framework, by combing the derived parametric linear programs relaxation problem with the interval deleting rule, a branch-and-bound algorithm is established for globally solving the (QICQP). Thirdly, compared with the existent methods, some numerical examples in existent literatures are used to verify the computational efficiency of the proposed algorithm in Section 3. Thirdly, compared with the existent methods, some numerical examples in existent literatures are used to verify the computational efficiency of the proposed algorithm in Section 3. Fourthly, the proposed algorithm is convergent to the global optima of the initial problem (QICQP) by successively partitioning of the initial box and by solving those derived parametric linear programs relaxation problems. Finally, compared with some existent algorithms, numerical results demonstrate the computational efficiency of the proposed algorithm.

The remaining sections of this article are organized as follows. First of all, we present a new parametric linearizing technique. By utilizing this method, we can convert the (QICQP) into a parametric linear programs relaxation problem, which can be used to compute the lower bounds of the optimal values of the initial problem (QICQP) and its subproblems. Secondly, based on the branch-and-bound framework, by successive partitioning of the initial box and by solving those derived parametric linear programs relaxation problems, a new branch-and-bound algorithm is designed for globally solving the (QICQP).}

2 New parametric linearizing approach

In this section, we propose a new parametric linearizing approach for deriving the parametric linear programs relaxation problem of the (QICQP). The detailed parametric linearizing approach is presented as follows:

Assume that $Z = \{ (z_1, z_2, \ldots, z_n)^T \in \mathbb{R}^n : l_i \leq z_j \leq u_i, j = 1, \ldots, n \} \subseteq \mathbb{Z}^n$, $\lambda = (\lambda_{jk})_{n \times n} \in \mathbb{R}^{n \times n}$ is a symmetric matrix, and $\lambda_{jk} \in \{0, 1\}$. For convenience, for any $z \in Z$, for any $k \in \{1, 2, \ldots, n\}$, some expressions are introduced as follows:

$$
\begin{align*}
  z_k(\lambda_{kk}) & = l_k + \lambda_{kk}(u_k - l_k), \\
  z_k(1 - \lambda_{kk}) & = l_k + (1 - \lambda_{kk})(u_k - l_k), \\
  h_{kk}(z) & = z_k^2, \\
  \bar{h}_{kk}(z, Z, \lambda_{kk}) & = [z_k(\lambda_{kk})]^2 + 2z_k(\lambda_{kk})[z_k - z_k(\lambda_{kk})], \\
  \bar{h}_{kk}(z, Z, \lambda_{kk}) & = [z_k(\lambda_{kk})]^2 + 2z_k(1 - \lambda_{kk})[z_k - z_k(\lambda_{kk})].
\end{align*}
$$

Obviously, we have $z_k(0) = l_k$, $z_k(1) = u_k$.

**Theorem 2.1.** For any $k \in \{1, 2, \ldots, n\}$, for any $z \in Z$, consider the functions $h_{kk}(z)$, $\bar{h}_{kk}(z, Z, \lambda_{kk})$ and $\bar{h}_{kk}(z, Z, \lambda_{kk})$, then, the following conclusions hold:

\begin{align*}
  h_{kk}(z, Z, \lambda_{kk}) & \leq \bar{h}_{kk}(z) \leq \bar{h}_{kk}(z, Z, \lambda_{kk}); \\
  \lim_{|u_k - l_k| \to 0} [h_{kk}(z) - \bar{h}_{kk}(z, Z, \lambda_{kk})] &= 0 \\
  \lim_{|u_k - l_k| \to 0} [\bar{h}_{kk}(z, Z, \lambda_{kk}) - h_{kk}(z)] &= 0.
\end{align*}
Proof. (i) By the mean value theorem, for any $z \in Z$, there exists a point $\xi_k = \alpha z_k + (1 - \alpha)z_k(\lambda_{kk})$, where $\alpha \in [0, 1]$, such that

$$z_k^2 = [z_k(\lambda_{kk})]^2 + 2\xi_k[z_k - z_k(\lambda_{kk})].$$

If $\lambda_{kk} = 0$, then we have

$$\xi_k \geq l_k = z_k(\lambda_{kk})$$

and $z_k - z_k(\lambda_{kk}) = z_k - l_k \geq 0$.

If $\lambda_{kk} = 1$, then it follows that

$$\xi_k \leq u_k = z_k(1 - \lambda_{kk})$$

and $z_k - z_k(\lambda_{kk}) = z_k - u_k \leq 0$.

Thus, we can get that

$$h_{kk}(z) = z_k^2 = [z_k(\lambda_{kk})]^2 + 2\xi_k[z_k - z_k(\lambda_{kk})] 
\geq [z_k(\lambda_{kk})]^2 + 2z_k(1 - \lambda_{kk})[z_k - z_k(\lambda_{kk})] = \overline{h}_{kk}(z, Z, \lambda_{kk}).$$

Similarly, if $\lambda_{kk} = 0$, then we have

$$\xi_k \leq u_k = z_k(1 - \lambda_{kk})$$

and $z_k - z_k(\lambda_{kk}) = z_k - u_k \geq 0$.

If $\lambda_{kk} = 1$, then it follows that

$$\xi_k \geq l_k = z_k(1 - \lambda_{kk})$$

and $z_k - z_k(\lambda_{kk}) = z_k - u_k \leq 0$.

