Vector fields satisfying the barycenter property

Abstract: We show that if a vector field $X$ has the $C^1$ robustly barycenter property then it does not have singularities and it is Axiom A without cycles. Moreover, if a generic $C^1$-vector field has the barycenter property then it does not have singularities and it is Axiom A without cycles. Moreover, we apply the results to the divergence free vector fields. It is an extension of the results of the barycenter property for generic diffeomorphisms and volume preserving diffeomorphisms [1].

Keywords: Barycenter property, Singular point, Generic, Hyperbolic, Axiom A, Anosov

MSC: 37D20, 37C75

1 Introduction

Let $M$ be a closed $n$-dimensional smooth Riemannian manifold, and let $d$ be the distance on $M$ induced from a Riemannian metric $\left\| \cdot \right\|$ on the tangent bundle $TM$, and denote by $C^1(M)$ the set of $C^1$-vector fields on $M$ endowed with the $C^1$-topology. Then every $X \in C^1(M)$ generates a $C^1$-flow $X_t : M \times \mathbb{R} \to M$; that is a $C^1$-map such that $X_t : M \to M$ is a diffeomorphism satisfying $X_0(x) = x$ and $X_{t+s}(x) = X_t(X_s(x))$ for all $t, s \in \mathbb{R}$ and $x \in M$. Let $X_t$ be the flow of $X \in C^1(M)$, and let $\Lambda$ be a $X_t$-invariant compact set. The set $\Lambda$ is called hyperbolic for $X_t$ if there are constants $C > 0$, $\lambda > 0$ and a splitting $T_M = E^s_x \oplus (X(x)) \oplus E^u_x$ such that the tangent flow $DX_t : TM \to TM$ leaves the invariant continuous splitting and

$$\|DX_t|_{E^s_x}\| \leq Ce^{-\lambda t} \text{ and } \|DX_t|_{E^u_x}\| \leq Ce^{\lambda t}$$

for $t > 0$ and $x \in \Lambda$, where $(X(x))$ is the subspace generated by $X(x)$. If $\Lambda = M$, then we say that $X$ is Anosov.

We say that $p \in M$ is a periodic point if it is not a singularity and there exists $T > 0$ such that $X_T(p) = p$. Then the smallest positive $T$ is the period of $p$ and is denoted by $\pi(p)$. The orbit of a periodic point is called a periodic orbit. Denote the set of periodic orbits by $P(X)$. Then the set of critical orbits of $X$ is defined as the set of critical orbits of $X$ is the set $\text{Crit}(X) = \text{Sing}(X) \cup P(X)$, where $\text{Sing}(X)$ is the set of singularities of $X$. We say that $X$ is transitive if there is a point $x \in M$ such that $\omega(x) = M$, where $\omega(x)$ is the omega limit set of $x$.

Remark 1.1. In the study of smooth dynamics, there are many results about the diffeomorphisms which also hold for vector fields without singularity but do not hold for vector fields with singularities (see [2-6]). For example, if a diffeomorphism is a star diffeomorphism then it is $\Omega$-stable (see [2, 4]). However, we know that Lorenz attractor is a star flow, but its non-wandering set is not hyperbolic (see [7]).

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Note that if a vector field $X$ satisfies a star vector field and $\text{Sing}(X) = \emptyset$ then it satisfies both Axiom A and the no-cycle condition (see [3]). Recently, many authors have used the dynamical properties to control of singularities of vector fields (see [6, 8, 9]).

The stability theory is a main topic in differentiable dynamical systems. For instance, Mañé [10] proved that if a diffeomorphism $f$ on a compact smooth manifold $M$ with $\dim M = 2$ is robustly transitive then it is Anosov. For vector fields, Doering [11] proved that if a vector field $X$ on a compact smooth manifold $M$ with $\dim M = 3$ is robustly transitive then it is Anosov. For the types of the pseudo orbit tracing properties (shadowing property, specification property, limit shadowing property, ...), there are close relations between these properties and structural stability hyperbolicity. Lee and Sakai [5] proved that if a vector field without singularities has the $C^1$ robustly shadowing property then it is structurally stable. Arbieto et al. [8] proved that if a vector field $X$ has the $C^1$ robustly specification property then it is Anosov. Lee [6] proved that if a vector field $X$ has the $C^1$ robustly limit shadowing property then it is Anosov.

For a compact invariant set $\Lambda$ of a diffeomorphism $f$, we say that the set $\Lambda$ is robustly transitive if there are a $C^1$ neighborhood $U(f)$ of $f$ and a neighborhood $U$ of $\Lambda$ such that for any $g \in U(f)$, $\Lambda_g(U) = \bigcap_{n \in \mathbb{Z}} g^n(U)$ is transitive for $g$, where $\Lambda_g(U)$ is the continuation of $\Lambda$. Firstly, Abdenur et al. [12] introduced the barycenter property and later, Tian and Sun [13] introduced a new type of the barycenter property. For any two periodic points $p, q \in P(f)$, we say that $p, q$ have the barycenter property if for any $\epsilon > 0$ there exists an integer $N = N(\epsilon, p, q) > 0$ such that for any two integers $n_1 > n_2 > 0$ there is a point $x \in M$ such that

$$d(f^i(x), f^i(p)) < \epsilon, \ -n_1 \leq i \leq 0, \text{ and } d(f^{i+N}(x), f^i(q)) < \epsilon, \ 0 \leq i \leq n_2.$$ 

We say that $f$ has the barycenter property if the barycenter property holds for any two periodic points $p, q \in P(f)$. The barycenter property is not equal to the specification property, and the shadowing property (see [1, Remark 1.1]). In this paper, we use the definition of Tian and Sun [13]. Tian and Sun [13] proved that if a robustly transitive diffeomorphism $f$ on a compact smooth manifold has the barycenter property then it is hyperbolic. Very recently, Lee [1] proved that if a diffeomorphism $f$ has the $C^1$ robustly barycenter property then it is Anosov without cycles.

Using the barycenter property for vector fields, we study a stability theory ($\Omega$-stable) which is a very valuable subject by Remark 1.1. Now we introduce the barycenter property for vector fields. For any critical orbits $\gamma$ and $\eta$, we say that $p \in \gamma, q \in \eta$ have the barycenter property if for any $\epsilon > 0$, there is $T = T(\epsilon, p, q) > 0$ such that for any $\tau > 0$ there is $z \in M$ such that

$$d(X_t(z), X_t(p)) < \epsilon \text{ for } -\tau \leq t \leq 0 \text{ and } d(X_{T+t}(z), X_t(q)) < \epsilon \text{ for } 0 \leq t \leq \tau.$$ 

A vector field $X$ has the barycenter property if the barycenter property holds for any critical orbits $\gamma$ and $\eta$. We say that $X \in \mathcal{X}(M)$ has the $C^1$ robustly barycenter property if there is a $C^1$-neighborhood $U(X)$ of $X$ such that for any $Y \in U(X)$, $Y$ has the barycenter property. Then we have the following:

**Theorem A.** If a vector field $X$ has the $C^1$ robustly barycenter property, then $\text{Sing}(X) = \emptyset$ and $X$ is Axiom A without cycles.

A subset $\mathcal{G} \subset \mathcal{X}^1(M)$ is called residual if it contains a countable intersection of open and dense subsets of $\mathcal{X}^1(M)$. A dynamic property is called $C^1$ generic if it holds in a residual subset of $\mathcal{X}^1(M)$. Arbieto et al. [8] proved that $C^1$ generically, if a vector field $X$ has the specification property then it is Anosov. Ribeiro [14] proved that $C^1$ generically, if a transitive vector field has the shadowing property then it is Anosov and if a vector field has the limit shadowing property then it is Anosov. For that, we have the following which is a result of the paper.

**Theorem B.** For $C^1$ generic $X \in \mathcal{X}^1(M)$, if a vector field $X$ has the barycenter property then $\text{Sing}(X) = \emptyset$ and $X$ is Axiom A without cycles.
2 Proof of Theorem A

Let $M$ be as before, and let $X \in \mathfrak{X}(M)$. For $p \in \gamma \in P(X)$, the strong stable manifold $W^{ss}(p)$ of $p$ and stable manifold $W^s(\gamma)$ of $\gamma$ are defined as follows:

$$W^{ss}(p) = \{ y \in M : d(X_t(y), X_t(p)) \to 0 \ as \ t \to \infty \},$$

and

$$W^s(\gamma) = \bigcup_{t \in \mathbb{R}} W^{ss}(X_t(p)).$$

If $\eta > 0$ then the local strong stable manifold $W^{ss}_{\eta(p)}(p)$ of $p$ and the local stable manifolds $W^s_{\eta(\gamma)}(\gamma)$ of $\gamma$ are defined by

$$W^{ss}_{\eta(p)}(p) = \{ y \in M : d(X_t(y), X_t(p)) < \eta(p), \ if \ t \geq 0 \},$$

and

$$W^s_{\eta(\gamma)}(\gamma) = \{ y \in M : d(X_t(y), X_t(\gamma)) < \eta(\gamma), \ if \ t \geq 0 \}.$$

By the stable manifold theorem, there is $\epsilon = \epsilon(p) > 0$ such that

$$W^{ss}(p) = \bigcup_{t \geq 0} X_{-t}(W^{ss}_\epsilon(X_t(p))).$$

Analogously we can define the strong unstable manifold, unstable manifold, local strong unstable manifold and local unstable manifold. Denote by $\dim W^s(p)$.