Thus, we can get that

$$h_{kk}(z) = z_k^2 = [z_k(\lambda_{kk})]^2 + 2\xi_k[z_k - z_k(\lambda_{kk})] 
\leq [z_k(\lambda_{kk})]^2 + 2z_k(1 - \lambda_{kk})[z_k - z_k(\lambda_{kk})] = \underline{h}_{kk}(z, Z, \lambda_{kk}).$$

Therefore, for any $z \in Z$, we have that

$$\underline{h}_{kk}(z, Z, \lambda_{kk}) \leq h_{kk}(z) \leq \overline{h}_{kk}(z, Z, \lambda_{kk}).$$

(ii) Since

$$h_{kk}(z) - \overline{h}_{kk}(z, Z, \lambda_{kk}) = z_k^2 - \{[z_k(\lambda_{kk})]^2 + 2z_k(1 - \lambda_{kk})[z_k - z_k(\lambda_{kk})]\} = (z_k - z_k(\lambda_{kk}))^2 \leq (u_k - l_k)^2, \tag{4}$$

we have

$$\lim_{|u| \to 0} [h_{kk}(z) - \overline{h}_{kk}(z, Z, \lambda_{kk})] = 0.$$

Also since

$$\overline{h}_{kk}(z, Z, \lambda_{kk}) - h_{kk}(z) = [z_k(\lambda_{kk})]^2 + 2z_k(1 - \lambda_{kk})[z_k - z_k(\lambda_{kk})] - z_k^2
= (z_k(\lambda_{kk}) + z_k)(z_k(\lambda_{kk}) - z_k) + 2z_k(1 - \lambda_{kk})(z_k - z_k(\lambda_{kk}))
= [z_k - z_k(\lambda_{kk})][2z_k(1 - \lambda_{kk}) - z_k(\lambda_{kk}) - z_k]
= [z_k - z_k(\lambda_{kk})][z_k(1 - \lambda_{kk}) - z_k]
+ [z_k - z_k(\lambda_{kk})][z_k(1 - \lambda_{kk}) - z_k] \leq 2(u_k - l_k)^2. \tag{5}$$

Therefore, it follows that

$$\lim_{|u| \to 0} [\overline{h}_{kk}(z, Z, \lambda_{kk}) - h_{kk}(z)] = 0.$$

The proof is completed. \qed
Without loss of generality, for any $z \in Z$, for any $j \in \{1, 2, \ldots, n\}$, $k \in \{1, 2, \ldots, n\}$, $j \neq k$, we define
\[
\begin{align*}
  z_j(\lambda_{jk}) &= l_j + \lambda_{jk}(u_j - l_j), \\
  z_k(\lambda_{jk}) &= l_k + \lambda_{jk}(u_k - l_k), \\
  z_j(1 - \lambda_{jk}) &= l_j + (1 - \lambda_{jk})(u_j - l_j), \\
  z_k(1 - \lambda_{jk}) &= l_k + (1 - \lambda_{jk})(u_k - l_k), \\
  (z_j - z_k)(\lambda_{jk}) &= (l_j - u_k) + \lambda_{jk}(u_j - l_j - l_k - u_k), \\
  (z_j - z_k)(1 - \lambda_{jk}) &= (u_j - l_k - l_j - u_k).
\end{align*}
\]

Obviously, we have $(z_j - z_k)(0) = l_j - u_k$, $(z_j - z_k)(1) = u_j - l_k$.

In a similar way as in Theorem 2.1, we can get the following Theorem 2.2:

**Theorem 2.2.** For each $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, n$, for any $z \in Z$, we have:

(i) The following inequalities hold:
\[
\begin{align*}
  [z_j(\lambda_{jk})]^2 + 2z_j(\lambda_{jk})[z_j - z_j(\lambda_{jk})] &\leq z_j^2 \leq [z_j(\lambda_{jk})]^2 + 2z_j(1 - \lambda_{jk})[z_j - z_j(\lambda_{jk})], \\
  [z_k(\lambda_{jk})]^2 + 2z_k(\lambda_{jk})[z_k - z_k(\lambda_{jk})] &\leq z_k^2 \leq [z_k(\lambda_{jk})]^2 + 2z_k(1 - \lambda_{jk})[z_k - z_k(\lambda_{jk})], \\
  (z_j - z_k)^2 &\leq [(z_j - z_k)(\lambda_{jk})]^2 + 2(z_j - z_k)(1 - \lambda_{jk})[z_j - z_k - (z_j - z_k)(\lambda_{jk})], \\
  (z_j - z_k)^2 &\geq [(z_j - z_k)(\lambda_{jk})]^2 + 2(z_j - z_k)(\lambda_{jk})[z_j - z_k - (z_j - z_k)(\lambda_{jk})].
\end{align*}
\]

(ii) The following limitations hold:
\[
\begin{align*}
  \lim_{|u_j| \to 0} [z_j^2 - ([z_j(\lambda_{jk})]^2 + 2z_j(\lambda_{jk})[z_j - z_j(\lambda_{jk})])] &= 0, \\
  \lim_{|u_j| \to 0} [(z_j(\lambda_{jk})]^2 + 2z_j(1 - \lambda_{jk})[z_j - z_j(\lambda_{jk})] - z_j^2] &= 0, \\
  \lim_{|u_j| \to 0} [z_k^2 - ([z_k(\lambda_{jk})]^2 + 2z_k(\lambda_{jk})[z_k - z_k(\lambda_{jk})])] &= 0, \\
  \lim_{|u_j| \to 0} [z_k^2 - ([z_k(\lambda_{jk})]^2 + 2z_k(1 - \lambda_{jk})[z_k - z_k(\lambda_{jk})] - z_k^2] &= 0, \\
  \lim_{|u_j| \to 0} [[(z_j - z_k)(\lambda_{jk})]^2 + 2(z_j - z_k)(1 - \lambda_{jk})[z_j - z_k - (z_j - z_k)(\lambda_{jk})] - (z_j - z_k)^2] &= 0, \\
  \lim_{|u_j| \to 0} [(z_j - z_k)^2 - ([z_j - z_k)(\lambda_{jk})]^2 + 2(z_j - z_k)(\lambda_{jk})[z_k - (z_j - z_k)(\lambda_{jk})]] &= 0.
\end{align*}
\]

**Proof.** (i) From the inequality (1), replacing $\lambda_{kk}$ by $\lambda_{jk}$, and replacing $z_k$ by $z_j$, we can get that
\[
[z_j(\lambda_{jk})]^2 + 2z_j(\lambda_{jk})[z_j - z_j(\lambda_{jk})] \leq z_j^2 \leq [z_j(\lambda_{jk})]^2 + 2z_j(1 - \lambda_{jk})[z_j - z_j(\lambda_{jk})].
\]

From the inequality (1), replacing $\lambda_{kk}$ by $\lambda_{jk}$, we can get that
\[
[z_k(\lambda_{jk})]^2 + 2z_k(\lambda_{jk})[z_k - z_k(\lambda_{jk})] \leq z_k^2 \leq [z_k(\lambda_{jk})]^2 + 2z_k(1 - \lambda_{jk})[z_k - z_k(\lambda_{jk})].
\]

From (1), replacing $\lambda_{kk}$ and $z_k$ by $\lambda_{jk}$ and $(z_j - z_k)$, respectively, we can get that
\[
(z_j - z_k)^2 \leq [(z_j - z_k)(\lambda_{jk})]^2 + 2(z_j - z_k)(1 - \lambda_{jk})[(z_j - z_k) - (z_j - z_k)(\lambda_{jk})],
\]
\[
(z_j - z_k)^2 \geq [(z_j - z_k)(\lambda_{jk})]^2 + 2(z_j - z_k)(\lambda_{jk})[(z_j - z_k) - (z_j - z_k)(\lambda_{jk})].
\]

(ii) From the limitations (2) and (3), replacing $\lambda_{kk}$ and $z_k$ by $\lambda_{jk}$ and $z_j$, we have
\[
\lim_{|u_j| \to 0} [z_j^2 - ([z_j(\lambda_{jk})]^2 + 2z_j(\lambda_{jk})[z_j - z_j(\lambda_{jk})])] = 0
\]
and
\[
\lim_{|u_j| \to 0} [(z_j(\lambda_{jk})]^2 + 2z_j(1 - \lambda_{jk})[z_j - z_j(\lambda_{jk})] - z_j^2] = 0.
\]
From the limitations (2) and (3), replacing \( \lambda_{jk} \) by \( \lambda_{jk} \), it follows that
\[
\lim_{\|u-l\| \to 0} \left[ z_k^2 - \left( z_k(\lambda_{jk}) \right)^2 + 2z_k(\lambda_{jk})[z_k - z_k(\lambda_{jk})] \right] = 0
\]
and
\[
\lim_{\|u-l\| \to 0} \left[ (z_k(\lambda_{jk}))^2 + 2z_k(1 - \lambda_{jk})[z_k - z_k(\lambda_{jk})] - z_k^2 \right] = 0.
\]

By the limitations (2) and (3), replacing \( \lambda_{jk} \) and \( z_k \) by \( \lambda_{jk} \) and \( (z_j - z_k) \), respectively, we can get that
\[
\lim_{\|u-l\| \to 0} \left[ \left( (z_j - z_k)(\lambda_{jk}) \right)^2 + 2(z_j - z_k)(1 - \lambda_{jk})[z_j - z_k(\lambda_{jk})] - (z_j - z_k)^2 \right] = 0
\]
and
\[
\lim_{\|u-l\| \to 0} \left[ (z_j - z_k)^2 - \left( (z_j - z_k)(\lambda_{jk}) \right)^2 + 2(z_j - z_k)(\lambda_{jk})[z_j - z_k(\lambda_{jk})] \right] = 0.
\]

The proof is completed. \( \square \)

Without loss of generality, for any \( z \in Z \), for any \( j \in \{1, 2, \ldots, n\}, k \in \{1, 2, \ldots, n\}, j \neq k \), define
\[
h_{jk}(z) = z_jz_k = \frac{z_j^2 + z_k^2 - (z_j - z_k)^2}{2},
\]
\[
h_{jk}(z, Z, \lambda_{jk}) = \frac{1}{2} \left( (z_j(\lambda_{jk}))^2 + 2z_j(\lambda_{jk})[z_j - z_j(\lambda_{jk})] + [z_k(\lambda_{jk})]^2 + 2z_k(\lambda_{jk})[z_k - z_k(\lambda_{jk})] - \left( (z_j - z_k)(\lambda_{jk}) \right)^2 + 2(z_j - z_k)(1 - \lambda_{jk})[z_j - z_k(\lambda_{jk})] \right),
\]
\[
\tilde{h}_{jk}(z, Z, \lambda_{jk}) = \frac{1}{2} \left( (z_j(\lambda_{jk}))^2 + 2z_j(1 - \lambda_{jk})[z_j - z_j(\lambda_{jk})] + [z_k(\lambda_{jk})]^2 + 2z_k(1 - \lambda_{jk})[z_k - z_k(\lambda_{jk})] - \left( (z_j - z_k)(\lambda_{jk}) \right)^2 + 2(z_j - z_k)(\lambda_{jk})[z_j - z_k(\lambda_{jk})] \right).
\]