If $\sigma$ is a hyperbolic singularity of $X$ then there exists an $\epsilon = \epsilon(\sigma) > 0$ such that

$$W^s_\epsilon(\sigma) = \{ x \in M : d(X_t^s(x), \sigma) \leq \epsilon \ as \ t \geq 0 \}$$

and

$$W^s(\sigma) = \bigcap_{t \geq 0} X_t^s(W^s_\epsilon(\sigma)).$$

Analogous definitions hold for unstable manifolds.

**Lemma 2.1.** Let $\gamma$ and $\eta$ be hyperbolic critical points of $X$. If a vector field $X$ has the barycenter property then $W^s(\gamma) \cap W^u(\eta) \neq \emptyset$ and $W^u(\gamma) \cap W^s(\eta) \neq \emptyset$.

**Proof.** First, we consider periodic orbits. Take $p \in \gamma$ and $q \in \eta$ such that $p$ and $q$ are hyperbolic. Denote by $\epsilon(p)$ the size of the local strong unstable manifold of $p$ and by $\epsilon(q)$ the size of the local strong unstable manifold of $q$. Let $\epsilon = \min(\epsilon(p), \epsilon(q))$ and let $T = T(\epsilon, p, q)$ be given by the barycenter property. For $t > 0$, there is $x_t \in M$ such that $d(X_t(x_t), X_t(p)) \leq \epsilon$ for $-T \leq s \leq 0$ and $d(X_t(x_t), X_t(q)) \leq \epsilon$ for $0 \leq s \leq T$. Since $M$ is compact, there is a subsequence $\{x_{t_n}\} \subset \{x_t\}$ such that $x_{t_n} \to x$ as $t_n \to \infty (n \to \infty)$. Then we have that

$$d(X_{s}(x), X_{s}(p)) \leq \epsilon \ for \ -s \to \infty \ and$$

$$d(X_{s}(x), X_{s}(q)) \leq \epsilon \ for \ s \to \infty.$$

This means that $x \in W^{uu}_\epsilon(p)$ and $X_T(x) \in W^{ss}_\epsilon(q)$. Thus we have $W^u(p) \cap W^s(q) \neq \emptyset$. Similarly, we can show that $W^u(\gamma) \cap W^s(\eta) \neq \emptyset$. Consequently, $W^s(\gamma) \cap W^u(\eta) \neq \emptyset$ and $W^u(\gamma) \cap W^s(\eta) \neq \emptyset$.

Finally, we consider singular points. Let $\sigma$ and $\tau$ be hyperbolic singular points of $X$. As in the first case, we have $W^s(\sigma) \cap W^u(\tau) \neq \emptyset$ and $W^u(\sigma) \cap W^s(\tau) \neq \emptyset$. \hfill $\square$

A singularity $\sigma$ is a sink if all eigenvalues of $D_\sigma X$ have a negative real part. A periodic point $p$ is a sink if the eigenvalues of the derivative of the Poincaré map associated to $p$ have absolute value less than one. A source is a sink for the vector field $-X$.

**Lemma 2.2.** If a vector field $X$ has the barycenter property then $X$ dose not contains a sink and a source.
Lemma 2.5. Let $\mathfrak{M}$ be a $C^1$-neighborhood of $X$ and let $\sigma, \rho \in \text{Crit}(X)$. If $\dim W^s(\sigma) + \dim W^u(\rho) \leq \dim M$ then $W^s(\sigma) \cap W^u(\rho) = \emptyset$. Let $\sigma \in \text{Crit}(X)$ be hyperbolic. Then there exist a $C^1$-neighborhood $U(X)$ of $X$ and a neighborhood $U$ of $\sigma$ such that for any $Y \in U(X)$, there is a $Y$ such that $\gamma_Y$ is the continuation of $\sigma$ and $\text{index}(\gamma_Y) = \text{index}(\sigma) - \text{index}(\gamma_Y)$ (see [16]).

**Lemma 2.4.** Let $\sigma$ and $\tau$ be hyperbolic singular points and let $U(X)$ be a $C^1$ neighborhood of $X$. If a vector field $X$ has the $C^1$ robustly barycenter property, then for any $Y \in U(X)$, we have $\text{index}(\gamma_Y) = \text{index}(\gamma_Y)$, where $\gamma_Y$ and $\gamma_Y$ are the continuations of $\sigma$ and $\tau$, respectively.