**Theorem 2.3.** For each \( k = 1, 2, \ldots, n \), consider the functions \( h_{jk}(z, Z, \lambda_{jk}), h_{jk}(z) \) and \( \tilde{h}_{jk}(z, Z, \lambda_{jk}) \), then, for any \( z \in Z \), we have the following conclusions:
\[
h_{jk}(z) \leq h_{jk}(z) \leq \tilde{h}_{jk}(z, Z, \lambda_{jk}), \tag{6}
\]
\[
\lim_{\|u-l\| \to 0} [h_{jk}(z) - h_{jk}(z, Z, \lambda_{jk})] = 0 \tag{7}
\]
and
\[
\lim_{\|u-l\| \to 0} [\tilde{h}_{jk}(z, Z, \lambda_{jk}) - h_{jk}(z)] = 0. \tag{8}
\]

**Proof.** First of all, from the conclusions (i) of Theorem 2.2, it follows that
\[
h_{jk}(z) = z_jz_k = \frac{z_j^2 + z_k^2 - (z_j - z_k)^2}{2}
\]
\[
\geq \frac{1}{2} \left( (z_j(\lambda_{jk}))^2 + 2z_j(\lambda_{jk})[z_j - z_j(\lambda_{jk})] + [z_k(\lambda_{jk})]^2 + 2z_k(\lambda_{jk})[z_k - z_k(\lambda_{jk})] - \left( (z_j - z_k)(\lambda_{jk}) \right)^2 + 2(z_j - z_k)(1 - \lambda_{jk})[z_j - z_k(\lambda_{jk})] \right)
\]
\[
= h_{jk}(z, Z, \lambda_{jk})
\]
and
\[
h_{jk}(z) = z_jz_k = \frac{z_j^2 + z_k^2 - (z_j - z_k)^2}{2}
\]
\[
\leq \frac{1}{2} \left( (z_j(\lambda_{jk}))^2 + 2z_j(1 - \lambda_{jk})[z_j - z_j(\lambda_{jk})] + [z_k(\lambda_{jk})]^2 + 2z_k(1 - \lambda_{jk})[z_k - z_k(\lambda_{jk})] - \left( (z_j - z_k)(\lambda_{jk}) \right)^2 + 2(z_j - z_k)(\lambda_{jk})[z_j - z_k(\lambda_{jk})] \right)
\]
\[
= \tilde{h}_{jk}(z, Z, \lambda_{jk}).
\]
Secondly, from the inequalities (4) and (5), we have

\[ h_{jk}(z) - \bar{h}_{jk}(z, Z, \lambda_{jk}) = z_j z_k - \bar{h}_{jk}(z, Z, \lambda_{jk}) \]

\[ = \frac{1}{2} + z_j - z_k - z_j z_k \]

\[ = \frac{1}{2} \left( \left[ (z_j (\lambda_{jk}))^2 + 2z_j (1 - \lambda_{jk}) [z_j - z_k (\lambda_{jk})] \right] + 2z_k (1 - \lambda_{jk}) [z_k - z_k (\lambda_{jk})] \right) \]

\[ - \left( \left[ (z_j - z_k) (\lambda_{jk}) \right]^2 + 2(z_j - z_k) (\lambda_{jk}) [z_j - z_k - (z_j - z_k) (\lambda_{jk})] \right) \]

\[ \leq \left( (u_j - l_j)^2 + (u_k - l_k)^2 + (u_k + u_j - l_j - l_k)^2 \right). \]

Thus, we can get that \( \lim_{|u - l| \to 0} [h_{jk}(z) - \bar{h}_{jk}(z, Z, \lambda_{jk})] = 0. \)

Also from the inequalities (4) and (5), we get that

\[ \bar{h}_{jk}(z, Z, \lambda_{jk}) - h_{jk}(z) = \bar{h}_{jk}(z, Z, \lambda_{jk}) - z_j z_k \]

\[ = \frac{1}{2} \left( \left[ (z_j (\lambda_{jk}))^2 + 2z_j (1 - \lambda_{jk}) [z_j - z_k (\lambda_{jk})] \right] + 2z_k (1 - \lambda_{jk}) [z_k - z_k (\lambda_{jk})] \right) \]

\[ - \left( \left[ (z_j - z_k) (\lambda_{jk}) \right]^2 + 2(z_j - z_k) (\lambda_{jk}) [z_j - z_k - (z_j - z_k) (\lambda_{jk})] \right) \]

\[ \leq \left( (u_j - l_j)^2 + (u_k - l_k)^2 + (u_k + u_j - l_j - l_k)^2 \right). \]

Thus, it follows that \( \lim_{|u - l| \to 0} [\bar{h}_{jk}(z, Z, \lambda_{jk}) - h_{jk}(z)] = 0. \) \( \square \)

Without loss of generality, for any \( Z = [l, u] \subseteq Z^d, \) for any parameter matrix \( \lambda = (\lambda_{jk})_{n \times n}, \) for any \( z \in Z \) and \( i \in \{0, 1, \ldots, m\}, \) we let

\[ f_{i,k}^{i}(Z, Z; \lambda_{kk}) = \begin{cases} p_{ik}^{i,k} h_{i,k}^{i}(Z, Z; \lambda_{kk}), & \text{if } p_{ik}^{i,k} > 0, \\ p_{ik}^{i,k} h_{i,k}^{i}(Z, Z; \lambda_{kk}), & \text{if } p_{ik}^{i,k} < 0, \end{cases} \]

\[ f_{i,k}^{i}(Z, Z; \lambda_{kk}) = \begin{cases} p_{ik}^{i,k} h_{i,k}^{i}(Z, Z; \lambda_{kk}), & \text{if } p_{ik}^{i,k} > 0, \\ p_{ik}^{i,k} h_{i,k}^{i}(Z, Z; \lambda_{kk}), & \text{if } p_{ik}^{i,k} < 0, \end{cases} \]

\[ f_{j,k}^{i}(Z, Z; \lambda_{kk}) = \begin{cases} p_{jk}^{i,k} h_{j,k}^{i}(Z, Z; \lambda_{kk}), & \text{if } p_{jk}^{i,k} > 0, j \neq k, \\ p_{jk}^{i,k} h_{j,k}^{i}(Z, Z; \lambda_{kk}), & \text{if } p_{jk}^{i,k} < 0, j \neq k, \end{cases} \]

\[ f_{j,k}^{i}(Z, Z; \lambda_{kk}) = \begin{cases} p_{jk}^{i,k} h_{j,k}^{i}(Z, Z; \lambda_{kk}), & \text{if } p_{jk}^{i,k} > 0, j \neq k, \\ p_{jk}^{i,k} h_{j,k}^{i}(Z, Z; \lambda_{kk}), & \text{if } p_{jk}^{i,k} < 0, j \neq k. \end{cases} \]

\[ H_{i}^{\ell}(Z, Z, \lambda) = \sum_{k=1}^{n} \left( d_{ik}^{i} z_k + f_{i,k}^{i}(Z, Z; \lambda_{kk}) \right) + \sum_{j=1}^{n} \sum_{k=1}^{n} f_{i,k}^{i}(Z, Z; \lambda_{kk}). \]

\[ H_{i}^{\ell}(Z, Z, \lambda) = \sum_{k=1}^{n} \left( d_{ik}^{i} z_k + f_{i,k}^{i}(Z, Z; \lambda_{kk}) \right) + \sum_{j=1}^{n} \sum_{k=1}^{n} f_{i,k}^{i}(Z, Z; \lambda_{kk}). \]

**Theorem 2.4.** For \( \forall z \in Z = [l, u] \subseteq Z^d, \) for any given parameter matrix \( \lambda = (\lambda_{jk})_{n \times n}, \) for each \( i = 0, 1, \ldots, m, \) we have the following conclusions:

\[ H_{i}^{\ell}(Z, Z, \lambda) \leq H_{i}(z) \leq H_{i}^{U}(Z, Z, \lambda), \]
From the inequalities (1) and (6), for any $j, k \in \{1, \ldots, n\}$, we have
\begin{equation}
\begin{aligned}
\frac{f_i^{ij}}{f_i^{ij}}(z, Z, \lambda_{kk}) & \leq p_{kk} z_k^2 + \frac{f_i^{ij}}{f_i^{ij}}(z, Z, \lambda_{kk}), \\
\frac{f_i^{ij}}{f_i^{ij}}(z, Z, \lambda_{jk}) & \leq p_{jk} z_j z_k + \frac{f_i^{ij}}{f_i^{ij}}(z, Z, \lambda_{jk}).
\end{aligned}
\end{equation}
By the above inequalities (9) and (10), for any $z \in Z \subseteq Z^0$, we can get that
\begin{align}
H_i^{(1)}(z, Z, \lambda) = & \sum_{k=1}^{n} (d_k^i z_k + f_i^{ij}(z, Z, \lambda_{kk})) + \sum_{j=1}^{n} p_{jk} z_j z_k \\
& \leq \sum_{k=1}^{n} (d_k^i z_k + f_i^{ij}(z, Z, \lambda_{kk})) + \sum_{j=1}^{n} p_{jk} z_j z_k = H_i(z) \\
& \leq \sum_{k=1}^{n} (d_k^i z_k + f_i^{ij}(z, Z, \lambda_{kk})) + \sum_{j=1}^{n} f_i^{ij}(z, Z, \lambda_{jk}) \\
& = H_i^{(1)}(z, Z, \lambda).
\end{align}
Therefore, we have $H_i^{(1)}(z, Z, \lambda) \leq H_i(z) \leq H_i^{(1)}(z, Z, \lambda)$.

Secondly,
\begin{align}
H_i(z) - H_i^{(1)}(z, Z, \lambda) &= \sum_{k=1}^{n} d_k^i z_k + \sum_{k=1}^{n} p_{kk} z_k^2 + \sum_{j=1}^{n} p_{jk} z_j z_k \\
& - \left[ \sum_{k=1}^{n} d_k^i z_k + \sum_{k=1}^{n} f_i^{ij}(z, Z, \lambda_{kk}) + \sum_{j=1}^{n} \sum_{k=1, k \neq j}^{n} p_{jk} z_j z_k \right] \\
& = \sum_{k=1}^{n} \left[ p_{kk} z_k^2 - f_i^{ij}(z, Z, \lambda_{kk}) \right] \\
& + \sum_{j=1}^{n} \sum_{k=1, k \neq j}^{n} \left[ p_{jk} z_j z_k - f_i^{ij}(z, Z, \lambda_{jk}) \right] \\
& = \sum_{k=1}^{n} \left[ p_{kk}^i \{ h_{kk}(z) - h_{kk}(z, Z, \lambda_{kk}) \} \right] \\
& + \sum_{k=1, p_{kk}^i < 0}^{n} \left[ p_{kk}^i \{ h_{kk}(z) - \bar{h}_{kk}(z, Z, \lambda_{kk}) \} \right] \\
& + \sum_{j=1}^{n} \sum_{k=1, k \neq j}^{n} \left[ p_{jk}^i \{ h_{jk}(z) - \bar{h}_{jk}(z, Z, \lambda_{jk}) \} \right] \\
& + \sum_{j=1, p_{jk}^i < 0}^{n} \sum_{k=1, k \neq j}^{n} \left[ p_{jk}^i \{ h_{jk}(z) - \bar{h}_{jk}(z, Z, \lambda_{jk}) \} \right].
\end{align}
From the limitations (2),(3),(7) and (8), $\lim_{\|u-l\| \to 0} [h_{kk}(z) - h_{kk}(z, Z, \lambda_{kk})] = 0$, $\lim_{\|u-l\| \to 0} [\bar{h}_{kk}(z, Z, \lambda_{kk}) - h_{kk}(z)] = 0$, $\lim_{\|u-l\| \to 0} [h_{jk}(z) - h_{jk}(z, Z, \lambda_{jk})] = 0$ and $\lim_{\|u-l\| \to 0} [\bar{h}_{jk}(z, Z, \lambda_{jk}) - h_{jk}(z)] = 0$.