**Proof.** Let $\sigma$ and $\tau$ be hyperbolic singular points, and let $U(X)$ be a $C^1$-neighborhood of $X$. Then there is $Y \in U_1(X) \subset U(X)$ such that $\gamma_Y$ and $\gamma_Y$ are the continuations of $\sigma$ and $\tau$, respectively. Since $X$ has the barycenter property, by Lemma 2.2, $X$ has neither sinks nor sources. Thus we may assume that $\sigma$ has index $i$ and $\tau$ has index $j$ with $i \neq j$. If $j < i$ then $W^s(\sigma) + W^s(\tau) < \dim M$. Take $Y \in K \cap U_1(X)$. Then we have $\dim W^s(\sigma_Y) + \dim W^s(\gamma_Y) < \dim M$. Since $Y$ is Kupka-Smale, by Lemma 2.3 we know $W^s(\gamma_Y) \cap W^s(\tau_Y) = \emptyset$. Since $X$ has $C^1$ robustly barycenter property, this is a contradiction by Lemma 2.1. If $j > i$ then $\dim W^u(\sigma) + \dim W^u(\tau) < \dim M$. As in the case of $j < i$, we can take $Y \in K \cap U_1(X)$. Then we have $\dim W^u(\sigma_Y) + \dim W^u(\gamma_Y) < \dim M$. Since $Y$ is Kupka-Smale, by Lemma 2.3 we know $W^u(\gamma_Y) \cap W^u(\tau_Y) = \emptyset$. Since $X$ has $C^1$ robustly the barycenter property, this is a contradiction by Lemma 2.1.

The following was proved by [17, Lemma 1.1] which is Franks’ lemma for singular points.

**Lemma 2.5.** Let $X \in \mathfrak{M}$ and $\sigma \in \text{Sing}(X)$. Then for any $C^1$ neighborhood $U(X)$ of $X$ there are $\delta > 0$ and $\alpha > 0$ such that if $O(\delta) : T_{\sigma} M \to T_{\sigma} M$ is a linear map with $\|O(\delta) - D_{\sigma}X\| < \delta < \delta_0$ then there is $Y^0 \in U(X)$ satisfying

$$Y^0(x) = \begin{cases} (D_{\exp_\sigma}(x)\exp_\sigma \circ O(\delta) \circ \exp_\sigma^{-1}(x), & \text{if } x \in B_{\alpha/\delta}(x) \\ X(x), & \text{if } x \notin B_{\alpha/\delta}(x). \end{cases}$$

Furthermore, $d_0(Y^0, Y^0) \to 0$ as $\delta \to 0$. Here $Y^0$ is the vector field for $O(0) = D_{\sigma}X$ and $d_0$ is the $C^0$ metric.

By Lemma 2.5, $Y^0|_{B_{\alpha/\delta}(\sigma)}$ is regarded as a linearization of $X_{\beta_{\alpha/\delta}(\sigma)}$ with respect to the exponential coordinates. If there is an interval $I \subset \mathbb{R}$ and integral curve $\zeta(t) (t \in I)$ of the linear vector field $O(\delta)$ in $\exp_\sigma^{-1}(B_{\alpha/\delta}(\sigma)) \subset T_{\sigma} M$ then the composition $\exp_\sigma \circ \zeta : I \subset \mathbb{R} \to M$ is an integral curve of $Y^0$ in $B_{\alpha/\delta}(\sigma) \subset M$ (see [17]).

**Lemma 2.6.** Let $U(X)$ be a $C^1$ neighborhood of $X$. If a singular point $\sigma$ is not hyperbolic then there is $Y \in U(X)$ such that $X$ has two hyperbolic singular points with different indices.

**Proof.** Let $U_1(X) \subset U(X)$ be a $C^1$ neighborhood of $X$. Since a singular point $\sigma$ is not hyperbolic we have that $D_{\sigma}X$ has an eigenvalue $\lambda$ with $\Re(\lambda) = 0$. By Lemma 2.5, there is $Y \in U_1(X)$ such that $\sigma_Y$ is a singular point of $Y$ and $\mu$ is the only eigenvalue of $D_{\sigma_Y}Y$ with $\Re(\mu) = 0$. Then $T_{\sigma_Y}M = E_{\sigma_Y}^s \oplus E_{\sigma_Y}^u \oplus E_{\sigma_Y}^u$, where, $E_{\sigma}^u$ is the eigenspace of $D_{\sigma}Y$ associated with real part less than zero $E_{\sigma}^u$ is the eigenspace of $D_{\sigma}Y$ associated with real part greater than zero, and $E_{\sigma}^s$ is the eigenspace of $D_{\sigma}Y$ associated to $\mu$. 