Therefore, it follows that
\begin{align}
\lim_{\|u-l\| \to 0} [H_i(z) - H_i^{(1)}(z, Z, \lambda)] &= 0.
\end{align}
Similarly, we can prove that
\begin{align}
\lim_{\|u-l\| \to 0} [H_i^{U}(z, Z, \lambda) - H_i(z)] &= 0.
\end{align}
The proof is completed. □

By Theorem 2.4, we can construct the parametric linear programs relaxation problem (PLPRP) of the (QICQP) over $Z$ as follows:
\begin{align}
\text{PLPRP} : \quad & \min H_i^{U}(z, Z, \lambda), \\
& \text{s.t.} \quad H_i^{(1)}(z, Z, \lambda) \leq b_i, \quad i = 1, \ldots, m, \\
& \quad z \in Z = \{ z : l \leq z \leq u \}.
\end{align}
Proof. In a similar way as in the proof of Theorem 3 in [14], we may draw the conclusions for Theorem 3.1, so here it is omitted.

From Theorem 3.1, we can construct an interval deleting step to compress the investigated box for improving the convergent speed of the proposed branch-and-bound algorithm.
3.1 New branch-and-bound algorithm

For any sub-box $Z^s \subseteq Z^0$, we denote by $LB(Z^s)$ the optimal value of the (PLPRP) over the sub-box $Z^s$, and denote by $z^s = z(Z^s)$ the optimal solution of the (PLPRP) over the sub-box $Z^s$. Based on the branch-and-bound framework, combining the former branching step, bounding the lower bound step, bounding upper bound step and interval deleting step together, a new branch-and-bound algorithm is designed as follows.

Branch-and-Bound Algorithm Steps

**Initializing step.** Given the initial convergent error $\epsilon$, the initial randomly generated parameter matrix $\lambda$.

Solve the (PLPRP) over the initial box $Z^0$ to obtain its optimal solution $z^0$ and optimal value $LB(Z^0)$, denote by the initial lower bound $LB_0 = LB(Z^0)$. If $z^0$ is a feasible solution of the (QICQP), we denote by the initial upper bound $UB_0 = H_0(z^0)$. Otherwise, we denote by the initial upper bound $UB_0 = +\infty$.

If $UB_0 - LB_0 \leq \epsilon$, the proposed algorithm terminates, $z^0$ is a global $\epsilon$-optimal solution of the initial problem (QICQP). Otherwise, set $\Omega_0 = \{Z^0\}$, $\Lambda = \emptyset$, $s = 1$.

**Branching step.** Let $UB_s = UB_{s-1}$. Partition the investigated sub-box $Z^{s-1}$ into two sub-boxes $Z^{s-1}, Z^{s-2}$ by the selected branching rule, and denote by $\Lambda = \Lambda \cup \{Z^{s-1}\}$ the set of the deleting sub-boxes.

**Interval deleting step.** For each investigated sub-box $Z^{s,t}$, $t = 1, 2$, use the former interval deleting rule to compress the investigated sub-box, still denote by $Z^{s,t}$ the remaining sub-box.

**Bounding the lower bound step.** For each remaining sub-box $Z^{s,t}$, where $t = 1, 2$, solve the (PLPRP) over $Z^{s,t}$ to obtain its optimal solution $z^{s,t}$ and optimal value $LB(Z^{s,t})$, and let $\Omega_s = \{Z|Z \in \Omega_{s-1} \cup \{Z_{s-1}, Z_{s-2}\}, Z \notin \Lambda\}$ and $LB_s = \min\{LB(Z)|Z \in \Omega_s\}$.

**Bounding the upper bound step.** For each sub-box $Z^{s,t}$, if its midpoint $z^{mid}$ is the feasible point of the initial problem (QICQP), let $\Theta := \Theta \cup \{z^{mid}\}$, denote by the new upper bound $UB_s = \min_{z \in \Theta} H_0(z)$; if the optimal solution $z^{s,t}$ of the (PLPRP) is the feasible point of the initial problem (QICQP), denote by the new upper bound $UB_s = \min\{UB_s, H_0(z^{s,t})\}$, and denote by $z^s$ the best existent feasible point such that $UB_s = H_0(z^s)$.

**Terminating judgement step.** If $UB_s - LB_s \leq \epsilon$, the proposed algorithm terminates, $z^s$ is a global $\epsilon$-optimal solution of the initial problem (QICQP). Otherwise, denote by $s = s + 1$, and go to the Branching step.