Note that if \( \dim E^c_{\sigma_Y} = 2 \) then there are no singularities of \( Y \) near around \( \sigma_Y \) in the neighborhood of \( \sigma_Y \) (see [8, Theorem 6.2]).

Thus we consider \( \dim E^c_{\sigma_Y} = 1 \). Then there is \( r > 0 \) such that for all \( v \in E^c_{\sigma_Y}(r) \), \( Y(\exp_{\sigma_Y} v) = 0 \), where \( E^c_{\sigma_Y}(r) = E^c_{\sigma_Y} \cap T_{\sigma_Y} M(r) \). We can take \( \tau \in \exp_{\sigma_Y}(E^c_{\sigma_Y}(r)) - \{ \sigma_Y \} \) such that \( \tau \) is sufficiently close to \( \sigma_Y \) and \( \tau \) is not a hyperbolic singularity for \( Y \). We assume that \( \text{index}(\sigma_Y) = \text{index}(\tau) = j \). Then we can make a hyperbolic singular point \( \tau \) which index is different from \( \text{index}(\sigma_Y) = \text{index}(\tau) = j \). By Lemma 2.5, take \( 0 < \alpha < d(\sigma_Y, \tau)/2, 0 < \delta < \delta_0 \) and a linear map \( O : T_{\sigma_Y} M \to T_{\sigma} M \) such that \( O(v) = -\delta v \), for all \( v \in E^c_{\sigma_Y} \), and \( O(v) = D_{\sigma_Y} Y(v) \), for all \( v \in E^r_{\sigma_Y} \oplus E^s_{\sigma_Y} \). By Lemma 2.5, there is \( Z \in U_1(X) \) such that

\[
Z(x) = (D_{\exp^1_{\sigma_Y}(x)} \exp_{\sigma_Y}) \circ O \circ \exp^{-1}_{\sigma_Y}(x), \quad \text{if} \ x \in B_{\alpha/4}(\sigma_Y).
\]

Then there is the singular point \( \sigma_Z \) such that \( \sigma_Z \) is hyperbolic and \( \text{index}(\sigma_Z) = j + 1 \). Since \( Z(x) = Y(x) \) for all \( x \in B_\alpha(\sigma_Y) \), \( \tau \) is a non-hyperbolic singular point for \( Z \) which index is \( j \). Using Lemma 2.5, there are \( W \in C^1 \) close to \( Z(W \in U_1(X)) \) and a linear map \( L : T_{\tau} M \to T_{\tau} M \) such that for some \( 0 < \alpha_1 \leq \alpha \), \( W(x) = (D_{\exp^1_{\sigma_Y}(x)} \exp_{\sigma_Y}) \circ L \circ \exp^{-1}_{\sigma_Y}(x), \quad \text{if} \ x \in B_{\alpha_1/4}(\tau), \) and \( W(x) = Z(x) \) if \( x \not\in B_{\alpha_1}(\tau) \). Then \( \tau \) is hyperbolic singularity for \( W \) which index is \( j \). Thus the vector filed \( W \) has two hyperbolic singular points \( \sigma_Z \) with \( \text{index}(\sigma_Z) = j + 1 \) and \( \tau \) with \( \text{index}(\tau) = j \).

**Proposition 2.7.** If a vector field \( X \in X^1(M) \) has the \( C^1 \) robustly barycenter property then every singular points is hyperbolic.

**Proof.** Suppose, by contradiction, that there is a \( \sigma \in \text{Sing}(X) \) such that \( \sigma \) is not hyperbolic. By Lemma 2.6, there is \( Y \in C^1 \) close to \( X \) such that \( Y \) has two hyperbolic singular points \( \sigma_Y \) and \( \tau \) with different indices which is a contradiction by Lemma 2.4. Thus if a vector filed \( X \) has the \( C^1 \) robustly barycenter property then every singular points are hyperbolic.

**Lemma 2.8.** Let \( \gamma \) and \( \eta \) be hyperbolic periodic orbits and let \( U(X) \) be a \( C^1 \) neighborhood of \( X \). If a vector field \( X \) has the \( C^1 \) robustly barycenter property, then for any \( Y \in U(X) \), we have \( \text{index}(\gamma_Y) = \text{index}(\eta_Y) \), where \( \gamma_Y \) and \( \eta_Y \) are the continuations of \( \gamma \) and \( \eta \), respectively.

**Proof.** Let \( \gamma \) and \( \eta \) be hyperbolic closed orbits, and let \( U(X) \) be a \( C^1 \) neighborhood of \( X \). Then there is \( Y \in U_1(X) \subset U(X) \) such that \( \gamma_Y \) and \( \eta_Y \) are the continuations of \( \gamma \) and \( \eta \), respectively.

Suppose that \( \gamma \) has index \( i \) and \( \eta \) has index \( j \) with \( i \neq j \). If \( j < i \) then \( \dim W^s(\gamma_Y) + \dim W^u(\eta_Y) \leq \dim M \). Take \( Y \in K_S \cap U_1(X) \). Then we have \( \dim W^s(\gamma_Y) + \dim W^u(\eta_Y) \leq \dim M \). Since \( Y \) is Kupka-Smale, by Lemma 2.3 we know \( W^s(\gamma_Y) \cap W^u(\eta_Y) = \emptyset \). Since \( X \) has the \( C^1 \) robustly barycenter property, this is a contradiction by Lemma 2.1. Other case is similar.

The following was proved by [8, Theorem 4.3]. They used the \( C^1 \) robustly specification property. But, the result can be obtained similarly without any properties.

**Lemma 2.9.** Let \( U(X) \) be a \( C^1 \) neighborhood of \( X \) and let \( \gamma \) be a periodic orbit of \( X \). If a periodic point \( p \in \gamma \) is not hyperbolic then there is \( Y \in \mathcal{U}(X) \) such that \( Y \) has two hyperbolic periodic orbits with different indices.

**Proposition 2.10.** Let \( U(X) \) be a \( C^1 \) neighborhood of \( X \). Suppose that \( X \) has the \( C^1 \) robustly barycenter property. Then for any \( Y \in U(X) \), every periodic orbits of \( Y \) is hyperbolic.

**Proof.** Let \( U(X) \) be a \( C^1 \) neighborhood of \( X \). To derive a contradiction, we may assume that there is \( Y \in \mathcal{U}(X) \) such that \( Y \) has not hyperbolic periodic orbits. By Lemma 2.9, \( Y \) has two hyperbolic periodic orbits with different indices. Since \( X \) has the \( C^1 \) robustly barycenter property, this is a contradiction by Lemma 2.8.

**Theorem 2.11.** If a vector field \( X \) has the \( C^1 \) robustly barycenter property then \( \text{Sing}(X) = \emptyset \).

**Proof.** Let \( U(X) \) be a \( C^1 \) neighborhood of \( X \). Suppose that \( \text{Sing}(X) \neq \emptyset \). Then there are a hyperbolic \( \sigma \in \text{Sing}(X) \) with index \( i \) and a hyperbolic periodic orbit \( \gamma \) with index \( j \). Then there is a \( C^1 \) neighborhood \( U_1(X) \subset \mathcal{U}(X) \).
\(\mathcal{U}(X)\) of \(X\) such that for any \(Y \in \mathcal{U}(X)\), there are the continuations \(\sigma_Y\) and \(\gamma_Y\) of \(\sigma\) and \(\gamma\), respectively. Thus we know that \(\dim W^u(\sigma) = \dim W^u(\sigma_Y), \dim W^s(\sigma) = \dim W^s(\sigma_Y), \dim W^s(\gamma) = \dim W^s(\gamma_Y)\) and \(\dim W^u(\gamma) = \dim W^u(\gamma_Y)\).

If \(j < i\) then \(\dim W^u(\sigma) + \dim W^u(\gamma) \leq \dim M\). Take a vector field \(Z \in \mathcal{K}\mathcal{S} \cap \mathcal{U}_i(X)\) such that \(\dim W^u(\sigma_Z) + \dim W^u(\gamma_Z) \leq \dim M\). By Lemma 2.3, \(W^u(\sigma_Z) \cap W^u(\gamma_Z) = \emptyset\). This is a contradiction by Lemma 2.1.

If \(j \geq i\) then \(\dim W^u(\sigma) + \dim W^u(\gamma) \leq \dim M\). As in the case \(j < i\), we can take a vector field \(Y \in \mathcal{K}\mathcal{S} \cap \mathcal{U}_i(X)\) such that \(\dim W^u(\sigma_Y) + \dim W^u(\gamma_Y) \leq \dim M\). By Lemma 2.3, \(W^u(\sigma_Y) \cap W^u(\gamma_Y) = \emptyset\). This is a contradiction. Thus if a vector field \(X\) has the \(C^1\) robustly barycenter property then \(X\) has no singularities. \(\square\)

**Proof of Theorem A.** Suppose that \(X\) has the \(C^1\) robustly barycenter property. Then by Theorem 2.11, \(\text{Sing}(X) = \emptyset\). By Lemma 2.8 and Proposition 2.10, every periodic orbit of \(X\) is hyperbolic. Then by Gan and Wen [3], \(X\) satisfies Axiom A without cycles. \(\square\)

If a vector field \(X\) is transitive, then it is clear that \(\Omega(X) = M\). Thus if a nonsingular vector field satisfies Axiom A then it is Anosov. Then we have the following:

**Corollary 2.12.** If a transitive vector field \(X\) has the \(C^1\) robustly barycenter property then \(X\) is Anosov.