3.2 Global convergence of the proposed algorithm

Without loss of generality, we assume that $v$ is the global optimal value of the initial problem (QICQP). If the proposed algorithm terminates after $s$ finite iterations, where $s$ is a finite number such that $s \geq 0$, then it follows that $UB_s \leq LB_s + \epsilon$.

From the bounding the upper bound step of the proposed algorithm, we know that there must exist a feasible point $z^s$ of the initial problem (QICQP) such that $v \leq UB_s = H_0(z^s)$.

By the branch-and-bound structure of the proposed algorithm, we have $LB_s \leq v$.

Combining the above several inequalities together, it follows that $v \leq UB_s = H_0(z^s) \leq LB_s + \epsilon \leq v + \epsilon$. 

Therefore, $z^*$ is an $\epsilon$-global optimal solution of the initial problem (QICQP).

If the proposed algorithm does not terminate after finite iterations, for this case, the detailed convergent conclusions are given as follows.

**Theorem 3.2.** If the proposed algorithm does not terminate after finite iterations, then it will generate an infinite partitioning sequence $\{Z^n\}$ of the initial box $Z^0$, and any accumulation point of the sequence $\{Z^n\}$ will be a global optimum solution of the initial problem (QICQP).

**Proof.** First of all, in the proposed algorithm the selected branching method is the bisection of box, so that the branching process is exhaustive, that is to say, the branching step will ensure that the intervals of all variables tend to 0, i.e., $\|u - l\| \to 0$.

Secondly, from Theorem 2.4, the optimal solution of the (PLRP) will sufficiently approximate the optimal solution of the (QICQP) as $\|u - l\| \to 0$, and this ensures that the limitation $\lim_{n \to \infty} (UB_n - LB_n) = 0$ holds. So that the bounding operation is consistent.

Thirdly, in the proposed algorithm the subdivided box which attains the actual lower bound is selected for further partition at the later immediate iteration, so that the used selecting operation is bound improving.

From [26, Theorem IV.3], the sufficient condition of global convergence of the branch-and-bound algorithm is that the branching method is exhaustive, the bounding method is consistent and the selecting method is improvement, therefore, the proposed algorithm is convergent to the global optimal solution of the initial (QICQP). □

## 4 Numerical experiments

Given the convergent error $\epsilon = 10^{-6}$ and the parameter matrix $\lambda = (\lambda_{jk})_{n \times n} \in \mathbb{R}^{n \times n}$, where $\lambda_{jk} \in \{0, 1\}$, compared with the existing methods, several numerical examples in existing literature are tested on microcomputer, the procedure is coded in C++ software, the parametric linear programs relaxation problems are solved by the simplex method. These examples and their numerical results are listed as follows. In the following Tables 1 and 2, the number of iteration and running time in seconds for the algorithm are represented by “Iteration” and “Time(s)”, respectively.

**Example 4.1 ([16]).**

$$
\begin{align*}
\text{min } H_0(z) &= z_1 \\
\text{s.t. } H_1(z) &= \frac{1}{2} z_1 + \frac{1}{2} z_2 - \frac{1}{2} z_1^2 - \frac{1}{2} z_2^2 \leq 1, \\
H_2(z) &= \frac{1}{4} z_1^2 + \frac{1}{4} z_2^2 - \frac{1}{2} z_1 - \frac{3}{2} z_2 \leq -1, \\
1 &\leq z_1 \leq 5.5, \quad 1 \leq z_2 \leq 5.5.
\end{align*}
$$

**Example 4.2 ([16]).**

$$
\begin{align*}
\text{min } H_0(z) &= z_1 z_2 - 2z_1 + z_2 + 1 \\
\text{s.t. } H_1(z) &= 8z_2^2 - 6z_1 - 16z_2 \leq -11, \\
H_2(z) &= -z_2^2 + 3z_1 + 2z_2 \leq 7, \\
1 &\leq z_1 \leq 2.5, \quad 1 \leq z_2 \leq 2.225.
\end{align*}
$$

**Example 4.3 ([4,5,17]).**

$$
\begin{align*}
\text{min } H_0(z) &= z_1^2 + z_2^2 \\
\text{s.t. } H_1(z) &= 0.3z_1z_2 \geq 1, \\
2 &\leq z_1 \leq 5, \quad 1 \leq z_2 \leq 3.
\end{align*}
$$
Example 4.4 ([5,14,17,18]).

\[
\begin{aligned}
\text{min } H_0(z) &= z_1 \\
\text{ s.t. } H_1(z) &= 4z_1^2 - 4z_1^2 \leq 1, \\
H_2(z) &= -z_1 - z_2 \leq -1, \\
0.01 \leq z_1, z_2 &\leq 15.
\end{aligned}
\]

Example 4.5 ([4,6,14]).

\[
\begin{aligned}
\text{min } H_0(z) &= 6z_1^2 + 4z_2^2 + 5z_1z_2 \\
\text{ s.t. } H_1(z) &= -6z_1z_2 \leq -48, \\
0 \leq z_1, z_2 &\leq 10.
\end{aligned}
\]

Example 4.6 ([19]).

\[
\begin{aligned}
\text{min } H_0(z) &= -z_1 + z_1z_2^{0.5} - z_2 \\
\text{ s.t. } H_1(z) &= -6z_1 + 8z_2 \leq 3, \\
H_2(z) &= 3z_1 - z_2 \leq 3, \\
1 \leq z_1, z_2 &\leq 1.5.
\end{aligned}
\]

Example 4.7 ([14,20]).

\[
\begin{aligned}
\text{min } H_0(z) &= -4z_2 + (z_1 - 1)^2 + z_2^2 - 10z_1^2 \\
\text{ s.t. } H_1(z) &= z_1^2 + z_2^2 + z_3^2 \leq 2, \\
H_2(z) &= (z_1 - 2)^2 + z_2^2 + z_3^2 \leq 2, \\
2 - \sqrt{2} \leq z_1 \leq \sqrt{2}, &\quad 0 \leq z_2, z_3 \leq \sqrt{2}.
\end{aligned}
\]

**Table 1.** Numerical comparisons for Examples 4.1-4.7

<table>
<thead>
<tr>
<th>Example</th>
<th>Refs.</th>
<th>Optimal value</th>
<th>Optimal solution</th>
<th>Iteration</th>
<th>Time(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>our</td>
<td>1.177124344</td>
<td>(1.177124344, 2.177124344)</td>
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<td>0.0103</td>
</tr>
<tr>
<td></td>
<td>[16]</td>
<td>1.177124327</td>
<td>(1.177124327, 2.177124353)</td>
<td>434</td>
<td>1.0000</td>
</tr>
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<td>2</td>
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<td>(2.0000000, 1.0000000)</td>
<td>21</td>
<td>0.0079</td>
</tr>
<tr>
<td></td>
<td>[16]</td>
<td>-1.0</td>
<td>(2.0000000, 1.0000000)</td>
<td>24</td>
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</tr>
<tr>
<td>3</td>
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<td>(2.000000000, 1.666667279)</td>
<td>12</td>
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<td>(2.000000000, 1.666666667)</td>
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<td>0.0068</td>
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<tr>
<td></td>
<td>[5]</td>
<td>6.77782016</td>
<td>(2.000000000, 1.666666667)</td>
<td>40</td>
<td>0.0320</td>
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<tr>
<td></td>
<td>[17]</td>
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<td>(2.00003, 1.66665)</td>
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<td>0.1800</td>
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<tr>
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<td>(0.500000000, 0.5000000000)</td>
<td>25</td>
<td>0.0070</td>
</tr>
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<tr>
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<td>0.0193</td>
</tr>
<tr>
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<td>(0.5, 0.5)</td>
<td>91</td>
<td>0.8500</td>
</tr>
<tr>
<td></td>
<td>[18]</td>
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<td>(0.5, 0.5)</td>
<td>96</td>
<td>1.0000</td>
</tr>
<tr>
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<td>(2.560178568, 3.125000000)</td>
<td>46</td>
<td>0.0294</td>
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<td>[4]</td>
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<td>(2.555409888, 3.130613160)</td>
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<td>[6]</td>
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<td>(2.555745855, 3.130201688)</td>
<td>59</td>
<td>0.0385</td>
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<tr>
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<td>(1.499977112, 1.5)</td>
<td>37</td>
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</tr>
<tr>
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<td>(1.5, 1.5)</td>
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<td>0.1257</td>
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<tr>
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<td>(1.0, 0.181815071, 0.983332741)</td>
<td>98</td>
<td>0.1672</td>
</tr>
<tr>
<td></td>
<td>[14]</td>
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<td>(1.0, 0.181818470, 0.9833322113)</td>
<td>420</td>
<td>0.2845</td>
</tr>
<tr>
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<td>[20]</td>
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<td>(0.998712, 0.196213, 0.979216)</td>
<td>1648</td>
<td>0.3438</td>
</tr>
</tbody>
</table>
Example 4.8 ([3,14]).

\[
\begin{align*}
\max \ H_0(z) &= \sum_{i=1}^{n} z_i^2 \\
\text{s.t. } H_j(z) &= \sum_{i=1}^{j} z_i \leq j, \ j = 1, 2, \ldots, n, \\
& \ z_i \geq 0, \ i = 1, 2, \ldots, n.
\end{align*}
\]

Table 2. Numerical results for Example 4.8

<table>
<thead>
<tr>
<th>Refs.</th>
<th>Dimension n</th>
<th>Optimal value</th>
<th>Iteration</th>
<th>Time(s)</th>
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<td>0.22646</td>
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<td></td>
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<td></td>
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<td>300</td>
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</tr>
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<td>47.00</td>
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<td>965</td>
<td>106.33</td>
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<tr>
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</table>

Compared with the existing algorithms, the numerical results for examples 1-8 show that the proposed algorithm can be used to globally solve the quadratically inequality constrained quadratic programs with higher computational efficiency.

5 Concluding remarks

In this paper, we propose a new branch-and-bound algorithm for globally solving the quadratically inequality constrained quadratic programs. In this algorithm, we present a new parametric linearizing technique, which can be used to derive the parametric linear programs relaxation problem of the investigated problem (QICQP). To accelerate the computational speed of the proposed branch-and-bound algorithm, an interval deleting rule is used to reduce the investigated box. By subsequently partitioning the initial box and solving a sequence of parametric linear programs relaxation problems, the proposed algorithm is convergent to the global optima of the initial problem (QICQP). Finally, compared with some existing algorithms, numerical results show higher computational efficiency of the proposed algorithm.

Acknowledgement: This paper is supported by the National Natural Science Foundation of China (11671122, 11471102), the China Postdoctoral Science Foundation (2017M622340), the Basic and Frontier Technology Research Program of Henan Province (152300410097), the Cultivation Plan of Young Key Teachers in Colleges and Universities of Henan Province (2016GGJS-107), the Higher School Key Scientific Research Projects of Henan Province (18A110019,17A110021), the Major Scientific Research Projects of Henan Institute of Science and Technology (2015ZD07), the High-level Scientific Research Personnel Project for Henan Institute of Science and Technology (2015037), Henan Institute of Science and Technology Postdoctoral Science Foundation.
References