### 3 Proof of Theorem B

In this section, we are going to prove that \(C^1\) generically, if a vector field has the barycenter property, then we show that the vector field satisfies Axiom A and does not contain singularities.

**Theorem 3.1.** There is a residual set \(\mathcal{G}_0 \subset \mathcal{X}^1(M)\) such that for any \(X \in \mathcal{G}_0\), if a vector field \(X\) has the barycenter property then \(\text{Sing}(X) = \emptyset\).

**Proof.** Let \(X \in \mathcal{G}_0 = \mathcal{K}\mathcal{S}\) have the barycenter property. Suppose, by contradiction, that \(\text{Sing}(x) \neq \emptyset\). Then as in the proof of Theorem 2.11, there exist a hyperbolic \(\sigma \in \text{Sing}(X)\) with index \(i\) and a hyperbolic periodic orbit \(\gamma\) with index \(j\). If \(j < i\) then \(\dim W^u(\sigma) + \dim W^u(\gamma) \leq \dim M\). Since \(X \in \mathcal{K}\mathcal{S}\), by Lemma 2.3

\[W^u(\sigma) \cap W^u(\gamma) = \emptyset,\]

which is a contradiction by Lemma 2.1. If \(j \geq i\) then \(\dim W^s(\sigma) + \dim W^u(\gamma) \leq \dim M\). By the previous argument, we get a contradiction. \(\square\)

**Lemma 3.2.** There is a residual set \(\mathcal{G}_0 \subset \mathcal{X}^1(M)\) such that for any \(X \in \mathcal{G}_0\), if a vector field \(X\) has the barycenter property then for any hyperbolic periodic orbits \(\gamma\) and \(\eta\),

\[\text{index}(\gamma) = \text{index}(\eta)\]

**Proof.** Let \(X \in \mathcal{G}_0 = \mathcal{K}\mathcal{S}\) have the barycenter property, and let \(\gamma\) be a hyperbolic periodic orbit with index \(i\) and \(\eta\) be a hyperbolic periodic orbit with index \(j\). Assume that \(i \neq j\). If \(j < i\) then \(\dim W^u(\gamma) + \dim W^s(\eta) \leq \dim M\). Since \(X \in \mathcal{K}\mathcal{S}\), by Lemma 2.3, we have \(W^u(\gamma) \cap W^s(\eta) = \emptyset\). This is a contradiction by Lemma 2.1. If \(j \geq i\) then \(\dim W^u(\gamma) + \dim W^u(\eta) \leq \dim M\). Then the previous argument, we get a contradiction. \(\square\)

**Lemma 3.3 ([8, Lemma 5.1]).** There is a residual set \(\mathcal{G}_1 \subset \mathcal{X}^1(M)\) such that for any \(X \in \mathcal{G}_1\), if for any \(C^1\) neighborhood \(\mathcal{U}(X)\) of \(X\) there is \(Y \in \mathcal{U}(X)\) such that \(Y\) has two distinct hyperbolic periodic orbits with different indices then \(X\) has two distinct hyperbolic periodic orbits with different indices.

We say that a point \(p\) in a hyperbolic periodic orbit of \(X\) has a \(\delta\text{-weak hyperbolic eigenvalue}\) if there is a characteristic multiplier \(\lambda\) of the orbit of \(p\) such that

\[(1 - \delta) < |\lambda| < (1 + \delta).\]
Proposition 3.4. There is a residual set $G_2 \subset \mathcal{X}^1(M)$ such that for any $X \in G_2$, if a vector field $X$ has the barycenter property then there is $\delta > 0$ such that $X$ has no $\delta$-weak hyperbolic eigenvalue.

Proof. Let $X \in G_2 = G_0 \cap G_1$ have the barycenter property. To derive a contradiction, we may assume that for any $\delta > 0$ there is a periodic orbit $\gamma$ of $X$ such that $\gamma$ has a $\delta$-weak hyperbolic eigenvalue. Then there is $Y \in C^1$ close to $X$ such that $Y$ has a non-hyperbolic periodic orbit $\eta$. By Lemma 2.9, there is $Z \in C^1$ close to $Y$ such that $Z$ has two distinct hyperbolic periodic orbits with different indices. By Lemma 3.3, $X$ has two distinct hyperbolic periodic orbits with different indices. This is a contradiction by Lemma 3.2.

Lemma 3.5 ([8, Lemma 5.3]). There is a residual set $G_3 \subset \mathcal{X}^1(M)$ such that for any $X \in G_3$, if for any $\delta > 0$ and for any $C^1$-neighborhood $U(X)$ of $X$ there is $Y \in U(X)$ such that $Y$ has a hyperbolic periodic orbit $\gamma$ which has a $\delta$-weak hyperbolic eigenvalue then $X$ has a hyperbolic periodic orbit $\eta$ which has a $2\delta$-weak hyperbolic eigenvalue.

Proof of Theorem B. Let $X \in G_2 \cap G_3$. Suppose that $X$ has the barycenter property. By Lemma 3.1, $\text{Sing}(X) = \emptyset$. By the result of Gan and Wen [3], we show that every periodic orbits of $X$ is hyperbolic. Assume that there is a periodic orbit $\gamma$ of $X$ such that for any $\delta > 0$, $\gamma$ has a $\delta/2$-weak hyperbolic eigenvalue. Since $X \in G_3$, $X$ has a hyperbolic periodic orbit $\eta$ which has a $2\delta$-weak hyperbolic eigenvalue. Since $X$ has the barycenter property, $X$ has no $\delta$-weak hyperbolic eigenvalue. This is a contradiction by Proposition 3.4. Since $\text{Sing}(X) = \emptyset$ and every periodic orbits of $X$ is hyperbolic, by Gan and Wen [3], $X$ is Axiom A without cycle.

Corollary 3.6. For $C^1$ generic $X \in \mathcal{X}^1(M)$, if a transitive vector field $X$ has the barycenter property then $X$ is Anosov.

Let $M$ be a closed, connected and smooth $n(\geq 3)$-dimensional Riemannian manifold endowed with a volume form, which has a measure $\mu$, called the Lebesgue measure. Given a $C^r (r \geq 1)$ vector field $X : M \to TM$ the solution of the equation $x' = X(x)$ generates a $C^r$ flow, $X_t$; by the other side given a $C^r$ flow we can define a $C^{r-1}$ vector field by considering $X(x) = \frac{dX_t(x)}{dt}|_{t=0}$. We say that $X$ is divergence-free if its divergence is equal to zero. Note that, by Liouville formula, a flow $X_t$ is volume preserving if and only if the corresponding vector field, $X$, is divergence free. Let $\mathcal{X}^1_{\mu}(M)$ denote the space of $C^r$ divergence free vector fields and we consider the usual $C^1$ Whitney topology on this space. A vector field $X \in \mathcal{X}^1_{\mu}(M)$ is a divergence-free star vector field if there exists a $C^1$ neighborhood $U(X)$ of $X$ in $\mathcal{X}^1_{\mu}(M)$ such that if $Y \in U(X)$ then every point in $P(X) \cup \text{Sing}(X)$ is hyperbolic.

Theorem 3.7. If a divergence-free vector field $X \in \mathcal{X}^1_{\mu}(M)$ has the $C^1$ robustly barycenter property then $\text{Sing}(X) = \emptyset$ and $X$ is Anosov.

Proof. By Ferreira [18, Theorem 1], if a divergence free vector field $X \in \mathcal{X}^1_{\mu}(M)$ satisfies star vector fields then $\text{Sing}(X) = \emptyset$ and it is Anosov. To prove Theorem 3.7 we show that a divergence free vector field $X$ satisfies a star condition. It is almost similar to prove of Theorem A. Thus as in the proof of Theorem A, if a divergence free vector field $X$ has the $C^1$ robustly barycenter property then $X$ is Axiom A without cycles, that is, $X$ satisfies a star condition. This is a proof of Theorem 3.7.

Bessa et al [19] proved that $C^1$ generically, if a divergence free vector field $X$ has the shadowing property(expansive, specification property) then it is Anosov. From the results, we are going to prove $C^1$ generic divergence free vector fields when it has the barycenter property.

Theorem 3.8. For $C^1$ generic $X \in \mathcal{X}^1_{\mu}(M)$, if $X$ has the barycenter property then $\text{Sing}(X) = \emptyset$ and $X$ is Anosov.

Proof. By Ferreira [18, Theorem 1], if a divergence-free vector field $X \in \mathcal{X}^1_{\mu}(M)$ satisfies star vector fields then $\text{Sing}(X) = \emptyset$ and it is Anosov. By Bessa [20], $C^1$ generically, a divergence free vector field $X \in \mathcal{X}^1_{\mu}(M)$ is transitive. Therefore, we show that $C^1$ generically, if a divergence-free vector field $X$ has the barycenter
property then $X$ satisfies star vector fields. Thus as in the proof of Theorem B, $C^1$ generically, if a divergence free vector field $X \in \mathfrak{X}^1_\mu(M)$ has the barycenter property then it satisfies a star vector field, and so, $\text{Sing}(X) = \emptyset$ and it is Anosov.

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References